L18

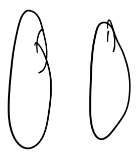
Back to S-T

Lists: same as Alon, but w/ bigger t (& better anal)

- $t \sim \log_2 d$ ¬BA, $s = t^2$, $\Gamma = [s]$,
  L_v's unif (indept) $\in \binom{\Gamma}{t}$ → whp ∄ L-col'g ↓

Persp: $X := |\{L\text{-colorings}\}|$ → whp $X = 0$?

how ab. $\mathbb{E}X$ small?

NO:  $K_{d,d}$

X Y fix $\Gamma = \Gamma_X \sqcup \Gamma_Y$ equipartn

→ $(s/2)^n$ potential cols → $\mathbb{E}X > \left(\frac{s}{2}\right)^n \left(\frac{t}{s}\right)^n = \left(\frac{t}{2}\right)^n$

→ ??

vs. (for too small t) $\exists L_x \subseteq \Gamma_Y$ or v.v. → $X = 0$

but <u>if not</u> then $X \approx (t/2)^n$ is typ.

[sim (e.g.) threshold for p.m.s]

Apply container thm w/ arb. (fixed, pos) $\varepsilon \longrightarrow \mathcal{C}$

- L-coloring is $(I_1, \dots, I_s)$ w/ I_j indept &

  $v \in I_j \Rightarrow j \in L_v$ (*)

- L-collaring: $C = (C_1, \dots, C_s)$ w/ $C_j \in \mathcal{C}$ &

  $\forall v \ \exists j \in L_v \ \ v \in C_j$ (**)

ETS whp. ∄ L-collaring ($\exists$ L-coloring $\Rightarrow \exists$ L-collaring)

loss: (**) is easier than (*)

gain: # potential L-collarings < $\exp[B \cdot \frac{n}{d} \log d \cdot s]$

(vs. possibly $\exp[\Omega(n \cdot s)]$ pot. L-colorings)

Ex. $K_{d,d}$ again. Here can take $\mathcal{C} = \{X, Y\}$ $\longrightarrow$

potential coll's $= 2^s$ (not using thm)

$C = (X, X, \dots, X, Y, \dots, Y)$ actual coll'g $\Leftrightarrow$

Γ_X (the X's), Γ_Y (the Y's)

$x \in X \Rightarrow L_x \not\subseteq \Gamma_Y$ & ... $\longrightarrow$ prob $\lesssim (1-2^{-t})^{2d}$ (smaller if $|\Gamma_X| \neq |\Gamma_Y|$)

$\sum \mathbb{P}(C \text{ coll'g}) \approx 2^{t^2} \exp[-2^{-t} 2d]$ (2^{t^2} negl.)

$= o(1)$ if $t = \log_2 d - 2\log_2 \log_2 d$ (as in Erdős-Rubin-Taylor)

(Back to thm)

$$\mathbb{P}(X \neq 0) \le \sum_C \mathbb{P}(\underbrace{C \models ⊛⊛}_{\text{good}})$$

given C: $F_v = \{ j : v \notin C_j \}$

$\alpha_v = |F_v| / s$

C good $\iff L_v \notin F_v \;\; \forall v$

$$\alpha = n^{-1} \sum \alpha_v = n^{-1} s^{-1} \sum |F_v| = n^{-1} s^{-1} \sum_j (n - |C_j|) > \tfrac{1}{2} - \varepsilon$$

$$\mathbb{P}(C \text{ gd}) \overset{⊛}{\approx} \prod (1 - \alpha_v^t) < \exp[-\sum \alpha_v^t]$$

Janssen ← $\le$ $\exp[-n \alpha^t] < \exp[-n(\tfrac{1}{2} - \varepsilon)^t]$

$$\sum_C \mathbb{P}(C \text{ good}) \lesssim \exp[B \tfrac{n}{d} \log d \cdot s - n(\tfrac{1}{2} - \varepsilon)^t]$$

$\underset{\text{check}}{=} o(1)$ for some $t = (1 - O(\varepsilon)) \log_2 d$

∎

⊛ larger if α_v small

Back to AKS [my int'g conditioning + more LLL, list-col'g, neg. corr.]

recall (from lec 2)

- AKS '81: G Δ-free, $\bar{d} \le d \Rightarrow \alpha(G) > c \frac{n \ln d}{d}$
  - mot: Sidon sets, $R(3,k) = O(k^2/\log k)$
  - pf in 1 lie: random greedy
  - Shearer '83: $c \sim 1$ ("easy" ind. pf)
    can't beat $c \approx 2$ (typ. for $G_{n,p}$)
    & random greedy does work
  - nat: ~~what ab. K_r-free~~ ($r > 3$ fixed)?
    AEKS '81: $\alpha > c_r \frac{n \log\log d}{d}$

(WE DO) Shearer '95: $c_r \frac{n \log d}{d \log\log d}$ → get rid of this?

(w amazingly simple pf of AKS – we'll see)

cf. AS p.336

Actually show:

$$\Delta_G \le d \Rightarrow \bar{\alpha}(G) > \begin{cases} c \frac{n \log d}{d} & \Delta\text{-free} \\ c_r \frac{n \log d}{d \log\log d} & K_r\text{-free } (r \geq 4) \end{cases}$$

- This is enough (why?)
  [e.g. $\bar{d} \le d \Rightarrow |\{v : d_v \le 2d\}| \geq n/2$]

- EX: statement for α → statement for $\bar{\alpha}$ (w diff c)

Pf $\mathbb{I}$ unif $\in \mathcal{I}(G)$

$$X_v := d\,\mathbb{1}_{\{v \in \mathbb{I}\}} + |\mathbb{I} \cap N_v|$$

MAIN $\forall v$

$$\mathbb{E} X_v = \begin{cases} \Omega(\log d) & r=3 \\ \Omega_r\left(\frac{\log d}{\log\log d}\right) & \text{in gen'l} \end{cases}$$

then done (why?):

$$\sum X_v = d\,|\mathbb{I}| + \sum_{w \in \mathbb{I}} d_w \leq 2d\,|\mathbb{I}|$$

$$\rightarrow \mathbb{E}\,|\mathbb{I}| \geq (2d)^{-1} \sum X_v \qquad \text{etc.}$$

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