$\boxed{L22}$

Pf of ① ($\sigma_M = s_1, \dots, s_\ell$)

ind. on M $\longrightarrow$ ETS recover σ_{M-1}

case 1: $d_M = +1 \longrightarrow \sigma_M = s_1, \dots, s_{\ell-1}$

case 2: $d_M = -(t-1) \longrightarrow$

$$\sigma_M = \underbrace{s_1, \dots, s_{\ell-t+1}, \dots, s_\ell}_{\sigma_M}, s_{\ell-t+1}, \dots s_{\ell-1}$$

Pf of ② repl. D_M by seq. of $+$'s & $-$'s:

$d_j = 1 \longrightarrow +$; $d_j = -(t-1) \longrightarrow + \underbrace{- \cdots -}_{t}$

$\longrightarrow$ M $+$'s & $< M$ $-$'s

$\xrightarrow{EX}$ # poss's $< 4^M$

End ① + ② $\longrightarrow$ # unsuccessful runs $< 4^M 5^n$

(or even runs w $|\sigma_M| < n$)

$\longrightarrow$ $\mathbb{P}$(unsuccessful) $\begin{cases} < 1 & \text{if } (5/4)^M > 5^n \\ \longrightarrow 0 & \dots \end{cases}$

Remk ($\longleftrightarrow$ AS Prob. 5.8.8) same for $|L_j| = 4$ if

$\exists$ forbidden substring in pf of ②

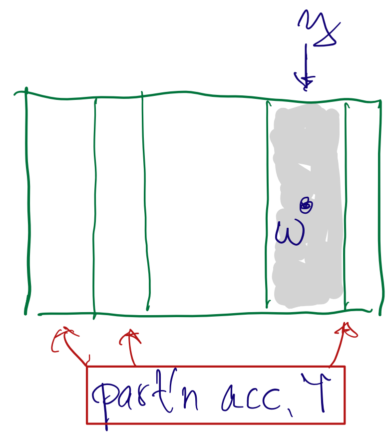
Back to concentration

[ref: AS, Ch. 7]

(saw Chernoff for ind's, NC. Now hoping for something sim for more int'g r.v.s)

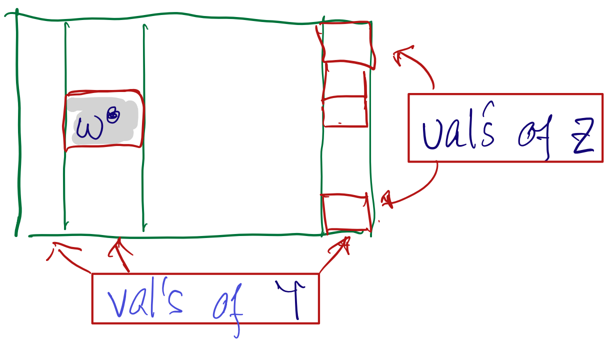
back to $\mathbb{E} X = \mathbb{E}\{\mathbb{E}[X \mid Y]\}$

$\mathbb{E}[X \mid Y](\omega)$ = avg of X on col Y $\longrightarrow$



sim:

$$\mathbb{E}[X \mid Y] = \mathbb{E}\{\mathbb{E}[X \mid Y, Z] \mid Y\}$$



X and $\mathbb{E}[X \mid Y, Z]$ have same avg on each $\left\{ \begin{array}{l} \text{cell} \\ \text{column} \end{array} \right.$

<u>martingales</u> $X_k \in \mathbb{R}$, Y_k r.v.s, X_k fn. of $Y_0, \ldots, Y_k$ $(k \geq 0)$

$\{X_k\}$ is a: <u>martingale</u> if $\mathbb{E}[X_{k+1} \mid X_0, \ldots, X_k] = X_k$

(more gen'l:) <u>martingale wrt</u> $\{Y_k\}$ if $\mathbb{E}[X_{k+1} \mid Y_0, \ldots, Y_k] = X_k$

We usually have:
- $k = 0, \ldots, m$ (finite length)
- mart. wrt $\{Y_k\}$
- X_0 fixed (& skip Y_0)

Ex. 0. Y_k : indept sym. Bernoullis, $X_k = \sum_{i=1}^{k} Y_i$ $(= S_k)$

Ex 1 same Y_k's ; $X_0 \equiv 1$

$$X_{k+1} = \begin{cases} 2X_k & \text{if } Y_{k+1} = 1 \\ 0 & \text{if } Y_{k+1} = -1 \end{cases}$$

Prop $\forall m > n$ $\mathbb{E}[X_m \mid Y_1, \dots, Y_n] = X_n$

[interp: fair game]

Pf ind. on $m-n$; $m = n+1$ is by defn. $\rightarrow$ $m > n+1$:

$$\text{l.h.s.} = \mathbb{E}\{\mathbb{E}[X_m \mid Y_1, \dots, Y_{m-1}] \mid Y_1, \dots, Y_n\}$$

$$\overset{\text{mart}}{=} \mathbb{E}[X_{m-1} \mid Y_1, \dots, Y_n] \overset{\text{ind}}{=} X_n$$

∎

[alt: repl. $m-1$ by $l \in (n, m)$ & use "ind" twice.]

Doob (or Doob's) mart:

$$X_k = \mathbb{E}[X \mid Y_1, \dots, Y_k]$$

where $X = X(Y_1, \dots)$

$\longrightarrow$

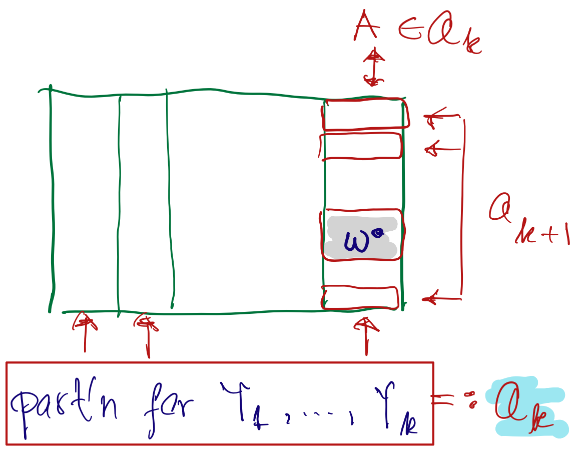
- $X_0 = \mathbb{E}X$
- if $X = X(Y_1, \dots, Y_m)$ then $X_m = X$

think of slowly revealing X

Prop $\{X_k\}$ mart. wrt $\{Y_k\}$

Pf $\mathbb{E}[X_{k+1} \mid Y_1,\dots,Y_k] = \mathbb{E}\{\mathbb{E}[X \mid Y_1,\dots,Y_{k+1}] \mid Y_1,\dots,Y_k\}$

$$= \mathbb{E}[X \mid Y_1,\dots,Y_k\} = X_k \qquad \blacksquare$$



$X_{k+1}(\omega)$: avg over $\boxed{}$

$\mathbb{E}[X_{k+1} \mid Y_1,\dots,Y_k]$ avg these over A

("slowly reveal X")

Converse: every finite length mart is a Doob mart

Pf $\{X_k\}_{k=1}^m$ mart wrt $\{Y_k\}_{k=1}^m$ $\to$ $X := X_m$ $\to$

$$\mathbb{E}[X \mid Y_1,\dots,Y_k] = \mathbb{E}[X_m \mid Y_1,\dots,Y_k] \overset{\text{Prop}}{=\!=} X_k \qquad \blacksquare$$

Another (equiv) view:

- T (finite) tree, leaves at depth m
- $X : \{\text{leaves}\} \to \mathbb{R}$
- $\forall\, u \neq$ leaf p_u prob dist on $\{$children of $u\}$
- $Y_1,\dots,Y_m$ RW root $\rightsquigarrow$ $\{$leaves$\}$ acc. to p_u's
- $X_k = \mathbb{E}[X(Y_m) \mid Y_1,\dots,Y_k]$

Concentration (what we mainly want from mart's)

$\{Z_k\}$ <u>mart. diff. seq</u> (mds) wrt $\{Y_k\}$ if

Z_k fn of $Y_1, \dots, Y_k$ & $\mathbb{E}[Z_{k+1} \mid Y_1, \dots, Y_k] = 0$

- for fin. length $Z = \sum Z_k$
- <u>nothing new</u>: $\{X_k\}$ mart wrt $\{Y_k\}$ $\rightarrow$

  $\{Z_k = X_k - X_{k-1}\}$ mds wrt $\{Y_k\}$

  & $Z = X - X_0$ if fin. length

  & $\{Z_k\}$ mds $\rightarrow \{X_k = \underset{\text{arb}}{X_0} + \sum_{i=1}^{k} Z_i\}$ mart

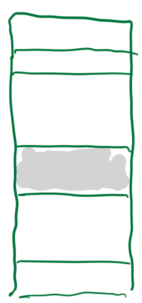
"<u>Azuma's Ineq</u>" (Azuma '67, Hoeffding '63)

$\{Z_k\}_1^m$ mds (wrt $\{Y_k\}$), $\boxed{|Z_k| \le c_k}$ ⊛ $\forall k \implies$

$$\mathbb{P}(Z > t) < \exp\left[-t^2 / 2\sum c_k^2\right]$$

- think: $\sum c_k^2 \longleftrightarrow \operatorname{var}(Z)$ (EX: $\operatorname{var} \le \sum c_k^2$)
- e.g. even $c_k = 1$ $\forall k$ $\supseteq$ Chernoff

pic. of ⊛:



$\leftarrow B \in \mathcal{Q}_k$

$A \in \mathcal{Q}_{k-1}$

$|(\text{avg on } B) - \text{avg on } A| \le c_k$

Pf of Azuma

Obs: W $\mathbb{R}$-val'd r.v., $\mathbb{E}W = 0$, $|W| \le a$ $\Rightarrow$

$$\mathbb{E}e^{W} \le \frac{1}{2}(e^{-a} + e^{a}) \le e^{a^2/2}$$

saw w/ Chernoff

EX: h cvx $\Rightarrow$ $\mathbb{E}h(W) \le \frac{1}{2}(h(-a) + h(a))$

Azuma (as usual):

$$\mathbb{P}(Z \ge t) = \mathbb{P}(e^{\varsigma Z} \ge e^{\varsigma t}) \le e^{-\varsigma t}\, \mathbb{E}e^{\varsigma Z}$$

MAIN: $\mathbb{E}e^{\varsigma Z} \le \exp[\varsigma^2 \sum c_k^2 / 2]$ ($\leadsto$ QED)

Pf Ind. on k to show

$$\mathbb{E}\, e^{\varsigma(Z_1 + \cdots + Z_k)} \le \exp[\varsigma^2 (c_1^2 + \cdots + c_k^2)/2]$$

$k = 1$: Obs. ($W = \varsigma Z_1$; note $\mathbb{E}W = 0$)

$\boxed{\text{L23}}$

$k > 1$:

$$\text{l.h.s.} = \mathbb{E}\prod_{i=1}^{k} e^{\varsigma Z_i} \quad \left(\text{us. } \neq \prod_{i=1}^{k} \mathbb{E}e^{\varsigma Z_i}\right)$$

$$= \mathbb{E}\left\{\mathbb{E}\left[\prod^{k} e^{\varsigma Z_i} \,\Big|\, Z_1, \dots, Z_{k-1}\right]\right\}$$

$$= \mathbb{E}\left\{ e^{\varsigma(Z_1 + \cdots + Z_{k-1})}\, \mathbb{E}\left[e^{\varsigma Z_k} \,\Big|\, Z_1, \dots, Z_{k-1}\right]\right\} \quad \text{why?}$$

$$\le e^{\varsigma^2 c_k^2/2}\, \mathbb{E}\, e^{\varsigma(Z_1 + \cdots + Z_{k-1})} \quad \text{why?}$$

$$\overset{\text{ind.}}{\le} e^{\varsigma^2 (c_1^2 + \cdots + c_k^2)/2}$$

∎