L20 slight change: given lists S_v $\left(|S_v| = q = (1+\varepsilon)\frac{\Delta}{\ln \Delta}\right)$

[always: σ, τ partial col's, γ color]

given σ:

- $V^\sigma = \sigma^{-1}(\Lambda)$
- $L_v^\sigma = S_v \setminus \sigma(N_v)$ (colors still allowed at v)
- $N_\gamma^\sigma(v) = \left|\{ w \in N_v \cap V^\sigma : \gamma \in L_w^\sigma \}\right|$

$\boxed{\ell = \Delta^{\varepsilon/2}}$ (we'll see why) (Haxell $\Longrightarrow$)

ETS $\exists$ p.c. σ s.t.

$$\boxed{\forall v \in V^\sigma \quad (1)\ |L_v^\sigma| > \ell \quad (2)\ \gamma \in L_v^\sigma \Rightarrow |N_\gamma^\sigma(v)| < \ell/2} \quad (*)$$

Expt: σ unif p.c. (cf. Shearer)

SHOW: $\mathbb{P}(\sigma \models (*)) > 0$

- using LLL ("lopsided" version, recalled below
  - note no literal independence)

[Molloy (v. rough): GOAL again $\sigma \models (*)$

- initial: any σ (e.g. $\sigma \equiv \Lambda$)
- iterate: flaw: v violating (1) or (2) $\longrightarrow$

$\boxed{\text{recolor}}$ N_v for some carefully chosen flaw v

$\rightsquigarrow$ "ent. compression" (e.g. AS T 5.7.6)

Back to Bernshteyn

[Recall LLL: ($A_1,\dots,A_m$; Γ gph on $[m]$, $\Delta_\Gamma \le d$)

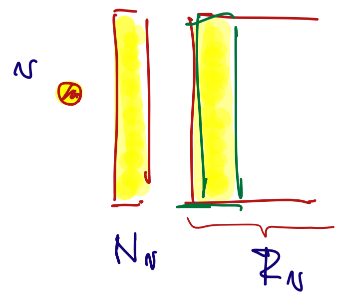
If $\forall i \not\sim S \subseteq [m]\setminus\{i\}$ $\quad \mathbb{P}(A_i \mid \bigcap_{j\in S} \bar{A}_j) \le p$ ⊛

and $\quad e(d+1)\,p \le 1$

then $\quad \mathbb{P}(\bigcap \bar{A}_i) > 0$]

Now (bad) events:

$A_v = \{ \underbrace{v \in V^\sigma}_{\sigma_v = \Lambda}$ s.t. violates ① or ② $\}$



▷ det. by $\sigma|_{\{z \,:\, \mathrm{dist}(v,z) \le 2\}}$

Γ (on V): $\quad v \sim w$ iff $\mathrm{dist}(v,w) \le 3$ $\quad$ (we'll see why)

$\Delta_\Gamma < \Delta^3 \longrightarrow$

For LLL ETS ⊛ w/ $\boxed{p = \Delta^{-3}/e}$

stronger:

▷ ML: $\forall v$ s.t. p.c. τ of R_v $\quad$ (again cf. Shearer)

$$\mathbb{P}(A_v \mid \sigma|_{R_v} = \tau) < p$$

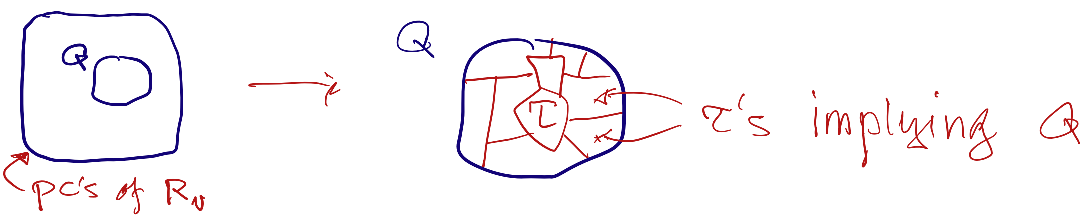
why stronger?

$$Z_v := \{ w \in V : \mathrm{dist}(v,w) \geq 4 \}$$
$$= \{ w \in V \setminus \{v\} : w \not\sim_\Gamma v \}$$

ML $\Rightarrow$ $\forall J \subseteq Z_v$

$$\mathbb{P}\Big(A_v \,\Big|\, \underbrace{\boxed{\bigcap_{w \in J} A_w \cap \bigcap_{w \in Z_v \setminus J} \overline{A_w}}}_{Q}\Big) < p \qquad (\text{why?})$$

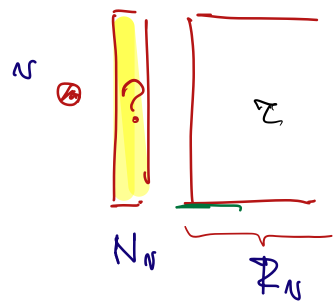
$\longleftrightarrow$ law of total prob. (τ is finer info than Q)



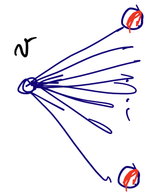
Pf of ML

Obs On $\overbrace{\{\sigma_v = \Lambda\}}^{\text{prereq. for } A_v} \cap \{\sigma|_{R_v} = \tau\}$

✱ $\sigma|_{N_v}$ is unif $\in \prod_{w \in N_v} \left(L_w^\tau \cup \{\Lambda\} \right)$



$\longrightarrow$ back to independence



$L_w := L_w^\tau \ (w \in N_v)$, $L_v := S_v$ (≠ done to τ)

▶ ML' (≈ Malloy) L_w's arb

$\sigma : N_v \to \Gamma \cup \{\Lambda\}$ as in $\boxed{*}$ (det. L_v^σ) $\Longrightarrow$

① $\mathbb{P}(|L_v^\sigma| \leq \ell) < \Delta^{-3}/2e$

② $\mathbb{P}(\exists\, \gamma \in L_v^\sigma \;\; |N_\gamma^\sigma(v)| \geq \ell/2) < \Delta^{-3}/2e$ (won't be close)

▶ Rmk think of the rest as just technical:

ML' – once found – eventually provable (if true)

not: $W_\gamma = \{ w \in N_v : \gamma \in L_w \}$

② (easier): Fix γ & show (which is more than enough):

$$\mathbb{P}(\underbrace{\gamma \in L_v^\sigma , |N_\gamma^\sigma(v)| > \ell/2}_{\gamma \text{ bad}}) \ll \Delta^{-4}$$

note:
- $\gamma \in L_v^\sigma \Leftrightarrow \sigma_w \neq \gamma \;\; \forall\, w \in W_\gamma$
- $N_\gamma^\sigma(v) = \{ w \in W_\gamma : \sigma_w = \Lambda \}$

▶ Rough idea (tradeoff):

W_γ small $\longrightarrow$ $N_\gamma^\sigma(v)$ small (probably)

W_γ large $\longrightarrow$ $\gamma \in L_v^\sigma$ unlikely

concrete:
$$\underbrace{\mathbb{P}(\sigma_w = \Lambda)}_{w \text{ contrib to } N_\gamma^\sigma(v)} = \frac{1}{|L_w| + 1} = \underbrace{\mathbb{P}(\sigma_w = \gamma)}_{\Rightarrow\, \gamma \notin L_v^\sigma}$$

$\longrightarrow$ 2.

(a bit like Haussler-Welzl :)

choose: (a) $\{w : \sigma_w \in \{\gamma, \Lambda\}\} =: Z$

(b) val's for $w \in Z$

$$\{\gamma \text{ bad}\} = \{|Z| > \ell/2,\ \sigma_w = \gamma\ \forall w \in Z\}$$

$\longrightarrow\ \mathbb{P}(\gamma \text{ bad}) \leq 2^{-\ell/2}$ tiny (1)

Now (1): (γ always $\in L_v$)

Examples

(a) $L_w = L_v\ \forall w \in N_v \longrightarrow \forall \gamma \in L_v$

$$\mathbb{P}(\gamma \in L_v^\sigma) = \left(1 - \frac{1}{q+1}\right)^{d_v} \geq \left(1 - \frac{1}{q+1}\right)^{\Delta}$$

$$\approx e^{-\Delta/q} = \Delta^{-\frac{1}{1+\varepsilon}} \approx \Delta^{-1+\varepsilon}$$

$\longrightarrow\ \mathbb{E}\,|L_v^\sigma| \gtrsim q\,\Delta^{-1+\varepsilon} \gg \ell$ $\quad$ [$\rightarrow$ choices of q & ℓ

not the end

L21

(b) (other extreme) $\forall w \in N_v\ L_w = \{\gamma_w\} \Leftarrow \gamma_w \in L_v$

& $\forall \gamma \in L_v\ \{w \in N_v : \gamma_w = \gamma\} = \frac{\Delta}{q} = \frac{\ln \Delta}{1+\varepsilon}$

or $d(v)/q \cdots$

$\longrightarrow\ \mathbb{P}(\gamma \in L_v^\sigma) = 2^{-\Delta/q}$ vs $e^{-\Delta/q}$ in (a)