L15

<u>Pf of LLL</u> (note $\mathbb{P}(A|BC) = \mathbb{P}(AB|C)/\mathbb{P}(B|C)$)

$$\mathbb{P}(\cap \bar{A}_i) = \mathbb{P}(\bar{A}_1)\,\mathbb{P}(\bar{A}_2|\bar{A}_1)\cdots \longrightarrow$$

<u>ETS</u>: $\forall\, S \subseteq [m]$ & $i \in [m]\setminus S$

$$\mathbb{P}(A_i \mid \bigcap_{j\in S} \bar{A}_j) < \boxed{ep} \to \text{QED, but } \underline{\text{NB}}\text{: supports induction}$$

<u>Pf</u> (ind on $|S|$)

Say $S = S_1 \cup S_2$ w $i \begin{cases} \sim j & j \in S_1 \\ \not\sim j & j \in S_2 \end{cases} \longrightarrow$

$$\mathbb{P}(A_i \mid \bigcap_{j\in S} \bar{A}_j) = \frac{\mathbb{P}(A_i \cap \bigcap_{j\in S_1} \bar{A}_j \mid \underbrace{\boxed{\bigcap_{j\in S_2} \bar{A}_j}}_{Q})}{\mathbb{P}(\bigcap_{j\in S_1} \bar{A}_j \mid Q)}$$

NUM $\le \mathbb{P}(A_i|Q) = p_i$

DENOM: WMA $S_1 = [r]$ $(r \le d) \longrightarrow$

$$\text{DENOM} = \mathbb{P}(\bar{A}_1|Q)\,\mathbb{P}(\bar{A}_2|Q\bar{A}_1)\cdots$$

$$\underset{\text{ind}}{\ge} (1-ep)^r \ge \left(1-\frac{1}{d+1}\right)^d \ge e^{-1}$$

$\square$

notes

① don't need independence: pf same ass'g only

$\forall i \; \forall S \subseteq [m] \setminus \{i\} \quad \mathbb{P}(A_i \mid \bigcap_{j \in S} \bar{A}_j) \leq p$

— $\exists$ appl. in AS § 5.6 & we'll see another

② asym. ver. (G still dep. gph)

if $x_1, \dots, x_m \in [0,1]$ sat.

$$\mathbb{P}(A_i) \leq x_i \prod_{j \sim i} (1 - x_j) \qquad \forall i$$

then $\mathbb{P}(\bigcap \bar{A}_i) \geq \prod (1 - x_i)$

- EX: (a) $\Rightarrow$ ver. above
  (b) pf of asym. ver.

- again, enough if $\mathbb{P}(A_i \mid \bigcap_{j \in S} \bar{A}_j) \leq \dots$

- useful (of c.) when p_i's vary
  sample appl's: Zhao § 6.2

③ directed ver. (AS, L5.1.1)

④ algorithmic ver: Moser, Moser-Tardos (!); AS § 5.7

(heading for another appl.; excuse to mention l.-c.)

list coloring (review?) G gph on V [ref: any GT book

$\forall v \in V$ $S_v \subseteq \Gamma = \{\text{"colors"}\}$, $S = (S_v : v \in V)$
"list"

$\sigma : V \to \Gamma$ is $\left\{\begin{matrix}\text{S-legal} \\ \text{an S-col'g}\end{matrix}\right\}$ if proper & $\sigma_v \in S_v \ \forall v$

▷ list-chromatic # (choice #, choosability)

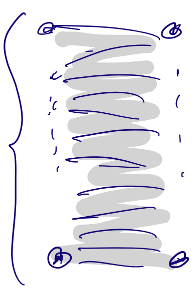
$\chi_\ell(G) := \min\{t : |S_v| = t \ \forall v \Rightarrow \exists\ S\text{-col'g}\}$ $(\geq \chi(G))$
$(ch(G))$ ↳ or $\geq$

Ex 1 (usual first ex): $\chi_\ell(K_{33}) = 3$ $(> 2 = \chi(K_{33}))$

Ex 0: $\chi_\ell(K_n) = ?$

l.-c. natural . . .

(Ex 1 $\subseteq$) Ex 2: $d = \binom{2k-1}{k} \Rightarrow \chi_\ell(K_{dd}) > k$ why?
$(\sim \frac{1}{2}\log_2 d)$



lists on each side: all k-sets

each color used on only one side $\Rightarrow$
$\exists$ side using $\leq k-1$ colors (etc) ☒

actually (EX) $\chi_\ell(K_{dd}) \sim \log_2 d$. (Erdős-Rubin-Taylor '76)

① if $\exists$ non 2-colorable, k-unif $\mathcal{H}$ w $|\mathcal{H}| = d$, then $\chi_\ell(K_{dd}) > k$

② $\exists$ such an $\mathcal{H}$ w $d = O(k^2 2^k)$ [cf. AS §§ 1.3 & 3.6]

In fact (Saxton-Thomason '15; AS §1.6):

▶ $\forall G \quad \chi_\ell(G) \geq (1-o(1)) \log_2 \delta_G$

we'll come back; for now just:

▶ Prop G bip $\Longrightarrow$ $\chi_\ell(G) = O(\Delta / \log \Delta)$ $\quad (\Delta = \Delta_G)$

$\longrightarrow \quad \max\{\chi_\ell(G) : \text{bip}, \Delta \leq d\} \in (\Omega(\log d), O(d/\log d))$

▶ what's the truth? (any improvement ...)

Pf of Prop. $t = c\frac{d}{\log d}$ (c TBA), $|S_v| = t \;\forall v$

choose $\sigma : \Upsilon \longrightarrow \Gamma'$, σ_y unif $\in S_y$ (indeptly)

$A_x = \{S_x \subseteq \sigma(N_x)\}$

ETS $\quad \mathbb{P}(\cap \bar{A}_x) > 0$

Dep. gph H on X: $\quad x \sim x' \iff N_x \cap N_{x'} \neq \emptyset$

by "Mut. Ind. Princ."

• $\Delta_H \leq d(d-1)$

▶ $\mathbb{P}(A_x)$?

WMA (the worst) $\quad d_x = d, \; S_y = S_x \;\forall y \sim x$

$\gamma \in S_x \longrightarrow \mathbb{P}(\gamma \notin \sigma(N_x)) = (1 - 1/t)^d \longrightarrow$

▶ $\mathbb{P}(A_x) \overset{?}{\leq} \prod_{\gamma \in S_x} \mathbb{P}(\gamma \in \sigma(N_x))$

$= \left[1 - (1 - 1/t)^d\right]^t =: p$

$\xrightarrow{\text{take } c > 1}$ $e(d^2 - d + 1)p$ $\quad$ tiny $\quad$ (t large) $\quad$ ∎