587 Problems

[First installment; at a guess: #1 is easy; # 2 is not hard and easy once found; #3 still easy, with some thought; #4 getting more ... interesting.]

1. Slightly modifying the notation from class, define a *tournament on* V to be $T = (V, A)$ with A (for "arcs") a subset of $V \times V$ and $xy \in A$ iff $yx \notin A$. A *subtournament* of T is then $T|_W = (W, A \cap (W \times W))$ for some $W \subseteq V$.

A tournament (V, A) is *transitive* if $xy, yz \in A \Rightarrow xz \in A$. Define $v(n)$ to be the largest k such that every tournament on n vertices has a transitive subtournament on k vertices. Show

(a) $v(n) \geq \lfloor \log_2 n \rfloor + 1$ (this part not random), and

(b) $v(n) \leq \lfloor 2 \log_2 n \rfloor + 1$.

2. Let $A_1, \ldots, A_n$ be events in a probability space, set $\mu = \sum \mathbb{P}(A_i)$, and let Q_l be the event that some l *independent* A_i's occur. Show $\mathbb{P}(Q_l) \leq \mu^l / l!$.

3. A *dominating set* in a graph G on V is $U \subseteq V$ such that each vertex of $V \setminus U$ has at least one neighbor in U. For any n and $1 < \delta \in \mathbb{N}$, find $\alpha = \alpha(\delta)$ as small as you can such that every G on n vertices with minimum degree at least δ has a dominating set of size at most αn.

[This is AS Theorem 1.2.2, but a good exercise. To make things a bit easier, check AS for α and then try to show it works.]

4. Let $G = (V, E)$ be a graph with $|V| = n$ and minimum degree δ. Show there is a partition $V = A \cup B$ such that each vertex of B has at least one neighbor in each of A, B, and $|A| < O(\frac{n \ln \delta}{\delta})$.

[AS, Problem 1.7.4; note they give it a $(*)$.]

[Second installment; just a little practice with calculations, as mentioned in class.]

5. (a) Use the "deletion method" to improve Erdős' lower bound on $R(k, k)$.

(b) Prove $R(k, 2k) = \Omega(k\alpha^k)$ with α as large as you can make it.

(c) Bound $R(3, t)$ and $R(4, t)$ from below, using the deletion method and trying to optimize the constant factors. You should get

$$R(3, t) > (1 - o(1)) \tfrac{2\sqrt{2}}{3} \left(\tfrac{t}{\ln t}\right)^{3/2} \quad \text{and} \quad R(4, t) > (1 - o(1)) \tfrac{3\sqrt[3]{6}}{16} \left(\tfrac{t}{\ln t}\right)^{2}.$$

[Third installment. Problem 7 (if new to you) is a lovely use of clever conditioning.]

6. Recall (or see AS, Sec. 1.3) that a hypergraph $\mathcal{H}$ on V has *Property B*—or is *2-colorable*—if there's a partition $V = R \cup B$ such that

$$A \cap R \neq \emptyset \neq A \cap B \ \ \forall A \in \mathcal{H}.$$

Let $m(n)$ be the least size of an n-uniform $\mathcal{H}$ that does *not* have Property B, and let $g(n)$ be the least size of an n-uniform $\mathcal{H}$, on some V (say, to avoid irrelevancies, with $m := |V|$ even), such that each $S \in \binom{V}{m/2}$ contains a member of $\mathcal{H}$. (So $m(n) \leq g(n)$, right?) Show

(a) $g(n) = O(n^2 2^n)$ (so also $m(n) = O(n^2 2^n)$;

(b) $g(n) = \Omega(n 2^n)$.

[The more interesting of these is (b). The bound in (a), from Erdős 1964, is still the best known upper bound on $m(n)$. (See AS, Thm. 1.3.2 for a more precise version, but note some of the effort there is unnecessary for (a).)

Estimating $m(n)$ is a classic problem. See AS, Cor. 3.5.2 for the current lower bound (a beautiful argument that was already mentioned in class). To appreciate that the current gap is large, it's natural to consider $m(n)2^{-(n-1)}$; so we're asking, how large the *expected number of monochromatic edges in a random coloring* (namely $|\mathcal{H}|2^{-(n-1)}$) must be to guarantee that there's no good coloring, and we only know that the answer is somewhere between $\Omega(\sqrt{n/\log n})$ and $O(n^2)$.

Hints: for (a) choose a random $\mathcal{H}$; for (b) use something with deletions to produce a bad S.]

7. Let $\mathcal{H}$ be a hypergraph on $[n]$. Let the "weight" function $\mathsf{w} : [n] \to [k]$ be chosen uniformly at random, and let Q be the event that there's a unique edge of minimum weight; that is,

$$Q = \{\exists A \in \mathcal{H},\ \mathsf{w}(A) < \mathsf{w}(B)\ \forall A \neq B \in \mathcal{H}\}$$

(where the *weight* of A is $\mathsf{w}(A) = \sum_{x \in A} \mathsf{w}(x)$).

(a) Show $\mathbb{P}(Q) > 1 - n/k$.

(b) Improve this to $1 - n/(2k)$.

[Fourth installment.]

8. An important though trivial fact (and AS, Theorem 2.2.1): for any graph G on vertex set V, there is a partition $X \cup Y$ of V with

$$|\nabla_G(X,Y)| \geq |G|/2$$

(where $\nabla_G(X,Y) = \{e \in G : e \cap X \neq \emptyset \neq e \cap Y\}$ and $|G| = |E(G)|$).

Show that if G and H are graphs on V and $\min\{|G|,|H|\}$ is sufficiently large, then there is a partition $X \cup Y$ of V with

$$|\nabla_G(X,Y)| \geq .49|G| \quad \text{and} \quad |\nabla_H(X,Y)| \geq .49|H|.$$

9. For $i \in [n]$, let $v_i = (x_i, y_i) \in \mathbb{Z}^2$ with each of $|x_i|, |y_i|$ at most $2^{n/2}/(100\sqrt{n})$. Show that there are disjoint $I, J \subseteq [n]$ with $\sum_{i \in I} v_i = \sum_{i \in J} v_i$.

[AS, 4.8.5]

10. Show that there is a positive constant c for which the following holds. If $a_1, \ldots, a_n \in \mathbf{R}$ satisfy $\sum a_i^2 = 1$, and $\varepsilon_1, \ldots, \varepsilon_n$ are chosen uniformly and independently from $\{\pm 1\}$, then $\mathbb{P}(|\sum \varepsilon_i a_i| \leq 1) \geq c$.

[AS, 4.8.2 (with a $(*)$). The old conjecture that one can take $c = 1/2$ (note this is best possible) was proved by Keller and Klein in 2020 (in 76 pages).]

11. Let $D_{n,p}$ be the random digraph on $[n]$ in which each arc is present with probability p, independent of other choices. Show

$$\mathbb{P}(D_{n,p} \text{ is Hamiltonian}) \geq \mathbb{P}(G_{n,p} \text{ is Hamiltonian}).$$

[Of course for digraphs, "cycle" means *directed* cycle. This problem is a beautiful use of coupling that may look easier later in the term but is a nice challenge now. Cryptic hint: try a *sequence* of couplings.]

[Fifth installment.]

12. Let A_i be independent events, $X_i = \mathbf{1}_{A_i}$, $X = \sum X_i$, and $\mathbb{P}(A_i) = p_i$. Show that, for a fixed positive μ, $X \xrightarrow{\mathrm{d}} \mathrm{Po}(\mu)$ iff

$$\sum p_i \to \mu \text{ and } \max p_i \to 0.$$

[As usual, A_i is really $A_i^{(n)}$ and similarly for X_i, X and p_i. A proof—at least for "if"—isn't necessarily easy to find, but should be short once found.]

13. For $p = n^{-1} \ln n$, show

$$\mathbb{P}(G_{n,p} \text{ has no isolated vertices and is not connected}) \to 0.$$

[For a challenge you could try a different assertion from class: for n even, $\mathbb{P}(G_{n,p}$ has no isolated vertices and no perfect matching$) \to 0$.]

14. For events $A_1, \ldots, A_n$ in a probability space, with $\mu = \sum \mathbb{P}(A_i)$,

$$\begin{aligned}\mathbb{P}(\text{some } \mu + t \text{ independent } A_i\text{'s occur}) &\le \exp\left[-\mu\varphi(t/\mu)\right] \\ &\le \exp\left[-t^2/(2(\mu + t/3))\right],\end{aligned}$$

where $\varphi(x) = (1 + x)\ln(1 + x) - x$ for $x \ge -1$ (so $\varphi(-1) = 1$).

[Cf. Problem 2. What's interesting here is the first inequality; the second bound (a little calculus exercise, or see p. 27 of [JŁR]) is there to make sense of the first. Hint: consider, for suitable k, the number of sequences of k independent events that occur and use Markov. You'll eventually want to bound a sum by an integral.]

[Sixth installment. First two from class; I hope all are reasonably interesting.]

15. Recall Ajtai, Komlós and Szemerédi showed:

Theorem. There is a fixed $c > 0$ such that if G is triangle-free on n vertices with average degree at most d, then

$$\alpha(G) > cn \log d/d.$$

Show that this implies the same statement for $\overline{\alpha}(G)$ (with a different c).

[Hint: Consider $\alpha(G[V_p])$ for a suitable p. This involves minor details that can be annoying to write, but the main point is finding something that *should* work.]

16. Given G and $\lambda > 0$, let $\mathbf{I}$ be chosen from the independent sets of G according to the hard-core measure with fugacity λ (as defined in class). Show that $\mathbb{E}|\mathbf{I}|$ is increasing in λ.

17. With $S_1, \ldots, S_m \subseteq S$, let $X_1, \ldots, X_d \in S$ be chosen independently, with X_i uniform from S_i, and $X = \cup X_i$. Show that for any $A \subseteq S$ and $i \in S \setminus A$,

$$\mathbb{P}(i \in X | A \subseteq X) \leq \mathbb{P}(i \in X). \tag{1}$$

[It follows that for any $B \subseteq S$, $\mathbb{P}(B \subseteq X) \leq \prod_{i \in B} \mathbb{P}(i \in X)$. Hopefully (1) is intuitively clear, but proving it might need a little idea.]

18. [Recall that for a graph $G = (V, E)$ and $S = (S_v : v \in V)$ with $S_v \subseteq \Gamma$, a coloring $\sigma : V \to \Gamma$ is *S-legal* if it is proper in the usual sense and $\sigma(v) \in S_v$ $\forall v$. The *list-chromatic number* (or *choosability*), $\chi_l(G)$, of G is the least t such that *every* S as above with $|S_v| = t$ $\forall v$ admits an S-legal coloring.]

Show that for a *bipartite* G of maximum degree D, $\chi_l(G) = O(D/(\log D))$.

[An infuriating open problem: *what can one really say here?* At a guess, the truth is $\Theta(\log D)$ (which *is* a lower bound; see AS, Sec. 1.6), but *any* improvement on the bound in the problem would be very interesting.]

[Seventh installment. There's a beautiful soft proof of # 20, which maybe takes some thought to convince oneself it works. (You might find a slight echo of a trick we've used, but this problem isn't so much related to things we've been doing.)]

19. Show that there is a fixed C such that if $\mathcal{H}$ is a t-uniform, t-regular hypergraph on $V = [n]$, with n even and $t > 1$, then there is an $f : V \to \{\pm 1\}$ with

$$|f(H)| \leq C\sqrt{t \ln t} \quad \forall H \in \mathcal{H}$$

and

$$f(V) = 0.$$

[Easy once found.]

20. Let $G = (V, E)$ be a finite, connected graph and let H be the graph with vertex set $V \times \{0, 1\}$ and edge set

$$\{\{(x, \varepsilon), (y, \varepsilon)\} : \{x, y\} \in E, \varepsilon \in \{0, 1\}\} \cup \{\{(x, 0), (x, 1)\} : x \in V\}.$$

(Thus H consists of two disjoint copies of G plus, for each $v \in V$, an edge joining the two copies of v. Such an H is often called a *bunkbed graph*.)

Let $a, b \in V$ and let $(a, 0) = X_0, X_1 \ldots$ be the random walk on H starting from $(a, 0)$. [That is, if $X_{i-1} = v$ then X_i is uniform from N_v, independent of $X_0, \ldots, X_{i-2}$.] For the hitting times

$$S = \min\{t : X_t = (b, 0)\} \text{ and } T = \min\{t : X_t = (b, 1)\},$$

show $\mathbb{P}(S \leq T) \geq 1/2$.

[You might also try showing that $\mathbb{P}(S \leq t) \geq \mathbb{P}(T \leq t)$ $(\forall t)$ is *not* true.

In the background here is the notorious

Bunkbed Conjecture: For percolation (at any p) on H,

$$\mathbb{P}((a, 0) \leftrightarrow (b, 0)) \geq \mathbb{P}((a, 0) \leftrightarrow (b, 1)).$$

(*Percolation* means we keep edges of H with probability p (independently), and $u \leftrightarrow v$ means there's a path of retained edges between u and v.)]

[Eighth installment.]

21. Let $G = G(n, 1/2)$ and $k = k_0(n) - 3$ (k_0 as in class). Show that

$$\mathbb{P}(\alpha(G) < k) = \exp\left[-O\left(\frac{n^2}{\log^2 n}\right)\right].$$

22. Recalling that χ_l was defined in Problem 18, show that

$$\text{w.h.p. } \chi_l(G_{n,1/2}) \sim n/(2\log_2 n).$$

[Hint: do something for a while, then do something else.]

[Ninth installment. A few more than usual, partly because it's been a while, but also because there are some nice problems here.]

23. For a permutation $\pi = (\pi(1), \ldots, \pi(n))$ of $[n]$, let $X(\pi)$ be the length of a longest increasing subsequence in π, i.e. a sequence $i_1 < \cdots < i_k$ with $\pi(i_1) < \cdots < \pi(i_k)$. Show that if π is a (uniform) random permutation of $[n]$ then $X = X(\pi)$ is "concentrated in $O(\sqrt{n})$."

[A famous result of Logan and Shepp and (independently) Vershik and Kirov says $\mathbb{E}X \sim 2\sqrt{n}$. Ulam had asked for the value of $\mathbb{E}X$ and the asymptotic was conjectured by Hammersley. The actual concentration—a *hard* analytic result—turns out to be more like $n^{1/6}$.]

24. Let G be a simple d-regular graph on vertex set V and $(Y_0, Y_1, \ldots)$ the usual random walk on G starting from some fixed Y_0. Fix m and for $v \in V$ set $X(v) = |\{k \in [m] : Y_k \in N_v\}|$, where N_v is the neighborhood of v. Show that for any $v \in V$ and $\lambda > 1$,

$$\mathbb{P}(|X(v) - \tfrac{1}{d}\textstyle\sum_{w\sim v} X(w)| > \lambda) < 2e^{-(\lambda-1)^2/(2m)}.$$

25. Let G be a graph with $\chi(G) = 1000$ and U a uniform subset of $V = V(G)$. Show that

$$\mathbb{P}(\chi(G[U]) \le 400) < .01$$

(where, as usual, $G[U]$ is the subgraph induced by U).

[A-S, 7.9.2]

26. Let $G = G_{n,1/2}$. Prove that $\chi(G)$ is concentrated in $O(\sqrt{n}/\log n)$; that is: for any $\varepsilon > 0$ there are a constant C and a function $k(n)$ such that

$$\mathbb{P}(k(n) \le \chi(G) \le k(n) + C\sqrt{n}/\log n) > 1 - \varepsilon.$$

[A-S, 7.9.3]

27. Show that for any (fxied) $\varepsilon > 0$, there are infinitely many n for which

$$\max_k \mathbb{P}(\alpha(G) = k) < 1/2 + \varepsilon.$$

[Hint: for a given k, consider the least n satisfying $\mathbb{P}(\alpha(G_{n,1/2}) \ge k) > 1/2$.]

28. Show the Alon-Krivilevich theorem stated in class implies: for fixed $\beta < 1/2$ and $r : \mathbb{N} \to \mathbb{N}$ with $r(n) < n^\beta$, there is $p = p(n)$ for which $\chi(G_{n,p}) = r(n)$ a.s.

[Tenth installment. Problem 31 is also correct for trees, but probably a little easier to see if one thinks of forests. (So if you have it for forests you could try trees.)]

29. Suppose $\alpha : X \to \mathbb{R}^+$ satisfies $\sum \alpha_x \le 1/(2p)$ and let

$$\mathcal{F} = \{A \subseteq X : \textstyle\sum_{x \in A} \alpha_x \ge 1\}.$$

Thus $\mathcal{F}$ is weakly p-small, and this is shown by an α supported on singletons. Show that $\mathcal{F}$ is (Kp)-small for some universal K.

[This is a baby case, due to Talagrand, of his conjecture that $q_f(\mathcal{F}) < Kq(\mathcal{F})$ for all $\mathcal{F}$ and some fixed K.]

30. Show that there is a fixed K such that for any n-vertex forest H,

$$q_f(\mathcal{F}_H) < K\Delta_H/n.$$

31. Show that for any $D > \log n/\log\log n$, there is an n-vertex forest H with $\Delta_H \le D$ and

$$p_c(\mathcal{F}_H) = \Omega(D \log D/n).$$

[This is still true—but uninteresting—if $D \log D < \log n$.]