642.583 Problem Set 2 (final)

1. Show that for $A \subseteq \mathbb{Q}^n$ of size $a$ (and $\log = \log_2$),

$$|\nabla(A)| \geq a(n - \log a).$$

[Your proof should not give (or depend on) the exact bound stated in class.]

2. Let $G$ be the Hamming graph on $V = \{0, 1\}^n$ ($=: \mathbb{Q}^n$). Show: if $X \subseteq V$ has $|X| > 2^{n-1}$, then some component of $G[X]$ meets every facet.

[And note $X = \{\text{odd vertices}\}$ has size $2^n - 1$ and only singleton components.]

[As usual $G[Z]$ is the subgraph induced by $Z$; feel free to (abusively) use $Z$ for $G[Z]$. A facet is one of the sets $\{x : x_i = \varepsilon\}$ with $i \in [n]$ and $\varepsilon \in \{0, 1\}$. Possibly helpful: for $i \in [n]$ and $u \in V$, use $u^i$ for the $v$ with $v_j = u_j$ iff $j \neq i$, and note a connected $Z \subseteq V$ meets every facet iff

$$(\ast) \text{ for each } j \in [n], Z \text{ contains } \{u, u^j\} \text{ for some } u.$$]

This one may be relatively tough. You’ll again want some kind of shift. Once you find it, part of the challenge of the problem is figuring out a reasonable way to say why it does what you say it does—no essays please.]

[In problems 3 and 4 (and more generally) you can omit justification of “obvious” identities.]

3. Fix $0 \leq t \leq r \leq n$. [Also, let $M(i, j)$ be the incidence matrix used in class, and assume $R, T$ and $U$ range over subsets of $[n]$ of sizes $r, t$ and $t - 1$ respectively.] Finish Lovász’s proof of EKR by showing that if $M(t-1, t)x = 0$, then for any $R$,

$$\sum_{T \cap R = \emptyset} x_T = (-1)^t \sum_{T \subseteq R} x_T.$$  

4. Suppose $A_1, \ldots, A_m, B_1, \ldots, B_m \subseteq [n]$ satisfy

(a) $A_k \cap B_k = \emptyset \ \forall k \in [m]$,

(b) for all distinct $i, j \in [n]$ there is a unique $k \in [m]$ such that $(i, j)$ lies in one of $A_k \times B_k, B_k \times A_k$.

Show that $m \geq n - 1$. (You should also check that $m = n - 1$ is achievable, but this isn’t part of the problem.)
[Hint: Consider the \([n] \times [n]\) matrix \(M = \sum_{k \in [n]} M_k\), where \(M_k(i, j) = \begin{cases} 1 & \text{if } (i, j) \in A_k \times B_k, \\ 0 & \text{otherwise}. \end{cases}\)

5. Suppose \(X \subseteq \mathbb{Z}_3^n\) has the property that for all distinct \(x, y \in X\) there is an \(i\) such that \(y_i = x_i + 1\) (addition in \(\mathbb{Z}_3\) of course). Then \(|X| \leq 2^n\).

[More general (not required but you could try): Let \(q\) be a prime power and \(D\) a \(d\)-subset of \(\mathbb{F}_q^n\), and suppose \(X \subseteq \mathbb{F}_q^n\) satisfies: for all distinct \(x, y \in X\) there is some \(i\) for which \(y_i - x_i \in D\). Then \(|X| \leq (d + 1)^n\).]

6. Show that it’s not possible to cover \(\{0, 1\}^n \setminus \{0\}\) (\(\subseteq \mathbb{R}^n\)) by fewer than \(n\) affine hyperplanes not containing 0.

[An affine hyperplane is \(\{x \in \mathbb{R}^n : \langle a, x \rangle = b\}\) with \(a \in \mathbb{R}^n \setminus \{0\}\) and \(b \in \mathbb{R}\).

Cryptic comments: This shares some ideas with things we’ve seen, but not the idea of mapping to a set of linearly independent vectors. You may want to start by proving some general supporting claim.]

7. Let \(\mu\) be uniform measure on \(2^n\), let \(A_1, \ldots, A_r \subseteq 2^n\) be increasing with \(\max \mu(A_i) \leq 1/2\), and set 
\[
\mathcal{A} = \{X \subseteq [n] : \exists! i \in [r] \text{ with } X \in A_i\} \quad (= \cup_i (A_i \setminus \cup_{j \neq i} A_j)).
\]

Show that \(\mu(\mathcal{A}) < 1 - c\) for some fixed positive \(c\).

[Open and very interesting: if \(\max \mu(A_i) < \varepsilon\), is it true that \(\mu(\mathcal{A}) < e^{-1} + \delta\varepsilon\) with \(\delta\varepsilon \to 0\) as \(\varepsilon \to 0\)?

(Why would \(e^{-1}\) be best possible?)]