Thm (J&R T 3.19; Boll 81, Karoński–Ruciński 83, ...)

H SB (fixed), $X = |\{H\text{'s in } G = G_{n,p}\}|$

$p \asymp n^{-1/m(H)}$, say $\mathbb{E}X \left(= \underbrace{\frac{(n)_{v_H}}{\mathrm{aut}(H)}}_{m} p^{e_H} \right) \to \mu$

$\Longrightarrow \quad X \xrightarrow{d} \mathrm{Po}(\mu)$

E.g. $H = \triangle\,\triangle$? (how ab. $\Diamond\!-$?)

$\boxed{L9}$ **Pf** (sketch & see J&R.)

want $\mathbb{E}(X)_r \longrightarrow \mu^r$

$$\mathbb{E}(X)_r = \sum_{\alpha_1,\dots,\alpha_r} \mathbb{P}(H_{\alpha_1} \cup \cdots \cup H_{\alpha_r} \subseteq G)$$

$$= \sum{}' + \sum{}'' \quad \text{w/ } \Sigma' \longleftrightarrow H_\alpha\text{'s disjt}$$

ETS: ① $\Sigma' \to \mu^r$ (triv. EX)

$$\Sigma' = \underbrace{|\{(\alpha_1,\dots,\alpha_r) : \text{disjt}\}|}_{\to \text{ EX: this } \sim m^r} \cdot p^{e_H \cdot r}$$

② $\Sigma'' \to 0$ ("sketch" $k=2$ for conn. to earlier & see J&R)

recall for Cheb needed $\sum\sum_{\alpha\sim\beta} \mathbb{E}X_\alpha X_\beta \ll \mu^2$ [w $p \gg n^{-1/m(H)}$]

$\Diamond\!-$: needed $p \gg n^{-4/5}$ $(\asymp n^{-1/m(H)})$ only for diag

gen'l H: need $p \gg n^{-1/m(H)}$ when $d(\underbrace{H_\alpha \cap H_\beta}_{K}) = m(H)$

But now: ① SB $\Longrightarrow$ only K is H

② we deleted diag

Back to isds (briefly)

Q: th's for $\begin{cases} \text{conn ?} \\ \text{p.m. (n even) ?} \\ \text{H.c. ?} \end{cases}$

recall: $G = G_{n,p}$, $p = \dfrac{\ln n + c}{n}$ $\Longrightarrow$

$\mathbb{P}(\text{no isd's}) \longrightarrow e^{-e^{-c}}$

thm (E-R) same hyp's $\Longrightarrow$

$\mathbb{P}(G \text{ conn}) \longrightarrow e^{-e^{-c}}$ [→ sharp th. at $\dfrac{\ln n}{n}$]

equiv: $\mathbb{P}(\underbrace{\text{no isds \& disconn}}_{(A)}) \longrightarrow 0$ [iso's main obstruction → !]

EX/HW (?)

▷ stranger: "hitting time" vn (BT '85):

$e_1, \ldots$ unif. ord. of $E(K_n)$, $G_t = (V, \{e_1, \ldots, e_t\})$

$T = \min\{t : G_t \text{ has no iso's}\}$ ("H.T.")

$\longrightarrow$ a.s. G_T conn.

▷ why stranger?

couple G & G_T: choose: $\left.\begin{array}{l} \text{(i) ord (det } T) \\ \text{(ii) } m := |G| \end{array}\right\}$ ind.

$\longrightarrow G = G_m$ (right marginal)

(A) $\Longrightarrow m \geq T$ ($\{\text{no iso's in } G\} \equiv \{m \geq T\}$)

(A) + this $\Longrightarrow G_T$ disconn. ▢

▷ where did we use val of p? (we didn't)

Sim ① p.m. (n even): a.s. $G_T \models$ p.m. $\longrightarrow$

(G, p as above:) $\mathbb{P}(G \models \text{p.m.}) \to$ same

vs: $|\{\text{p.m.'s in } K_n\}| = \dfrac{n!}{(n/2)!\, 2^{n/2}} = (n-1)!! \approx (n/e)^{n/2}$ $\longrightarrow$

$X = |\{\text{p.m.'s in } G\}| \longrightarrow \mathbb{E}X \approx \left(\dfrac{np}{e}\right)^{n/2}$

— huge if e.g. $p = 3/n$ (typ: $X = 0$ or huge)

$\boxed{\mathbb{E} \text{ not the issue.}}$

② Ham: T: H.T. for $\delta \geq 2$ $\longrightarrow$

G_T Ham. a.s.

Cor (EX) $p = \dfrac{\ln n + \ln \ln n + c}{n} \Longrightarrow$

$\mathbb{P}(G \text{ Ham}) \longrightarrow e^{-e^{-c}}$

(though again $\mathbb{E}|\{\text{H.c.'s}\}|$ huge if $p > 3/n$)

One more $\bar{\text{w}}$ Poisson (ref: Wormald, Models of RRG's ch. 2, esp. § 2.2)

d given (here fixed) nd even (n large)

$\mathcal{G}_{n,d} := \{$(labelled) d-reg (simple) gphs on $[n]\}$

▶ Thm 1 $|\mathcal{G}_{n,d}| \sim e^{-(d^2-1)/4}(nd-1)!!\,(d!)^{-n}$

- terms will have meanings
- Bender-Canfield '78 via gen. fns. — here probabilistic

▷ $G_{n,d}$: unif $\in \mathcal{G}_{n,d}$ [e.g. $d=3$ ($d=2$ not so int'g)]

does what? e.g. Hamiltonian?

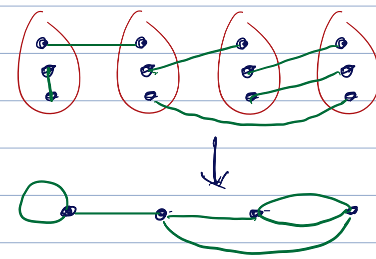
YES: Robinson–Wormald '94 (!)

▷ how to work w/ it?

▷ pairing (a.k.a. config) model (ver. of Boll. '80)

$W = \bigcup_{i=1}^{n} W_i \qquad |W_i| = d$

$\mathcal{P}^*_{n,d} := \{\text{pairings of } W\}$



$\downarrow \pi$ (nat.)

$\mathcal{G}^*_{n,d} := \{$(lab.) $\underbrace{\text{d-reg}}$ multigraphs on $[n]\}$

loops contrib 2

$\supseteq \mathcal{G}_{n,d}$

$\mathbb{P}$: unif $\in \mathcal{P}^*_{n,d}$

$\downarrow \pi$

$G \in \mathcal{G}^*_{n,d}$ not unif; e.g. $(d=2, n=3)$:

$|\pi^{-1}(\;)| = 1$

$|\pi^{-1}(\;)| = 2$

$|\pi^{-1}(\;)| = 8$ (why?)

▷ but G simple $\Rightarrow |\pi^{-1}(G)| = (d!)^n$ (why?)

a.k.a. $G \in \mathcal{G}_{n,d}$

$\mathcal{P}_{n,d} := \pi^{-1}(\mathcal{G}_{n,d})$ ("simple pairings")

$\to |\mathcal{G}_{n,d}| = (d!)^{-n} |\mathcal{P}_{n,d}| = (d!)^{-n} \underbrace{|\mathcal{P}^*_{n,d}|}_{(nd-1)!!} \boxed{\dfrac{|\mathcal{P}_{n,d}|}{|\mathcal{P}^*_{n,d}|}} = \mathbb{P}(\text{simple})$ ◁

(MP)