$\boxed{L7}$ refs: A-S §14.4; Matousek, "Lec's on D.G."; Pach-Agarwal

RECAP: ① Trace $(T(\mathcal{F}, S))$

VC-dim $(VC(\mathcal{F}))$

Th "Sauer-Shelah" $\Rightarrow$

$$VC(\mathcal{F}) = d \Rightarrow |Tr(\mathcal{F}, W)| \lesssim |W|^d \quad \forall\, W \subseteq V$$

② ($\mathcal{H}$ hypergraph)

cover, cover # $(\tau(\mathcal{H}))$

frac. cover, fr. cover # $(\tau^*(\mathcal{H}))$

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int'd in τ for $\mathcal{F}_\varepsilon = \{A \in \mathcal{F} : |A| \geq \varepsilon n\}$

ε-net (for $\mathcal{F}$) $\equiv$ cover of $\mathcal{F}_\varepsilon$

hope bd not dep'g on n

Scen: $VC(\mathcal{F}) = d$

think: d fixed; ε fixed, small rel d; $n \to \infty$

but gen'l for a bit!

persp: $1/\varepsilon$ nat. b'mark:

① $\tau^*(\mathcal{F}_\varepsilon) \leq 1/\varepsilon$ (Ex: $=$ if $\mathcal{F}$ εn-unif & reg)

② try random cover:

$X = \{x_1, \dots, x_m\}$ TBA $\qquad$ x_i's indept, unif $\in V$

(random multiset)

$|X \cap A| := \{i : x_i \in A\}$

Tail: (a) X unif $\in \binom{V}{m}$

(b) $X = V_p$, $p = m/n$ $\quad \perp$

$\mathbb{E}|X \cap A| = am/n \qquad (a = |A|)$

$\mathbb{P}(X \cap A = \phi) = (1 - a/n)^m < e^{-am/n}$

$\leq e^{-\varepsilon m}$ if $A \in \mathcal{F}_\varepsilon$

$\longrightarrow$ X tends to hit $A \in \mathcal{F}_\varepsilon$ if $m \gg 1/\varepsilon$

e.g. $\tau(\mathcal{F}_\varepsilon) < \frac{1}{\varepsilon} \log |\mathcal{F}_\varepsilon|$ (A) $\qquad$ ($\log = \ln$)

e.g. $\mathcal{F}_\varepsilon = \binom{V}{\varepsilon n} \rightarrow \tau \approx (1-\varepsilon)n$

bd in (A) $\approx (\log 1/\varepsilon) n \gg n$

e.g. $VC(\mathcal{F}) = d \Rightarrow |\mathcal{F}_\varepsilon| \leq |\mathcal{F}| < n^d \longrightarrow$

$\tau(\mathcal{F}_\varepsilon) < \frac{1}{\varepsilon} d \log n$ (still fn of n)

Thm (HW 87, foll'g VC) $VC(\mathcal{F}) = d \Rightarrow$

$\tau(\mathcal{F}_\varepsilon) < m := \frac{2d}{\varepsilon} \log_2 \frac{4d}{\varepsilon}$ (AS slightly diff)

Ex*: tightish

Pf: $\boxed{\text{show}}$ X as above covers whp.

AXIOM: ξ binom $\Rightarrow P(\xi \geq \lfloor \mathbb{E}\xi \rfloor) \geq \frac{1}{2}$

Ex: known? true?

Greenberg–Mohri '13: $\mathbb{E}\xi > 1 \Rightarrow \mathbb{P}(\xi \geq \mathbb{E}\xi) > 1/4$

aux: $Y = \{y_1, \ldots, y_m\}$ y_i's unif $\in V$ & indept (also of X)

events: $\mathcal{A} = \{\exists A \in \mathcal{H} : X \cap A = \phi\}$

$\cup$

$\mathcal{B}$ $\{\exists A \in \mathcal{H} : X \cap A = \phi, |Y \cap A| \geq \lfloor \varepsilon m \rfloor\}$

"coupling up" (choose X, then Y):

AXIOM

$P(\mathcal{B} | \mathcal{A}) \geq \frac{1}{2}$ (A as in $\mathcal{A}$ $\rightarrow$ $P(|Y \cap A| \geq \lfloor \varepsilon m \rfloor) \geq \frac{1}{2}$)

$\rightarrow P(\mathcal{B}) = P(\mathcal{A}\mathcal{B}) = P(\mathcal{A}) P(\mathcal{B}|\mathcal{A}) \geq P(\mathcal{A})/2$

$\rightarrow \boxed{P(\mathcal{A}) \leq 2 P(\mathcal{B})} \rightarrow$ ETS $\mathcal{B}$ unlikely

"coupling down" (ch. $Z := X \cup Y$, then X)

$\mathcal{B} \Rightarrow \exists A \in \mathcal{H}$ $|Z \cap A| \geq \lfloor \varepsilon m \rfloor$, $X \cap A = \phi$

given Z & A w $|Z \cap A| \geq \lfloor \varepsilon m \rfloor$

$P(X \cap A = \phi) < 2^{-\lfloor \varepsilon m \rfloor} \rightarrow \boxed{\text{U-bd ??}}$

$\sum_{A \in \mathcal{H}} P(|Z \cap A| \geq \lfloor \varepsilon m \rfloor, X \cap A = \phi) < ??$

— wrong question

$P(\mathcal{B}) = \sum_Z P(Z) P(\mathcal{B}|Z)$ $(= \mathbb{E}\{P(\mathcal{B}|Z)\})$

Given Z: $\tilde{Z}$ underlying set

rel. part of A is $A \cap \tilde{Z}$ $\rightarrow \# < (2m)^d$

$\rightarrow P(\mathcal{B}|Z) < (2m)^d \, 2^{-\varepsilon m}$

$= \left(\frac{4d}{\varepsilon} \log \frac{4d}{\varepsilon}\right)^d \left(\frac{4d}{\varepsilon}\right)^{-2d}$ = small ☒