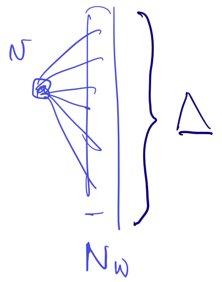
L21 RECAP



$|L_w| \le q = (1+\varepsilon)\frac{\Delta}{\ln \Delta} = |L_v|$

w always in N_v, γ always in L_v

$W_\gamma = \{w : \gamma \in L_w\}$

$\sigma : N_v \to \Gamma \cup \{\Lambda\}$ unif $\in \prod_w (L_w \cup \{\Lambda\})$

$\boxed{\ell = \Delta^{\varepsilon/2}}$

ML'

① $\mathbb{P}(|L_v^\sigma| < \ell) < \Delta^{-3}/(2e)$

② $\mathbb{P}(\underbrace{\exists \gamma \in L_v^\sigma \ \ |N_\gamma^\sigma| > \ell/2}_{\gamma \text{ bad}}) < \Delta^{-3}/(2e)$

Pf ML': • ess. Molloy

• pf of ML' int'g but eventually doable — IMO MP is <u>finding</u> statement & believing it cd be true

• showed ②: $\forall \gamma \quad \mathbb{P}(\gamma \text{ bad}) < 2^{-\ell/2}$

& (forgot to say) $\cup$ bd over γ

• now ①: PLAN: (a) $\mathbb{E}|L_v^\sigma|$ (somewhat $> \ell$)

(b) concentration

$$\boxed{\{\gamma \in L_v^\sigma\} = \{\sigma_w \ne \gamma \ \ \forall w \in W_\gamma\}}$$

Ex. 1 $L_w = L_v \ \forall w \rightsquigarrow \mathbb{E}|L_v^\sigma| > q e^{-\Delta/q} \gg \ell$

$\rightsquigarrow$ vals of q & ℓ

End RECAP

Ex. 2 $\forall w \quad L_w = \{x_w\} \to \mathbb{P}(x \in L_v^\sigma) = 2^{-|W_x|}$

$\mathbb{E}|L_v^\sigma| = \sum_x 2^{-|W_x|}$ $\quad$ ($\exists \sum |W_x| \leq \Delta$) min. when

$|W_x| = \Delta/q \;\forall x \to \mathbb{E} < q\,2^{-\Delta/q}$ (vs. $q e^{-\Delta/q}$ in (a))

(now gen'l)

Show: (i) $\mathbb{E}|L_v^\sigma| \gg l$ ($> (1+c)l$ wd suffice)

(ii) conc (some work but ≈ clear w experience):

natural: $\mathbb{P}(|L_v^\sigma| < l) < \exp[-\Omega(l)]$

but $\{x \in L_v^\sigma\}$'s dependent $\to$?

Pf of (i)

$$\mathbb{E}|L_v^\sigma| = \sum_x \mathbb{P}(x \in L_v^\sigma) = \sum_x \prod_{w \in W_x}\left(1 - \frac{1}{|L_w|+1}\right)$$

$$\overset{*}{\geq} \sum_x \exp\left[-\sum_{w \in W_x} 1/|L_w|\right]$$

AMGM or Jensen $\mapsto$ $\geq q \exp\left[-q^{-1}\sum_x \sum_{w \in W_x} 1/|L_w|\right]$

why not =? $\mapsto$ $\geq q\exp[-q^{-1}|N_v|]$

$$= q e^{-\Delta/q} \gg l \quad \text{(as in Ex. 1)}$$

* Ex (unimp.): $\forall \alpha > 0 \quad 1 - \frac{1}{\alpha+1} > e^{-1/\alpha}$

(ii) $X = |L_v^\sigma| = \sum_x X_x$,

$X_x = \mathbb{1}_{\{x \in L_v^\sigma\}} \sim \mathrm{Ber}$ (see **) won't use

$\mu := \mathbb{E}X \geq 2l$ (say)

SHOW: $\mathbb{P}(X \le \ell) \le \exp[-(\mu-\ell)^2/2\mu] \le e^{-\ell/4}$

$\uparrow$ hd for indept's (AS Thm A.1.13; cf. JŁR, T2.1)

Pf for indepts does:

$Y = q - |X| = \sum_\gamma Y_\gamma \quad (Y_\gamma = 1 - X_\gamma = \mathbb{1}_{\{\gamma \in G(N_v)\}})$

$\mathbb{P}(X \le \ell) = \mathbb{P}(Y \ge q-\ell) = \mathbb{P}(e^{sY} \ge e^{s(q-\ell)})$

$\le e^{-s(q-\ell)} \boxed{\mathbb{E}\, e^{sY}}$

& $\mathbb{E}\, e^{sY} = \prod \mathbb{E}\, e^{sY_\gamma} = \ldots$

Here we do better:

Claim $\mathbb{E}\, e^{sY} \le \prod \mathbb{E}\, e^{sY_\gamma}$ $\leadsto$ ∎

Pf ① Y_α's are negatively correlated

i.e. $\forall B \subseteq L_v \quad \mathbb{E} \prod_{\alpha \in B} Y_\alpha \le \prod_{\alpha \in B} \mathbb{E} Y_\alpha$ ⊛

[which we know:

$Z = \{\sigma_1, \ldots, \sigma_d\}$, σ_i's indept $\in S$, $B \subseteq S$

$\Rightarrow \mathbb{P}(Z \supseteq B) \le \prod_{j \in B} \mathbb{P}(j \in Z)$]

② $\mathbb{E}\, e^{sY} \overset{\text{l. of } \mathbb{E}}{=} \sum_k \frac{s^k}{k!} \mathbb{E}\, Y^k$ $\quad Y = \sum Y_\gamma$ $\quad \boxed{\text{SO?}}$

expand w l. of $\mathbb{E}$ $\leftarrow$ = pos. comb. of terms of form

$\mathbb{E} \prod_\gamma Y_\gamma^{s_\gamma} \overset{⊛}{\le} \prod_\gamma \mathbb{E}\, Y_\gamma^{s_\gamma}$ (= val. if indept etc)

∎

Quick look at entropy compression

A repetition in a seq. $a_1, a_2, \ldots$ is two identical consec. blocks; i.e. $a_{i+1}, \ldots, a_{i+d}, a_{i+d+1}, \ldots, a_{i+2d}$ with $a_{i+d+j} = a_{i+j}$ for $j \in [d]$

Thm (Thue 1906) $\exists$ infinite rep-free seq. in $\{1,2,3\}^{\mathbb{N}}$

Thm (Grytczuk–Kozik–Micek '13; AS Thm 5.7.6)

$\forall n$ & $L_1, \ldots, L_n$ of size 5

$\exists$ rep-free $s_1, \ldots, s_n$ w/ $s_i \in L_i$ $\forall i$

- $\xrightarrow{\text{cptness}}$ $\forall L_1, \ldots$ w/ $|L_i| = 5$ $\exists$ inf. ...
- AS, Prob 5.8.8 also for size 4

  OPEN (AFAIK): size 3?

Pf (more formal ver. in AS)

randomized alg $\longrightarrow$ seq. of rep-free strings $\sigma_1, \sigma_2, \ldots$

$\sigma_0 = \phi$ & for $j = 1, \ldots$:

step j $\sigma_{j-1} = s_1, \ldots, s_{i-1}$ (rep-free) $\longrightarrow$

s_i unif $\in L_i$ and

if $s_1, \ldots, s_i$ rep-free then this is σ_j

else: • EX: $\exists!$ rep. (in $s_1, \ldots, s_i$)

- remove the second block (so if length t then $\sigma_j = s_1, \ldots, s_{i-t}$)

choice at step j	σ_j
1	1
1	1
2	12
1	121
2	12

Alg succeeds if reaches σ_j of length n

Claim $\exists$ (universal) c s.t. $\forall n$ & $\boxed{M = Cn}$ (any M would be okay)

whp alg succeeds by time M

($\longrightarrow$)

Pf Let $d_j = |\sigma_j| - |\sigma_{j-1}|$ (so in the 1st case above $d_j = 1$ & in the 2nd $d_j = -t+1$)

Let $D_M = (d_1, \ldots, d_M)$ & for an M-step run of alg set

$$LOG = (D_M, \sigma_M)$$

ETS* ① LOG determines the run

② The # of possibilities for the LOG of an unsuccessful M-step run of alg is $< 4^M 5^n$

* why?