642.582 Problem Set 2 (final)

1. Show that there is a fixed $C > 1$ such that $|\text{End}(P)| > C^n$ for each $n > 1$ and poset P of size n (i.e. with ground set of size n).

[For simplicity let's restrict to P's in which each element is comparable to at least one other. (Of course "isolated" elements just make it easier, right?) Like all problems here, *this doesn't require a long solution.*]

2. [Recall from class (and probably elsewhere): For a graph G, the *matching number*, $\nu(G)$, is the maximum size of a *matching* (set of disjoint edges); the *(vertex) cover* number, $\tau(G)$, is the minimum size of a vertex cover (set of vertices meeting all edges); and $\nu(G) \le \tau(G)$ is trivial.]

Derive Dilworth's Theorem from

König's Theorem. If G is bipartite then $\tau(G) = \nu(G)$.

3. Determine the possibilities for equality in Sperner's Theorem.

4. Suppose G is a graph on $[n]$ with the property that each $e \in \binom{[n]}{2} \setminus E(G)$ belongs to a copy of K_s in $G + e$ ($= G$ with e added, of course). Show that

$$|E(G)| \ge \binom{n}{2} - \binom{n-s+2}{2}.$$

[Not to be handed in: why is this sharp?]

5. Given a positive integer k, let $\mathcal{A} = \binom{[2k-1]}{k}$ and $\mathcal{B} = \binom{[2k-1]}{k-1}$, and let Γ_k be the bigraph on $\mathcal{A} \cup \mathcal{B}$ with $A \sim B$ if $A \supseteq B$ (where $A \in \mathcal{A}$ and $B \in \mathcal{B}$). With $v(k) = |V(\Gamma_k)|$, let $a(k)$ be the maximum size of a set of vertices of $\Gamma(k)$ containing no hexagon and $\alpha(k) = a(k)/v(k)$. Show $\alpha(k+1) \le \alpha(k)$.

[This is pretty easy (eventually), but saying it nicely might take some thought. You can skip obvious justifications—I'm more interested in the idea.]

6. For $\mathcal{F} \subseteq 2^V$ and $X \subseteq V$, the *trace* of $\mathcal{F}$ on X is

$$\text{Tr}(\mathcal{F}, X) = \{A \cap X : A \in \mathcal{F}\},$$

and $\mathcal{F}$ *shatters* X if $\text{Tr}(\mathcal{F}, X) = 2^X$. Let $\mathcal{S}(\mathcal{F})$ be the family of sets shattered by $\mathcal{F}$. Prove that (for any $\mathcal{F}$)

$$|\mathcal{S}(\mathcal{F})| \ge |\mathcal{F}|. \tag{1}$$

[Note $\mathcal{S}(\mathcal{F}) = \mathcal{F}$ if $\mathcal{F}$ is an *ideal* (that is, if $A \subseteq B \in \mathcal{F} \;\Rightarrow\; A \in \mathcal{F}$). Your proof here should not use linear algebra (cf. Problem 8).]

7. Suppose $\mathcal{H} \subseteq 2^V$ is d-regular with $d \geq 2$, and

$$|A \cap B| = \lambda \text{ for all distinct } A, B \in \mathcal{H}.$$

Show that $\mathcal{H}$ is uniform.

8. Give another proof of (1), as follows. Let M be the $\mathcal{F} \times \mathcal{S}(\mathcal{F})$ *inclusion matrix*; that is,

$$M(F,S) = \begin{cases} 1 & \text{if } F \supseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Show that the rows of M are linearly independent over the reals; that is, there is no $\underline{0} \neq \mu : \mathcal{F} \to \mathbb{R}$ satisfying

$$\sum_{A \supseteq I} \mu_A = 0 \quad \forall I \in \mathcal{S}.$$