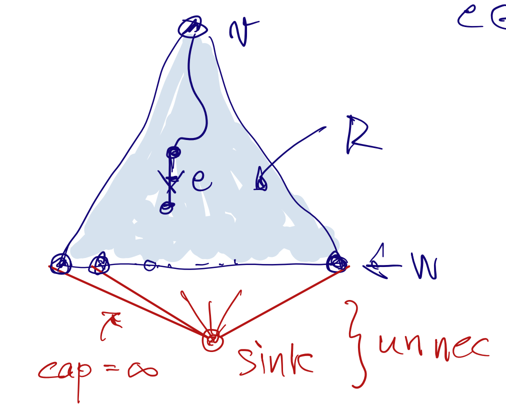
network:



$e \in R \to \mathrm{cap}(e) = C q^{|e|}$

[ref: Diestel or any book on GT]

$(*) + \text{mfmc}^{*} \longrightarrow \exists\ (v, W)\text{-flow } \varphi$ s.t.

(i) $\varphi(e) \leq C p^{|e|}$

(ii) $\boxed{\text{strength}} = 1$

a.k.a. "value" $= \sum_{e \ni v} \varphi(e) = \sum_{y \in W} \varphi(e_y)$

$*$ max-flow min-cut

Now μ: define on all of R & think: implementing "ROUGH"

$\mu(v) := 1$

$v \neq y \in R \to \mu(y) = \varphi(e_y) \leq C p^{|y|}$ flow $\left\{ \begin{matrix} \text{into} \\ \text{out of} \end{matrix} \right\}$ y [small if $|y|$ large (see below)

$\longrightarrow \sum_{y \in V_k} \mu(y) = 1 \quad \forall\, k \leq n$

(remk: same to $V_k \leftarrow$ any min'l cut in R, but not needed)

$\boxed{\text{L14}}$

$(\mathbb{E}X^2 =) \sum \sum \mu(y)\mu(z)\, q^{-|y \wedge z|}$ [always: $y, z \in W$, $w \in R$

$= \sum_{w} \sum_{y \wedge z = w} \mu(y)\mu(z)\, q^{-|w|}$

$\leq \sum_{k=0}^{n} q^{-k} \sum_{w \in V_k} \sum_{\boxed{y, z \leq w}} \mu(y)\mu(z)$ → see minor ⁘ ...

$= \sum_{k=0}^{n} q^{-k} \sum_{w \in V_k} \Big[\boxed{\sum_{y \leq w} \mu(y)} \Big]^2$ → $\mu(w)$ cf. "implementing"

$\leq \sum_{k=0}^{n} q^{-k} C p^{k} \leq C \sum_{k \geq 0} (p/q)^k =: K \ (< \infty)$

why?

Lovász Local Lemma (briefly; see also A-S, Zhao)

framework (cf. "Poisson Par"):

$A_1, \dots, A_m$ "bad" events, $p_i := \mathbb{P}(A_i)$ think $\begin{cases} p_i \text{ small} \\ m \text{ large} \end{cases}$

hope A_i's approx. ind (= ??) $\Longrightarrow$ $\boxed{\mathbb{P}(\cap \bar{A}_i) > 0}$

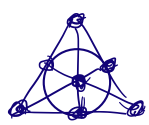
vs $\cup$-bd, where typ. $\mathbb{P} \to 1$

e.g. indept $\to$ $\mathbb{P}(\cap \bar{A}_i) = \prod (1-p_i) > 0$ but maybe tiny

▷ e.g. $\mathcal{H}$ hypergraph (on V) $\to$ discrepancy: [A-S, Ch. 13]

$$\text{disc}(\mathcal{H}) = \min_{\sigma: V \to \{\pm 1\}} \max_{H \in \mathcal{H}} |\sigma(H)|$$

$$\left(= \min_{V = R \oplus B} \max_{H \in \mathcal{H}} \big| |H \cap R| - |H \cap B| \big| \right)$$

e.g.  disc. = 3 (not 2-colorable)

▷ $f(t) := \max\{\text{disc}(\mathcal{H}) : t\text{-unif}, t\text{-reg}\} \approx ?$

e.g. $f(3) = 3$

$\to$ color at random?

$$\mathbb{P}(\underbrace{|\sigma(H)| > \lambda}_{A_H}) < 2e^{-\lambda^2/2t}$$ small if $\lambda \gg \sqrt{t}$

- want $\mathbb{P}(\cap \bar{A}_H) > 0$ ($\Leftrightarrow$ disc($\mathcal{H}$) $\le \lambda$)

  w/ λ as small as poss.

- $\cup$ bd? useless

- A_H's $\approx$ indept ?? ...

Def G on $[m]$ (or e.g. $\mathcal{H}$)

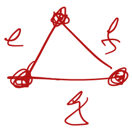
dependency / Lovász graph for $A_1,\dots,A_m$ if

$\forall i$ $\quad A_i$ indept of $\{A_j : j \not\sim i\}$ in G

[• A ind of $\{A_j : j \in J\}$ if

$\forall I \subseteq J \quad \mathbb{P}\left(A \mid \bigcap_{j\in I} A_j \cap \bigcap_{j\in J\setminus I} \bar{A}_j\right) = \mathbb{P}(A)$

• N.B. stronger than A_i, A_j ind $\forall j \in J$

e.g. 

$\sigma : V \to \{R, B\}$ unif
$A_e = \{e \text{ monochr}\}$ $\to$

A's pairwise indept but $A_e A_f \Rightarrow A_g$]

e.g. A_H's above $\to$ $H \sim H' \Leftrightarrow H \cap H'$ Lov. gph on $\mathcal{H}$

[most common source of Lov. gph: X_i's indept $\to$
A det by $(X_i : i \in I)$ ind. of all events det'd by $(X_j : j \notin I)$
– "Mutual Independence Princ" in Molloy-Reed]

Thm LLL (in Erdős–Lovász 75*) w not. as above

if ① $p_i \le p \quad \forall i$

② $\exists$ Lov. gph. w $\Delta \le d$

③ $ep(d+1) \le 1$

then $\mathbb{P}\left(\bigcap \bar{A}_i\right) > (1-ep)^m > 0$
$\to$ if $d > 0$; $d = 0 \to A_i$'s ind.

* EL: $4pd < 1 \Rightarrow \mathbb{P}\left(\bigcap \bar{A}_i\right) > (1-2p)$ – diff a.a. unimp.

persp: • e.g. $m = d+1$, $A_1, \dots, A_m$ disjt, $p_i = 1/d+1$, $G = K_m$

$\to$ $pd < 1$ & $\cap \bar{A}_i = \phi$ $\longrightarrow$

can't relax ③ by much $\xrightarrow{\text{in fact}}$ Shearer '85: can't shrink e

• or $\cup$ bd: $G = K_{d+1}$ & aim for $(d+1)p < 1$

back to discrep: $d \leq t(t-1)$ $\longrightarrow$

want $e(t^2-t+1)e^{-\lambda^2/2t} \leq 1$ $\rightsquigarrow$ $f(t) < (2+\varepsilon)\sqrt{t \ln t}$

$2+\varepsilon \leftrightarrow$ slightly large t (anyway, don't care)

Conj (Beck (?) ~81) $\Delta_{\mathcal{H}} \leq t \Rightarrow \operatorname{disc}(\mathcal{H}) = O(\sqrt{t})$

• is it obv. $\exists$ bd.?

Beck-Fiala Thm '81: $\operatorname{disc} \leq 2t-1$; AS §13.5

see also Komlós Conj. (Zhao C 5.1.5)