FACES PLEASE

RECORD

Prob's, handout

"easy" ⟷ not idea, just [illegible]

L2

l.b.'s for $R(k,t)$ (quickly, excuse to play & intro...)

$(V = [n])$ want $E(K_n) = R \cup B$ s.t.

used?

no $R\ K_k$, $B\ K_t$ → "good"

again color at random but now

$$\mathbb{P}(xy \in R) = p \quad \text{(still ind.)}$$

⊛ TBA; of c. $k < t \to p < 1/2$

⊛ no longer ↔ counting (# bad col's < # col's)

U-bd: if $\exists\, p$ s.t.

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1 \quad \text{then } R(k,t) > n \qquad \text{(A)}$$

E.g. (focus on) $\boxed{k=3}$

slight change of viewpt: $R \leftrightarrow G$ $(B \leftrightarrow \bar{G})$

random col $\longleftrightarrow G = G_{n,p}$ ▷ hidden par: $p = p(n)$

gd. col. $\longleftrightarrow G \begin{cases} \Delta\text{-free} \\ \alpha < t \end{cases}$ (⊛)

$R(3,t) = \min n$ s.t. $\nexists\, G \models$ (⊛)

• Ballpark $R(3,t) \leq \binom{t+2}{2} \approx t^2$

alt way to see: G on $[n]$ w/ (⊛):

deg's $< t$ (why?) ⟶

$t > \alpha \geq \frac{n}{t}$ ⟶ ⊠

U-bd: if $\exists\, p$ s.t.

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1 \quad \text{then } R(k,t) > n \quad Ⓐ$$

- what ab. Ⓐ ? (rough, reality check)

$\binom{n}{3}p^3 < 1 \to p \lesssim 1/n \longrightarrow 2^{nd}$ term?

$$\binom{n}{t}(1-p)^{\binom{t}{2}} \approx \Big[\underbrace{\tfrac{n}{t}\, e^{-pt/2}}\Big]^t$$

but for this < 1 need $t = \Omega(n)$ $(\longleftrightarrow \not\to \Phi)$

- Del. method (A-S, T 3.1.3)

$$\forall\, n,p \qquad R(k,t) > n - \underbrace{\left[\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}\right]}_{\mathbb{E}|\{flaws\}|}$$

because ... [$R(k,t) > |V(G)| - |\{flaws\}| \longrightarrow$

$R(k,t) > \mathbb{E}\big[|V(G)| - |\{flaws\}|\big] = n - \mathbb{E}|\{flaws\}|$]

Rough calc's for $k=3$:

lg. const's

- 1^{st} term: $p \approx n^{-2/3}$ ess. free (& that's all)
- 2^{nd} term $\approx \left[\frac{en}{t} e^{-tp/2}\right]^t \to$ ess: need [ ] < 1

$\to$ need $t \gtrsim n^{2/3}$ — enough?

NO, but $t \approx n^{2/3}\log n$ is

$\longleftrightarrow R(3,t) \gtrsim (t/\log t)^{3/2}$ ☑

- Erdős 61: $\Omega(t^2/\log^2 t)$ [reproved: Sp 77 via LLL]
- AKS 81: $O(t^2/\log t)$
- Kim '95: $\Theta(t^2/\log t)$

- AKS MP: G Δ-f $\Rightarrow \alpha(G) = \Omega\left(\frac{n \log \bar{d}}{\bar{d}}\right)$

  (vs. Turán ...)

- $R(3,t)$ (as before): $t > \alpha \geq d_v \quad \forall v \longrightarrow$

  $$t > \alpha > c \frac{n \log \bar{d}}{\bar{d}} \geq c \frac{n \log t}{t}$$

- sphere packing (!) (packing spheres of vol. 1)

  Campos - Jenssen - Michelin - Sahasrabudhe (arxiv '23):

  density $\gtrsim d\, 2^{-d} \log d$

  vs: $\Omega(2^{-d})$ "triv"

  $\approx 2 \cdot 2^{-d}$ Minkowski '05

  $c \cdot d\, 2^{-d}$ Rogers '47

  better cs: 1.68 Davenport-Rogers '47

  2 Ball '92

  6/e Vance '11

  65963 Venkatesh '13

[AKS MP: G Δ-f $\Rightarrow \alpha(G) = \Omega\left(\frac{n \log \bar{d}}{\bar{d}}\right)$]

Heur:

A. assume d-reg, $I :=$ unif ind. set, & pretend:

$$\forall v \quad p \approx \mathbb{P}(v \in I) \approx (1-p)^d \cdot \frac{1}{2} \approx e^{-pd} \qquad \text{(etc.)}$$

TBA — ignore

▶ Δ-free

B. "recall" Caro–Wei ($\Leftrightarrow$ Turán):

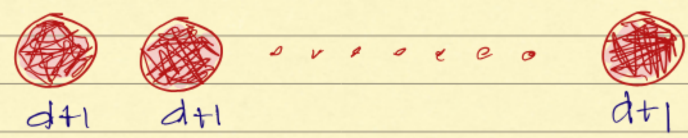
$$\alpha(G) \geq \sum \frac{1}{d_v + 1} \geq \frac{n}{\bar{d} + 1}$$

Jensen

Pf ("random greedy")

σ unif ord. of $[n]$ & I greedy:

$$\mathbb{P}(v \in I) \geq \mathbb{P}(v <_\sigma N_v) \geq \frac{1}{d_v + 1} \rightsquigarrow \blacksquare$$

— waste? 

$\longrightarrow$ NO

but Δ-free $\to$ YES? $\longrightarrow$

- AKS: random greedy tough to analyze* $\rightsquigarrow$

▶ "nibble" (KEY diff: concentration)

* but see Shearer '83

- Kim: again nibble [this one still hard]

$\to R(3,t) = \Omega(t^2/\log t)$

- sketch of Erdős $\left[\Omega(t^2/\log^2 t)\right]$

OBS $R(3,t) > n$ if $\exists$ G on n s.t.

$\forall T \in \binom{[n]}{t}$ $\exists\, xy \in G[T]$ $\boxed{N_x \cap N_y \subseteq T}$ $\Big\}$ "good"

- i.e. $\exists$ Δ-f. H on n w $\alpha(H) < t$

▶ why?

[H: greedy Δ-f subgph: $E(G) = \{e_1, \dots, e_m\}$

$H_0 = \emptyset$, $H_i = \begin{cases} H_{i-1} + e_i & \text{if } \Delta\text{-f} \\ H_{i-1} & \text{———} \end{cases}$

$\rightarrow$ H: Δ-f ✓

▶ $\alpha < t$ ——— why?]

a "pseudorandom" arg

$R(3,t) > n$ <u>if</u> $\exists\ G$ on n s.t.

$\forall T \in \binom{[n]}{t}$ $\exists\ xy \in G[T]$ $\boxed{N_x \cap N_y \subseteq T}$ "good"

<u>"calc's"</u> — just what <u>shd</u> happen → MANTRA ..

t, p still TBA, want <u>t small</u>

- $\mathbb{E}\,|\{\text{gd edges for } T\}| \approx (t^2 p/2)(1-p^2)^{n-t}$ ← given

$$\approx t^2 p \exp[-np^2] =: \mu$$

- (jump for now, later reflex:)

$\mathbb{P}(T \text{ bad}) \approx e^{-\mu}$ [familiar: Poisson binom.]

— need this $\lesssim \binom{n}{t}^{-1} \approx (n/t)^{-t} \approx \exp[-t \log n]$

$\longleftrightarrow$ $tpe^{-np^2} \gtrsim \log n$ →

- $t > p^{-1} \log n$ → want <u>p large</u>
- $p \approx 1/\sqrt{n}$ is free, larger kills

→ $t \gtrsim \sqrt{n} \log n$ etc.

- Larger (fixed) k
  - Ramsey (E-Sz ?) $O(t^{k-1})$
  - AKS (ind., EX*): $O(t^{k-1}/\log^{k-2} t)$
  - Sp $\begin{cases} (\text{'77}) & \Omega((t/\log t)^{(k+1)/2}) \quad (\text{LLL}) \\ (\text{'86}) & \$\$ \text{ for } t^{k-1+o(1)} \end{cases}$
- Mattheus-Verstraete '23: $R(4,t) = \Omega(t^3/\log^4 t)$