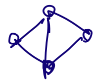
RECAP $H =$  , $\boxed{p \gg n^{-4/5}}$ $\quad [G = G_{n,p}]$

$H_1, \dots, H_m$ copies of H in K_n $\quad [m = \Theta(n^4)]$

$$X_\alpha = \mathbb{1}_{\{H_\alpha \subseteq G\}}, \quad X = \sum X_\alpha, \quad \mu = \mathbb{E}X = mp^5$$

$$\mathbb{E} X_\alpha X_\beta = p^{|H_\alpha \cup H_\beta|} = p^{10 - |H_\alpha \cap H_\beta|}$$

GOAL: $\boxed{(\sigma^2 =) \quad \mathbb{E}X^2 - \mu^2 \ll \mu^2} = m^2 p^{10}$

$$\mathbb{E}X^2 - \mu^2 = m \sum_{\beta \sim 1} \left(\mathbb{E}X_1 X_\beta - \mathbb{E}X_1 \mathbb{E}X_\beta \right)$$

$$\leq m \sum_{\beta \sim 1} \mathbb{E}X_1 X_\beta = mp^{10} \sum_{\beta \sim 1} p^{-|H_1 \cap H_\beta|}$$

$$= o(\mu^2) \iff \boxed{\sum_{\beta \sim 1} p^{-|H_1 \cap H_\beta|} = o(m)} \quad (?)$$

$$N_t := \left| \{ \beta : |H_1 \cap H_\beta| = t \} \right|$$

$$\text{l.h.s. of } (?) = \sum_{t=1}^{5} N_t p^{-t} \longrightarrow$$

$$(?) \iff \boxed{N_t p^{-t} = o(n^4) \quad t \in [5]}$$

$N_t := \left|\{\beta : |H_1 \cap H_\beta| = t\}\right|$

WANT: $\boxed{N_t p^{-t} = o(n^4) \quad t \in [5]}$

$\boxed{\text{L17}}$

MP: $|H_1 \cap H_\beta| = t \Rightarrow |V_1 \cap V_\beta| \geq ?$

$V_\alpha = V(H_\alpha)$

$\longrightarrow N_t = \Theta(n^{4-?})$

(CHECK:)

t	1	2	3	4	5
?	2	3	3	4	4

e.g. $t = 3$: $N_3 = O(n)$

$N_3 p^{-3} = o(n^{1+12/5})$ (okay)

e.g. $t = 5$ (diagonal)

$N_1 = 1, \quad N_1 p^{-5} = o(n^4)$

only point that needs $p \gg n^{-4/5}$

$(\longleftrightarrow \sum \mathbb{E} X_\alpha^2 \ll \mu^2$ for ind.$)$

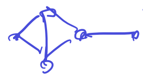
gen'l (fixed) H (briefly)

"density": $\rho(H) = e_H / v_H$ [e_H, v_H ...]

$\mathbb{E}[\#\text{ of } H\text{'s in } G_{n,p}] = \Theta(n^{v_H} p^{e_H})$

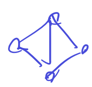
$= \Theta(1)$ if $p = \Theta(n^{-1/\rho(H)})$

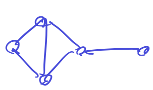
$\rho^*(H) = \max\{\rho(K) : \emptyset \neq K \subseteq H\}$

e.g. $\rho^*($  $) = 5/4$

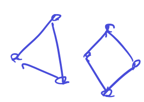
H is balanced if $\rho^*(H) = \rho(H)$

strictly bal. if $\rho(K) < \rho(H)$ $\forall K \subsetneq H$

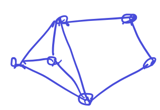
[str. bal: 

unbal: 

bal, not strict:



$\rho = 1$

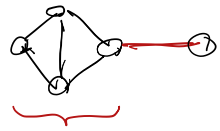


$\rho = 3/2$]

Thm (ER60 for bal.; Boll 81 in gen'l)

$\forall$ (fixed) H

$n^{-1/\rho^*(H)}$ is a threshold for {contain H}

e.g. $K =$  , $\quad p = n^{-\beta}$

H

β fixed $\in \left(\frac{4}{5}, \frac{5}{6}\right)$

$Y = \#$ of K's in $G := G_{n,p} \quad \longrightarrow$

$\mathbb{E}Y = \Theta(n^5 p^6) = \Theta(n^{5-6\beta}) \longrightarrow \infty$

but $\quad \mathbb{P}(\underbrace{Y \neq 0}_{G \supseteq K}) \le \mathbb{P}(X \neq 0) \longrightarrow 0$

$\hookrightarrow$ # of H's in G

[hard part of "$\supseteq K$" is "$\supseteq H$"

— if $G \supseteq H$ (rare) then typ many K's]

Pf of ER/B: (ess. same; MP :)

$$|H_1 \cap H_\beta| = t \Rightarrow |V_1 \cap V_\beta| \ge \ell \quad (= ?)$$

$$\longrightarrow N_t p^{-t} \ll n^{v_H - \ell + t/p^*} \qquad [p^* = p^*(H)]$$

$$\overset{?}{=} O(n^{v_H})$$

$$? \iff \boxed{p^* \ge t/\ell} \qquad (2)$$

Back to Erdős $R(k,k)$ "what's lost in $\cup$ bd?"

$$\mathbb{P}(\underbrace{\exists \text{ mono } K_k}_{X \neq 0}) \underset{\cup\text{-bd}}{\leq} \mathbb{E}\underbrace{|\{\text{mono } K_k\text{'s}\}|}_{X} = \binom{n}{k} 2 \cdot 2^{-\binom{k}{2}}$$

Erdős: $\underbrace{\mathbb{E} < 1}_{A} \Rightarrow \mathbb{P}(X \neq 0) < 1 \quad (\Rightarrow R(k,k) > n)$

EX (A-S, T4.5.1): $\underbrace{\mathbb{E} \to \infty}_{B} \Rightarrow \mathbb{P}(X \neq 0) \to 1$

A vs. B: tiny change in n

One more (already men'd): again $G_{n,p}$

$H =$ p.m. (I I ⋯ I ; $2 \mid n$) H NOT FIXED

$\mathcal{F} = \{\text{cont. } H\}$, $X = \#$ p.m.'s

EX: $\mathbb{E}X = \dfrac{n!}{(n/2)!\, 2^{n/2}}\, p^{n/2} \approx (np/e)^{n/2}$

e.g. $p = 100/n \rightarrow$ • $\mathbb{E}X$ huge

• $\mathbb{P}(X \neq 0) \rightarrow 0$ (why?)

(recall:) $\mathbb{P}(v \text{ isolated}) = (1-p)^n \approx e^{-\boxed{c}}$ $= 100$

$|\{\text{iso's}\}| \underset{\text{typ}}{\approx} \mathbb{E}|\{\text{iso's}\}| \approx e^{-c} n$

we "saw"

- TRUTH: threshold $= \Theta\left(\frac{\log n}{n}\right)$

  ($\mathbb{E}|\{\text{iso's}\}| \approx 1 \leftrightarrow p \approx \frac{\ln n}{n}$)

  [below th: • us. $\exists$ iso's $\rightarrow X = 0$

  • if no iso's then X us. large]

- "hitting time" version

- Sim (but harder): Ham. cycle (stop at $\delta_G = 2$)