$\boxed{L5}$ (RECAP)

Baranyais thm:

$r_1, \dots, r_k \in \mathbb{N}$ (repeats okay)

$\mathcal{H}_i$: copy of $\binom{[n]}{r_i}$

$\alpha_{ij} \in \mathbb{N}$ $\quad i \in [k],\ j \in [\ell]$

(i) $\sum_j \alpha_{ij} = \binom{n}{r_i} \quad \forall i$

(ii) $\sum_i \alpha_{ij} r_i = n \quad \forall j$

$\Rightarrow$

$\exists$ part'n $\bigcup \mathcal{H}_i = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_\ell$ w/

$\mathcal{F}_j =$ p.m. & $\boxed{|\mathcal{H}_i \cap \mathcal{F}_j| = \alpha_{ij}}$

Pf by ind on n

Claim $\boxed{\text{ETS}}$ $\exists\ \alpha'_{ij} \in \{0,1\}$ s.t.

(a) $\sum_j \alpha'_{ij} = \binom{n-1}{\boxed{r_i - 1}}$ r_i' $\quad \forall i$

(b) $\sum_i \alpha'_{ij} = 1 \quad \forall j$

(c) $\alpha_{ij} = 0 \Rightarrow \alpha'_{ij} = 0$

Pf.

Existence? (punchline)

$$\boxed{x_{ij} := \frac{r_i}{n}\,\alpha_{ij}} \quad (*) \longrightarrow$$

(a) $\sum_j x_{ij} = \frac{r_i}{n} \sum_j \alpha_{ij} = \frac{r_i}{n}\binom{n}{r_i} = \binom{n-1}{r_i-1}$

(b) $\sum_i x_{ij} = \frac{r_i}{n} \sum_i \alpha_{ij} = 1$

(c) $\alpha_{ij} = 0 \Rightarrow x_{ij} = 0$

(d) $x_{ij} \in [0,1]$ (by (b) or (*))

(*) v unif $\in [n] \to x_{ij} = \mathbb{P}($the $A \in \mathcal{F}_j$ cont'g v is from $\mathcal{H}_i)$

$\boxed{\text{so?}}$

Lemma (a_{ij}) $m \times n$ $\boxed{\mathbb{R}}$-mat w integer line sums r_i, c_j

$\Rightarrow \exists$ $m \times n$ $\boxed{\mathbb{Z}}$-mat. (b_{ij}) w

① same line sums

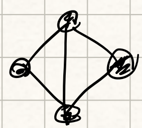
② $|b_{ij} - a_{ij}| < 1 \quad \forall i,j$

Thresholds & 2nd moment method

E.g. H fixed graph, $G = G_{n,p}$ ($p = p(n)$ TBA) $\longrightarrow$

when is G likely to contain (a copy of) H?

↳ p

e.g. $H =$  (then gen'l)

$X = \#$ of H's in G

↳ unlab. (unimp.)

$\longrightarrow$ $\underbrace{\mathbb{P}(X \neq 0)}_{G \supseteq H}$ does what?

start with $\mathbb{E}$:

$H_1, \dots, H_m$ copies of H in K_n

$$m = \frac{(n)_4}{4} \asymp \Theta(n^4)$$

↳ $|\mathrm{Aut}(H)|$

$X_\alpha = \mathbb{1}_{\{H_\alpha \subseteq G\}}$ $\quad (X = \sum X_\alpha)$

$$\mathbb{E}X = \sum \mathbb{E}X_\alpha = mp^5 \begin{cases} \asymp 1 & \text{if } p \asymp n^{-4/5} \\ \to 0 & p \ll n^{-4/5} \quad ① \\ \to \infty & p \gg n^{-4/5} \quad ② \end{cases}$$

$(n \to \infty)$

① $\Longrightarrow$ $\mathbb{P}(\underbrace{X\neq 0}_{G\supseteq H}) \to 0$ if $p \ll n^{-4/5}$

$\mathbb{P}$ $\underbrace{\mathbb{P}(X\neq 0) \to 1}_{G\supseteq H \left\{ \begin{array}{l} \text{w.h.p.} \\ \text{a.s.} \\ \text{a.a.s.} \end{array} \right.}$ if $p \gg n^{-4/5}$ ②

[More gen'l for a while, then back to ②]

- X fin. set — sorry
- μ_p : prob. measure on 2^X : $\mu_p(A) = p^{|A|}(1-p)^{|X-A|}$

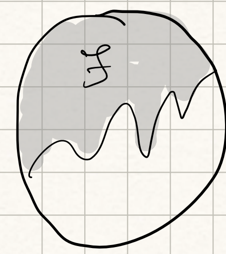
  $\longrightarrow$ random $A =: X_p$

  $\mathbb{P}$ e.g. $X = \binom{[n]}{2} = E(K_n) \to X_p = G_{n,p}$

- $\mathcal{F} \subseteq 2^X$: "fam," "prop"

  e.g. $X = \binom{[n]}{2}$: $\mathcal{F}$ graph. prop. if iso. invar

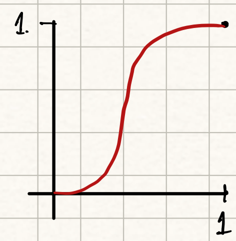
  e.g. {cont H}, {conn. (sp.)}, {planar}



- $\mathcal{F}$ incr (a.k.a. "mono") if ...

e.g. $\{\text{cont. } x\}$, $\{|A| \geq k\}$

e.g. gph prop's above are incr, incr, decr



Obs $\mathcal{F}$ incr $\Rightarrow$ $\mu_p(\mathcal{F})$ incr.

& strict if $\mathcal{F} \neq \emptyset, 2^X$

— "obv" but why?

thresholds: X_n fin sets

$$\mathcal{F}_n \subseteq 2^{X_n} \text{ incr}, \quad \mathcal{F} = \{\mathcal{F}_n\}$$

E-R 60: $p_0 = p_0(n)$ a $\left\{\begin{matrix}\text{th.} \\ \text{th. fn.}\end{matrix}\right\}$ for $\mathcal{F}$ if

$$\mu_p(\mathcal{F}_n) \longrightarrow \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases} \qquad [p = p(n)]$$

e.g. $\mathbb{P}(G_{n,p} \in \mathcal{F})$

E.g. $\mathbb{P}(G \supseteq H) \longrightarrow 1$ if $p \gg n^{-4/5}$?

is $n^{-4/5}$ a threshold for $\mathcal{F} = \{\text{contain } \}$?

us. $\mathcal{F}$ "nat" (e.g. doesn't men. n) but not nec'ly

ER'60 (e.g.) $n^{-4/5}$ is a th. for {cont. $H =$  }

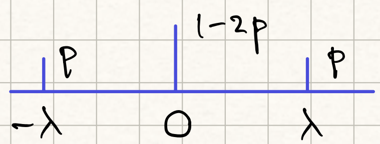
[a.k.a. $\mathbb{P}(G \supseteq H) \longrightarrow 1$ if $p \gg n^{-4/5}$]

Pf: "2^{nd} m.m." (can do better here, but ...)

▷ Cheb: $\forall$ r.v. X & $\lambda > 0$

$$\mathbb{P}(|X-\mu| \geq \lambda) \leq \frac{\sigma^2}{\lambda^2}$$

[$\mu = \mu_X$, $\sigma^2 = \sigma_X^2$

- makes sense
- sharp: 

$\sigma^2 = 2p\lambda^2$ $(\mu = 0)$

$\mathbb{P}(|X| \geq \lambda) = 2p$

Pf $\mathbb{P}(|X-\mu| \geq \lambda) = \mathbb{P}(\underbrace{(X-\mu)^2}_{\geq 0} \geq \lambda^2) \leq \lambda^{-2}\, \mathbb{E}(X-\mu)^2$ ☐

Remks
- Cheb ⟷ conc. of meas ◁ (...)
- weak for "nice" X (as here) but
  - always true & rel. easy
  - often enough

→ useful esp. for tougher X's

↓

some spectacular successes

[For me: e.g.
- Robinson-W
- Achlioptas & Co.
- Riordan]

- "2^{nd} m.m" = ?

[back to E-R]

$$\mathbb{P}(X=0) \le \mathbb{P}(|X-\mu| \ge \mu) \le \frac{\sigma^2}{\mu^2} = \frac{\mathbb{E}X^2}{\mu^2} - 1$$

hope for: $\left.\begin{array}{l} \sigma^2 = o(\mu^2) \\ \mathbb{E}X^2 \sim \mu^2 \end{array}\right\}$ equiv

$\boxed{\text{note hidden } n}$

– in which case $X \sim \mu$ a.s. i.e.

$$\forall \varepsilon > 0 \quad \mathbb{P}\left(\frac{X}{\mu} \in (1-\varepsilon, 1+\varepsilon)\right) \to 1$$

[Recall: $H_1, \dots, H_m$ copies of H in K_n $(m = \Theta(n^4))$

$X_\alpha = \mathbb{1}_{\{H_\alpha \subseteq G\}}$ $(G = G_{n,p})$

$(X = \sum X_\alpha, \ \mu = mp^5)$]

Aim for: $(\sigma^2 =)$ $\mathbb{E}X^2 - \mu^2 \ll \mu^2$ (*)

$$\mathbb{E}X^2 = \mathbb{E}\sum\sum X_\alpha X_\beta = \sum\sum \mathbb{E} X_\alpha X_\beta$$

$$\mu^2 = \sum\sum \mathbb{E}X_\alpha \mathbb{E}X_\beta = m^2 p^{10}$$

heur: pretend X_α's ind $\longrightarrow$

$$\mathbb{E}X_\alpha X_\beta = \mathbb{E}X_\alpha \, \mathbb{E}X_\beta \quad \forall \alpha \neq \beta \longrightarrow$$

$$\mathbb{E}X^2 - \mu^2 = \sum\sum(\underbrace{\mathbb{E}X_\alpha}_{=\mathbb{E}X_\alpha^2} - \mathbb{E}^2 X_\alpha) \leq \mu$$

$\Longrightarrow$ $\circledast$ if $\mu \to \infty$ (which we have)

IDEA/HOPE: this is $\approx$ right

Now just calc's

edges

$$\mathbb{E}X_\alpha X_\beta = p^{|H_\alpha \cup H_\beta|} = p^{10 - |H_\alpha \cap H_\beta|}$$

e.g. X_α, X_β ind. $\Longleftrightarrow$ $H_\alpha \cap H_\beta = \emptyset$

notation: $\alpha \sim \beta$: $H_\alpha \cap H_\beta \neq \emptyset$

$$\mathbb{E}X^2 = \sum\sum \mathbb{E}X_\alpha X_\beta = m \sum_\beta \mathbb{E}X_1 X_\beta$$

$$\mathbb{E}X^2 - \mu^2 = m \sum_{\beta \sim 1} (\mathbb{E}X_1 X_\beta - \underbrace{\mathbb{E}X_1 \mathbb{E}X_\beta}_{p^{10}})$$

$$\leq m \sum_{\beta \sim 1} \mathbb{E}X_1 X_\beta = m p^{10} \sum_{\beta \sim 1} p^{-|H_1 \cap H_\beta|}$$