582 Suggested problems

The first three problems are from Stanley, Chapter 1 (Problems 1(a,b,h,i,j), 2(a,b) and 12); number 4 is Lovász, Chapter 1, Problems 42(e,h); and 3(b) is van Lint-Wilson, Problem 13D. The other parts of Problems 1 and 2 of Stanley are also worth a look.

1. For each of the following find as simple a solution as possible.

(a) How many subsets of the set [10] contain at least one odd integer?

(b) In how many ways can seven people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same sides)?

(c) How many compositions of 19 use only the parts 2 and 3?

(d) In how many different ways can the letters of the word MISSISSIPPI be arranged if the four S's cannot appear consecutively?

(e) Find the number of sequences $(a_1, \ldots, a_{12})$ consisting of four 0's and eight 1's, and having no two consecutive 0's.

2. Give *combinatorial* proofs (assuming all variables are nonnegative integers):

(a) $\sum_{i=0}^{n} \binom{x+i}{i} = \binom{x+n+1}{n}$

(b) $\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$

3.(a) How many sequences $(T_1, \ldots, T_k)$ of subsets of $[n]$ satisfy $T_1 \subseteq \cdots \subseteq T_k$?

(b) If S is the set of all ordered k-tuples $\mathcal{A} = (A_1, \ldots, A_k)$ of subsets of $[n]$, what is

$$\sum_{\mathcal{A} \in S} |A_1 \cup \cdots \cup A_k|?$$

4. Find a closed form for each sum:

(a) $\displaystyle\sum_{i=m}^{n} \binom{n}{i}\binom{i}{m}$ (combinatorial proof preferred)

(b) $\displaystyle\sum_{i=0}^{m} (-1)^i \binom{n}{i}$

[Suggestion: examples will help you guess the answers—*of course this is good advice in general.*]

5. For fixed k, find (and justify) an asymptotic expression for $S(n,k)$.