642.582 Problem Set 1 (final installment)

[Please see the homework guidelines on the course page.
If something seems wrong, *please ask* before wasting a lot of time on it.]

1. Find (with justification) the number of sequences $\Pi_1, \ldots, \Pi_n$ satisfying

(i) for each $i$, $\Pi_i$ is an unordered partition of $[n]$ into $i$ nonempty blocks (so $\Pi_1 = \{[n]\}$ and $\Pi_n$ is the partition into singletons), and

(ii) for $i = 2, \ldots, n$, $\Pi_i$ is gotten from $\Pi_{i-1}$ by splitting some block (necessarily of size at least 2) into two nonempty blocks.

2. For $x, n \in \mathbb{N}$, give a *simple* (one sentence?) explanation for the identity

$$\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n}.$$

3. Let $X = \{x_1, \ldots, x_n\}$ with the $x_i$'s totally ordered by some *unknown* "$\prec$." For $A \subseteq X$, let $\max(A)$ be the largest (under $\prec$) of the $x_i$'s in $A$.

Show that for any given $A_1, \ldots, A_m \subseteq X$, the number of possibilities for

$$\underline{v} := (\max(A_1), \ldots, \max(A_m))$$

is less than $2^{m+n}$.

[If you find the solution you'll probably get a somewhat better bound. For perspective, note $n!$ is a trivial upper bound, but is much larger than $2^{m+n}$ if $m$ isn't too big relative to $n$. This one could be challenging, though the proof is pretty simple. In writing, remember you needn't justify the obvious, and aim for a clear, compact explanation of why it's true.]

4. Suppose $A_1, \ldots, A_m, B_1, \ldots, B_m \subseteq X$ satisfy

$$|A_I| = |B_I| \ \forall I \subset [m] \quad \text{and} \quad |A_{[m]}| \neq |B_{[m]}|$$

(where $A_I = \cap_{i \in I} A_i$ and similarly for $B_I$). **Use I-E** to show that $|X| \geq 2^{m-1}$.

[Exercise (not to be handed in): the bound is best possible (for every $m$).]

5. Let $V = V_1 \cup \cdots \cup V_k$ be a partition with $|V_i| = n \ \forall i$ and say $T \in \binom{V}{k}$ is a *transversal* if it meets every $V_i$. Show that if $h : \binom{V}{k} \to \mathbb{R}$ satisfies $h(T) = 1$ for each transversal $T$, then there is some $S \subseteq V$ with

$$|\overline{h}(S)| \geq c_k n^k,$$

where $c_k > 0$ depends only on $k$ and $\overline{h}(S) = \sum\{h(T) : T \subseteq S, |T| = k\}$.

[For $X \subseteq \binom{V}{k}$ please use $h(X) = \sum_{E \in X} h(E)$.]

6. Let $A_i = A_i^{(n)}$ be independent events with $X_i = \mathbf{1}_{A_i}$ and $\mathbb{P}(A_i) = p_i$, and set $X = \sum X_i$. Show that if $X \xrightarrow{\text{d}} \text{Po}(\mu)$ for a fixed positive $\mu$, then (i) $\sum p_i \to \mu$ and (ii) $\max p_i \to 0$. (As usual, $X$ and $p_i$ are really $X^{(n)}$ and $p_i^{(n)}$.)

[This isn't necessarily easy to find, but should *not* require a long argument. Try starting by writing expressions for $\mathbb{P}(X = 0)$ and $\mathbb{P}(X = 1)$.
For perspective (repeat of a remark from class and not needed here): for *general* $\mathbb{N}$-valued r.v.'s $X = X^{(n)}$, $X \xrightarrow{\text{d}} \text{Po}(\mu)$ does *not* imply $\mathbb{E}X \to \mu$.]