1. (a) For fixed \( k \) and \( n \to \infty \), give (and justify) asymptotics for \( |s(n, k)| \).
(b) (bonus) How large can \( k \) be without invalidating the answer in (a)?
[For this problem let’s say \( m! \) is legal in asymptotic expressions.]

2. Find a closed form for \( \sum_k c(n, k)2^k \) (where \( c(n, k) \) is the number of \( k \)-cycle permutations of \([n]\)).

3. Let \( F_n \) be the \( n \)th Fibonacci number (with \( F_0 = F_1 = 1 \)). With \((a_1, \ldots, a_k)\) ranging over compositions of \( n \), prove combinatorially that

\[
F_{2n-1} = \sum a_1a_2 \cdots a_k.
\]
[E.g. \( F_7 = 4 + 3 \cdot 1 + 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 1 \cdot 1 \). Here I’ll settle for the correspondence without a lot of justification.]

4. Use the generating function \( \prod_{j \geq 0} (1 + x^{2^j})^{-1} \) to show that for \( n \geq 2 \), the number of partitions of \( n \) into powers of 2 is even.
[E.g. for \( n = 4 \), the relevant partitions are 4, 22, 211 and 1111.]

5. For a partition \( \lambda \), let \( u(\lambda) \) and \( v(\lambda) \) be (respectively) the number of times 1 appears as a part of \( \lambda \) and the number of different integers that appear as parts of \( \lambda \). Use 2-variable generating functions to show that (for any \( n \))

\[
\sum_{\lambda \vdash n} u(\lambda) = \sum_{\lambda \vdash n} v(\lambda).
\]
[This one could be relatively tough and may take a little more space than most (e.g. a pretty compact half page in the solutions). A little differentiation may be helpful. It can also be done “combinatorially,” but that’s not what’s asked for here.]

6. Let \( V = V_1 \cup \cdots \cup V_k \) be a partition with \( |V_i| = n \ \forall i \) and say \( T \in \binom{V}{k} \) is a transversal if it meets every \( V_i \). Show that if \( h : \binom{V}{k} \to \mathbb{R} \) satisfies \( h(T) = 1 \) for each transversal \( T \), then there is some \( S \subseteq V \) with

\[
|h(S)| \geq c_k n^k,
\]
where $c_k > 0$ depends only on $k$ and $\widetilde{h}(S) = \sum \{ h(T) : T \subseteq S, |T| = k \}$.  
[For $X \subseteq \binom{V}{k}$ please use $h(X) = \sum_{E \in X} h(E)$.]

7. Suppose $p_I \in \mathbb{R}$ for $I \subseteq [n]$. Show that there are $A; A_1, \ldots, A_n \subseteq A$; and a probability measure on $A$ with

$$
\mathbb{P}(A_I) = p_I \forall I \subseteq [n]
$$

(where $A_I = \cap_{i \in I} A_i$) if and only if

$$
\sum_{K \supseteq I} (-1)^{|K|-|I|} p_K \geq 0 \quad \forall I \subseteq [n] \quad (1)
$$

and

$$
p_\emptyset = 1. \quad (2)
$$

[Let’s make this a 2-part problem: say necessity is (a) and sufficiency (b). You’ll want to use inclusion-exclusion for probability measures; we’ll get to this first thing on Monday, but you can also find it on page 11 of the posted Lecture 9 (and anyway it works exactly like I-E as already discussed in class).]