642.583 Problem Set 1 (first installment)

1. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to shatter $I \subseteq [n]$ if every subset of $I$ is $I \cap A$ for some $A \in \mathcal{F}$. Given $\mathcal{F}$, let $\mathcal{S}$ be the family of sets shattered by $\mathcal{F}$. Give two proofs of the inequality

$$|\mathcal{S}| \geq |\mathcal{F}|$$

(note this is exact whenever $\mathcal{F}$ is an ideal), as follows.

(a) Let $\mathcal{B} = \{ I \subseteq [n] : I \text{ not shattered by } \mathcal{F} \}$ and show $|\mathcal{B}| + |\mathcal{F}| \leq 2^n$:

Work in $R := \mathbb{Z}_2[x_1, \ldots, x_n]$. Regard $\mathcal{F}$ as a set of $\{0, 1\}$-vectors in the usual way, and for each $a \in \mathcal{F}$ set $f_a(x) = \prod(x_i - a_i + 1)$. Find polynomials $g_I$ ($I \in \mathcal{B}$) for which $\{f_a : a \in \mathcal{F}\} \cup \{g_I : I \in \mathcal{B}\}$ is independent.

(b) Let $M$ be the $\mathcal{F} \times \mathcal{S}$ inclusion matrix; that is,

$$M(F,S) = \begin{cases} 
1 & \text{if } F \supseteq S \\
0 & \text{otherwise.}
\end{cases}$$

Show that the rows of $M$ are linearly independent over the reals; in other words, there is no $0 \neq \mu : \mathcal{F} \to \mathbb{R}$ satisfying

$$\sum_{A \supseteq I} \mu_A = 0 \quad \forall I \in \mathcal{S}.$$  

[Hint: consider a minimal $I \subseteq [n]$ for which (2) fails.]

[Remark: neither of (a),(b) is the easiest way to prove (1).]