Pf I unif $\in \mathcal{I}(G)$

$$X_v := d\,\mathbb{1}_{\{v \in I\}} + |I \cap N_v|$$

MAIN $\forall v$

$$\mathbb{E}X_v = \begin{cases} \Omega(\log d) & r=3 \\ \Omega_r\left(\frac{\log d}{\log\log d}\right) & \text{in gen'l} \end{cases}$$

then done (why?):

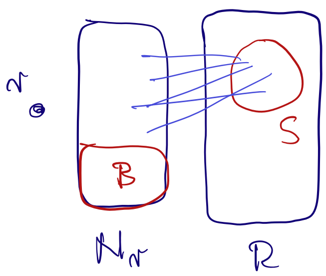
$$\sum X_v = d\,|I| + \sum_{w \in I} d_w \le 2d\,|I|$$

$$\to \mathbb{E}|I| \ge (2d)^{-1} \sum X_v \qquad \text{etc.}$$

$\boxed{\text{D}}$

$\boxed{\text{L19}}$

Pf of MAIN (fix v)



$R = V \setminus (N_v \cup \{v\})$

$S = I \cap R$

Claim $\forall$ indept $s \subseteq R$

$$\mathbb{E}[X_v \mid S = s] = \begin{cases} \Omega(\log d) & r=3 \\ \Omega_r\left(\frac{\log d}{\log\log d}\right) & \text{in gen'l} \end{cases}$$

(this is enough — why?)

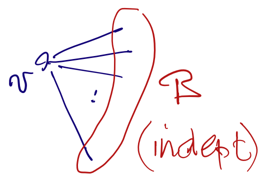
$B := N_v \setminus N_s$ (verts of N_v that can still be in I if $S = s$)

$\Rightarrow$ On $\{S = s\}$ $I \cap (N_v \cup \{v\})$ is unif from

$\{$ind. sets of $B\} \cup \{\{v\}\}$ $(= \{$ind sets of $B \cup \{v\}\})$

$\boxed{\log = \log_2}$

$r=3$: $G[B \cup \{v\}]$ = star at v $\longrightarrow$



ind. sets: $\{v\}$ & 2^B $\longrightarrow$

$$\mathbb{E}[X_v \mid S = S] = \frac{d}{2^b+1} + \frac{2^b}{2^b+1} \cdot \frac{b}{2} \qquad (b = |B|)$$

$$= \Omega(\log d) \qquad \square$$

if $b < \log d - \log\log d$ then 1^{st} term $\gtrsim \log d$

else 2^{nd} term $\gtrsim \frac{1}{2}\log d$

$r > 3$

Lemma H K_r-free, $i(H) = M \Longrightarrow$

$$\bar{\alpha}(H) > c_r \frac{\log M}{\log\log M}$$ (not the c_r of the thm)

notes ① $\alpha \le \log M$ triv.

② nonsense w/o K_r-free (e.g. for K_n)

③ typ. for $G_{n,p}$ (say $p = n^{-\gamma}$, $\gamma \in (0,1)$ fixed):

$$\alpha \approx \frac{2}{p}\ln(np), \quad i(G) = \exp[\Theta(np \log^2(np))]$$

(EX: this looks right)

then done: $H := G[B]$ K_{r-1}-free ($M = i(H)$)

$$\rightarrow \mathbb{E}[X_v \mid S = S] = \frac{d}{M+1} + \frac{M}{M+1}\bar{\alpha}(H)$$

$$= \frac{d}{M+1} + \Omega\left(\frac{\log M}{\log\log M}\right) \overset{*}{=} \Omega\left(\frac{\log d}{\log\log d}\right) \qquad \square$$

$*$ if $M < d/\log d$ then 1^{st} term $> \log d$; else ...

Pf of Lemma

① Obs H K_r-free (on n) $\Rightarrow$ $\alpha(H) > n^{\varepsilon}$ ($\varepsilon = \frac{1}{r-1}$; we need $\Omega(1)$)

[slightly rough: $k = \alpha(H)+1$ $\longrightarrow$ $n < R(r,k) \leq \binom{k+r-2}{r-1} < k^{r-1}$]

$\longrightarrow$ $n^{\varepsilon} < \log M \leq n$, say $\log M = n^{\beta}$ (defines β)

② aiming for $\bar{\alpha} \gtrsim \frac{\log M}{\log\log M} \simeq \frac{n^{\beta}}{\beta \log n}$ $\longrightarrow$

ETS $\left|\left\{ I \in \mathcal{I}(G) : |I| \leq k := \frac{n^{\beta}}{\log n} \right\}\right| \ll M$

just convenient

but

$$\text{l.h.s.} < \binom{n}{\leq k} < \exp_2\left[\frac{n^{\beta}}{\log n} \log\left(\frac{en}{k}\right)\right]$$

$$\approx \exp_2\left[(1-\beta) n^{\beta}\right] \ll 2^{n^{\beta}} = M$$

☒

How about χ, χ_ℓ for $\triangle$-free (or K_r-free) G?

AKS $\longrightarrow$ for $\triangle$-f may hope for $\chi, \chi_\ell < C \frac{\Delta}{\log \Delta}$

- this the truth for $G_{n,p}$ (p ...)
- avg deg r.g. – right?

▶ A. Johansson:

- (unpub '96) $\triangle$-f $\Rightarrow$ $(\chi(G) \leq)$ $\chi_\ell(G) < C \frac{\Delta}{\log \Delta}$

  nibble + ent & hard; see Molloy-Reed

- (unpub '96) K_r-f $\Rightarrow$ ... $< C_r \frac{\Delta \log\log \Delta}{\log \Delta}$

▶ Molloy '19: new proof(s)

- better const for $\triangle$-f (not MP for us)
- easier & v. int'g:

randomized recoloring & "ent. compr." ($\leftrightarrow$ Moser-Tardos)

$\boxed{\text{simpler ex: GKM = AS T 5.7.6}}$

▶ Bernshteyn '19: same w/o recoloring

– well do $\triangle$-free – surprisingly easy rel. previous

Thm G $\triangle$-free $\Rightarrow$ $\chi_\ell(G) < (1+o(1)) \frac{\Delta}{\ln \Delta}$

$(o(1) = o_\Delta(1), \ \Delta = \Delta_G)$

equiv: $\forall \varepsilon > 0$, G $\triangle$-f. $\Rightarrow$ $\chi_\ell(G) < (1+\varepsilon) \frac{\Delta}{\ln \Delta}$

for $\Delta > \Delta_\varepsilon$

Prelim: def: $V_1, \dots, V_m$ disjt sets →

X transversal (of the V_i's) if $|X \cap V_i| = 1 \ \forall i$

Alon '88 $V_1, \dots, V_m \subseteq V(H)$ disjt, $|V_i| > 2e\Delta_H \ \forall i$

$\Rightarrow \exists$ indept transv.

Pf (easy) EX $\bar{w}$ LLL; AS P5.5.3 $\square$

Remks

Haxell '01: $|V_i| \geq 2\Delta_H$ is enough

- Szabó-Tardos 06: $2\Delta_H - 1$ is not enough (!)
- Cor (why? ⊛) F gph, $L_v \subseteq \Gamma$ $(v \in V = V(F))$

If ① $|L_v| \geq 2d$ $\forall v \in V$ and

② $\forall v \in V$ & $\gamma \in L_v$, $\underbrace{|\{w \sim v : \gamma \in L_w\}|}_{\text{"color deg"}} \leq d$

then $\exists$ L-coloring

⊛ $V(H) = \{(v, \gamma) : v \in V(F),\ \gamma \in S_v\}$

$E(H)$: $(v, \gamma) \sim (w, \gamma)$ $\forall v \sim w$, $\gamma \in S_v \cap S_w$

$V_v = \{(v, \gamma) : \gamma \in S_v\}$ $(v \in V(H))$ $\square$

Reed-Sudakov '02: $|L_v| > (1 + o_d(1))\, d$ is enough

(via nibble; see also Glock-Sudakov '22)

Back to Bernshteyn's setup:

- lists L_v $(\subseteq \Gamma)$

$$|L_v| = q := (1+\varepsilon)\frac{\Delta}{\ln \Delta}$$ (think Δ large)

- partial col'g: $\sigma : V \to \Gamma \cup \{\Lambda\}$ s.t. (Λ → blank)

  ① $\sigma_v \in L_v \cup \{\Lambda\}$ $\forall v$

  ② proper: $\nexists$ $u \sim v$ w/ $\sigma_u = \sigma_v \neq \Lambda$

[always: σ, τ partial col's, γ color]

$\boxed{L20}$

given σ:

- $V^\sigma = \sigma^{-1}(\Lambda)$
- $L_v^\sigma = L_v \setminus \sigma(N_v)$ (colors still allowed at v)
- $N_\gamma^\sigma(v) = \left|\{ w \in N_v \cap V^\sigma : \gamma \in L_w^\sigma \}\right|$

$\boxed{\ell = \Delta^{\varepsilon/2}}$ (we'll see why)

(Haxell $\Longrightarrow$)

$\boxed{ETS}$ $\exists$ p.c. σ s.t.

$$\forall v \in V^\sigma \quad \left.\begin{array}{l} ① \ |L_v^\sigma| > \ell \\ ② \ \gamma \in L_v^\sigma \Rightarrow |N_\gamma^\sigma(v)| < \ell/2 \end{array}\right\} (*)$$