$\boxed{L8}$ Poisson approx [one ref: AS chap. 8]

recall (& see "handout"):

$X \sim Po(\mu)$ if $\mathbb{P}(X=k) = \underbrace{e^{-\mu}\mu^k/k!}_{}$ $\forall k \in \mathbb{N}$

note $\Sigma = 1$ & check $\mathbb{E}X = \mu$

e.g. $\mu > 0$ const, $p = p(n) = \mu/n$, $X\ (= X^{(n)}) \sim Bin(n,p) \rightarrow$

$\boxed{X \xrightarrow{d} Po(\mu)}$ i.e. $\mathbb{P}(X=k) \to e^{-\mu}\mu^k/k!$ $\forall k \in \mathbb{N}$

▷ "Poisson paradigm": $A_1, \ldots$ events, $X_i = \mathbb{1}_{A_i}$, $X = \sum X_i$

$p_i = \mathbb{P}(A_i) = \mathbb{E}X_i$, $\mu = \mathbb{E}X\ (= \sum p_i)$ $\rightsquigarrow$

if p_i's $\underset{=?}{\boxed{\text{small}}}$ & A_i's $\underset{=??}{\boxed{\approx \text{indept}}}$, then $X \approx Po(\mu)$

fixed pt

e.g. derangements: σ unif $\in \mathfrak{S}_n$, $A_i = \{\sigma_i = i\}$, $p_i = 1/n$, $\mu = 1$

$\mathbb{P}(\underbrace{X=0}_{\text{"derangement"}}) \xrightarrow{?} 1/e$; yes: $\mathbb{P}(X=0) = \sum_{k=0}^{n} (-1)^k/k! \xrightarrow{\text{fast}} 1/e$

▷ e.g. $G = G_{n,p}$ $(p = p(n))$

$A_v = \{v \text{ iso}\} \rightarrow p_v = (1-p)^{n-1} \approx e^{-pn}$

int'g range: $p = \dfrac{\ln n + c}{n}$ [think c fixed, not nec. pos.

$\rightarrow p_v \sim e^{-c}/n \longrightarrow \mu \to e^{-c}$

$\longrightarrow$ expect $X\ (= |\{\text{isols}\}|) \approx Po(e^{-c})$

– true, e.g. $\mathbb{P}(X=0) \to e^{-e^{-c}} \to \begin{cases} 0 & c \to -\infty \\ 1 & c \to \infty \end{cases}$

(a sharp threshold)

Remk imp. to know what to expect

(sim. discussion earlier: Erdős l.b. for $R(3,k)$)

Ex. EPIT (ev. pt. in Δ; AS §8.3, here just heur):

when shd this happen?

① $\mathbb{E}|\{\Delta\text{'s on } x\}| = \binom{n-1}{2} p^3 =: \mu$

$\xrightarrow{\text{expect}}$ $\mathbb{P}(x \notin \Delta) \approx e^{-\mu}$ $\longrightarrow$

② $\mathbb{E}|\{x : \text{not in } \Delta\}| \approx n e^{-\mu}$

$\xrightarrow{\text{expect}}$ $\mathbb{P}(\text{EPIT}) \approx \exp[-ne^{-\mu}]$

$\longrightarrow$ AS, Thm 8.3.2: choose p so $e^{-\mu} \overset{*}{=} c/n$

then $\mathbb{P}(\text{EPIT}) \longrightarrow e^{-c}$

$*$: $\mu = \ln(n/c) \longrightarrow p \approx 2^{1/3} \boxed{n^{-2/3} \ln^{1/3} (n}/c)$

ess =

Brun's sieve (AS p.132)

note: $Y \sim Po(\mu) \Rightarrow \underline{\mathbb{E}(Y)_r} = e^{-\mu} \sum (k)_r \frac{\mu^k}{k!} = \cdots = \mu^r$

"factorial moments"

Thm ("Brun's sieve") $\underbrace{A_i, X_i, X}_{\text{ac. } A_i^{(n)} \text{ etc.}}$ as above $\boxed{\mu \text{ fixed}}$

If $\forall$ (fixed) r $\mathbb{E}(X)_r \to \mu^r$

(equiv $\mathbb{E}\binom{X}{r} \to \mu^r/r!$) ⊛

then $X \xrightarrow{d} Po(\mu)$

notes ① equiv: $X \in \mathbb{N}$ $\overline{\omega}$ ⊛ ...

why? (A_i occurs iff $X \geq i$)

② Brun $\subseteq$ "method of moments":

under mild hyp's $\mathbb{E}X^r \to \mathbb{E}Y^r$ (or $\mathbb{E}(X)_r \to \dots$)

$\implies X \xrightarrow{d} Y$

see J&R th. 6.1 & 6.7; note $\xrightarrow{d}$ changes.

③ converse? ("no" — why?)

Pf ($k=0$; rest EX) say $A_1, \dots, A_n$

$I \subseteq [n]$ always; $A_I = \bigcap_{i \in I} A_i$, $p_I = \mathbb{P}(A_I)$

$$S_r = \sum_{|I|=r} p_I \overset{\triangle}{=} \mathbb{E}\binom{X}{r} \quad \text{(right?)}$$

$$\left[\binom{X}{r} = \sum_{|I|=r} \mathbb{1}_{A_I} = \binom{|Z|}{r}, \quad Z = \{i : A_i \text{ occurs}\} \right]$$

so $\boxed{S_r \to \mu^r / r! \quad \forall r}$

E.g. derangements

E.g. iso's (my $S_r \sim \frac{n^r}{r!}(1-p)^{nr}$ & $n(1-p)^n \to \mu$)

⊛ indept. A_i's: HW

I-E: $\mathbb{P}(X=0) = \mathbb{P}(\cap \overline{A_i}) = \sum_I (-1)^{|I|} \mathbb{P}(A_I) = \sum_{r \geq 0} (-1)^r S_r$

$\downarrow$

$\mu^r/r!$

$\to \mathbb{P}(X=0) - e^{-\mu} = \sum_{r \geq 0} \left(\underbrace{S_r - \mu^r/r!}_{\to 0 \text{: so?}} \right)$

$[S_r = 0 \text{ if } r > n]$

▷ <u>Bonf</u>: $\mathbb{P}(X=0) \begin{cases} \le \sum_{r=0}^{2t} (-1)^r S_r \\ \ge \sum_{r=0}^{2t+1} (-1)^r S_r \end{cases}$

$\longrightarrow$ $\mathbb{P}(X=0) - e^{-\mu} \le \underbrace{\sum_{r=0}^{2t} \left|S_r - \mu^r/r!\right|}_{A} + \underbrace{\sum_{r>2t} \mu^r/r!}_{B}$

$\longrightarrow$ choose t so B is small ($< \varepsilon/2 \ldots$)

then n so A is small

& sim'ly l.b. $\square$

gen'l K: Jordan & gen of Bonf – see handout p.3

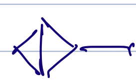
Back to $\mathbb{P}(G_{n,p} \supseteq H)$ for fixed H w/ $\mathbb{E}X \simeq 1$

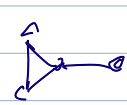
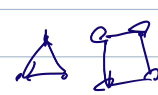
[recall density: $d(H) = e_H / v_H$

max'm d. of H: $m(H) = \max_{\emptyset \neq K \subseteq H} d(K)$]

H <u>bal</u> if $m = d$

<u>SB</u> if $d(K) < m$ $\forall K \subsetneq H$

<u>e.g.</u> unbal: 

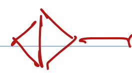
bal, not SB: , 

Thm (J&R T 3.19; Boll 81, Karoński–Ruciński 83, ...)

H SB (fixed), $X = |\{H\text{'s in } G = G_{n,p}\}|$

$p \asymp n^{-1/m(H)}$, say $\mathbb{E}X \left(= \dfrac{(n)_{v_H}}{\underbrace{\mathrm{aut}(H)}_{m}} p^{e_H} \right) \to \mu$

$\Rightarrow \quad X \xrightarrow{d} \mathrm{Po}(\mu)$

E.g. $H = \triangle\,\triangle$? ( ??)