1. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to **shatter** $I \subseteq [n]$ if every subset of $I$ is $I \cap A$ for some $A \in \mathcal{F}$. Given $\mathcal{F}$, let $\mathcal{S}$ be the family of sets shattered by $\mathcal{F}$. Give two proofs of the inequality

$$|\mathcal{S}| \geq |\mathcal{F}| \quad (1)$$

(note this is exact whenever $\mathcal{F}$ is an ideal), as follows.

(a) Let $\mathcal{B} = \{ I \subseteq [n] : I \text{ not shattered by } \mathcal{F} \}$ and show $|\mathcal{B}| + |\mathcal{F}| \leq 2^n$:

Work in $R := \mathbb{Z}_2 = \mathbb{Z} / 2\mathbb{Z}[x_1, \ldots, x_n]$. Regard $\mathcal{F}$ as a set of $\{0, 1\}$-vectors in the usual way, and for each $a \in \mathcal{F}$ set $f_a(x) = \prod (x_i - a_i + 1)$. Find polynomials $g_I$ ($I \in \mathcal{B}$) for which $\{f_a : a \in \mathcal{F}\} \cup \{g_I : I \in \mathcal{B}\}$ is independent.

(b) Let $M$ be the $\mathcal{F} \times \mathcal{S}$ inclusion matrix; that is,

$$M(F, S) = \begin{cases} 1 & \text{if } F \supseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Show that the rows of $M$ are linearly independent over the reals; in other words, there is no $0 \neq \mu : \mathcal{F} \to \mathbb{R}$ satisfying

$$\sum_{A \supseteq I} \mu_A = 0 \quad \forall I \in \mathcal{S}. \quad (2)$$

[Hint: consider a minimal $I \subseteq [n]$ for which (2) fails.]

[Remark: neither of (a),(b) is the easiest way to prove (1).]

2. Show it’s not possible to cover $\{0, 1\}^n \setminus \{0\}$ ($\subseteq \mathbb{R}^n$) by fewer than $n$ affine hyperplanes not containing $0$.

[Recall that an affine hyperplane is $H = \{ x \in \mathbb{R}^n : \langle a, x \rangle = b \}$ for some $0 \neq a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Of course $H \nsubseteq 0$ iff $b \neq 0$.]

3. For $X \subseteq [n]$ and $i \in [n]$, let $X + i = \{x + i : x \in X\}$, where addition is modulo $n$. Here are two old conjectures (the first reminiscent of EKR):

**Conjecture 1.** Let $X$ be a $k$-subset of $[n]$ and suppose $\mathcal{F} \subseteq 2^{[n]}$ satisfies

$$\forall A, B \in \mathcal{F}, \ A \cap B \supseteq X + i \text{ for some } i \in [n].$$

Then $|\mathcal{F}| \leq 2^{n-k}$. 

1
[Not to be handed in: this would be best possible.]

Conjecture 2. For any \( k \)-subset \( X \) of \([n]\), there is a \( k \times n \) \( \{0,1\} \)-matrix \( M \) such that for each \( i \in [n] \) the columns of \( M \) indexed by \( X + i \) are linearly independent over \( \mathbb{F}_2 \).

Show Conjecture 2 implies Conjecture 1. (Full credit—and some additional benefits—for proving Conjecture 1 without using Conjecture 2.)

4. (Recall that the \textit{convex hull}, \( \text{conv}(X) \), of \( X \subseteq \mathbb{R}^d \) is the intersection of all convex sets containing \( X \); equivalently, it is the set of all convex combinations of members of \( X \).) Use the fact that a system of \( m \) homogeneous linear equations in more than \( m \) variables has a nontrivial solution to prove

\textit{Radon’s Theorem}. For any \( x_1,\ldots,x_{d+2} \in \mathbb{R}^d \), there are disjoint \( I,J \subseteq [d+2] \) with

\[
\text{conv}(x_i : i \in I) \cap \text{conv}(x_i : i \in J) \neq \emptyset.
\]

5. Say \( \mathcal{A}, \mathcal{B} \subseteq 2^{[n]} \) are \textit{cross-intersecting} if

\[
A \cap B \neq \emptyset \quad \forall A \in \mathcal{A}, \ B \in \mathcal{B}.
\]

Use Kruskal-Katona to show that if \( k+l \leq n \) \( (k,l \geq 1) \) and \( \mathcal{A} \subseteq \binom{n}{k}, \mathcal{B} \subseteq \binom{n}{l} \) are cross-intersecting, then either \( |\mathcal{A}| \leq \binom{n-1}{k-1} \) or \( |\mathcal{B}| < \binom{n-1}{l-1} \).

[Another way (but not here please): shifting as in the proof of EKR.]