L13

▷ p.g.f. for $\mathbb{N}$-val'd X: $f_X(s) = \sum \mathbb{P}(X=k)s^k$ $(= \mathbb{E}s^X)$
(prob. gen. fn.)

we do: $f = f_1 = f_L$, $f_n = f_{Z_n}$

⊛ $f_n(0) = \mathbb{P}(Z_n = 0) \nearrow \mathbb{P}(\mathcal{E})$

▷ Obs $f_n = \underbrace{f \circ \cdots \circ f}_{n}$

Pf ETS: X, Y $\mathbb{N}$-val'd & $Z = X \circ Y := Y_1 + \cdots + Y_X$ with

nanstd

$Y_{1,\dots}$ copies of Y & $X, Y_1, \dots$ indept

$$\Rightarrow \boxed{f_Z = f_X \circ f_Y} \quad ⊛$$

(& $f_{Z_n} = f_{Z_{n-1}} \circ f = f \circ f_{Z_{n-1}} \longrightarrow$ ☐)

Pf of ⊛ $p_k := \mathbb{P}(X=k)$, $q_\ell := \mathbb{P}(Y=\ell)$

$$\mathcal{C}_m f_Z(s) = \sum p_k \sum_{\ell_1+\cdots+\ell_k = m} q_{\ell_1} \cdots q_{\ell_k}$$

coeff of s^m

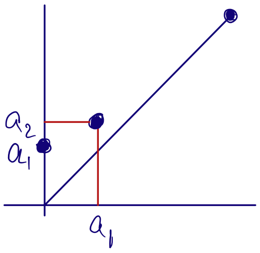
$$= \mathcal{C}_m \sum p_k \left(\sum q_\ell s^\ell\right)^k = \mathcal{C}_m f_X(f_Y(s))$$

☐

EX: reprove using $f_W(s) = \mathbb{E}s^W$

[$f_Z(s) = \mathbb{E}s^Z = \mathbb{E}\{\mathbb{E}[s^Z | X]\} = \mathbb{E}\{\mathbb{E}s^{Y_1+\cdots+Y_X}\}$

$= \mathbb{E}\{(\mathbb{E}s^Y)^X\} = f_X(f_Y(s))$ ☐]

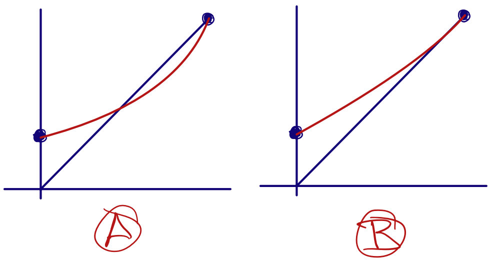


f: str. incr., concave, $f(0) > 0$, $f(1) = 1$

$$a_n := f_n(0) = \begin{cases} \mathbb{P}(Z_n = 0) \uparrow \mathbb{P}(\mathcal{E}) =: s^* \\ f(f_{n-1}(0)) = f(a_{n-1}) \end{cases}$$

$\longrightarrow$ $f(s^*) = s^*$ $\quad ((s^*, s^*) \in$ gph of $f)$

poss:



$s^* < 1 \iff$ A $\iff f'(1) > 1$

But $f'(1) = \mathbb{E}L$ $\longrightarrow$

UPSHOT: $\boxed{\mathbb{P}(\mathcal{E}) < 1 \iff \mathbb{E}L > 1}$

for T_r: $\mathbb{E}L = rp \longrightarrow p_c = 1/r$ & $\mu_{p_c}(\exists \text{ i.o.c.}) = 0$

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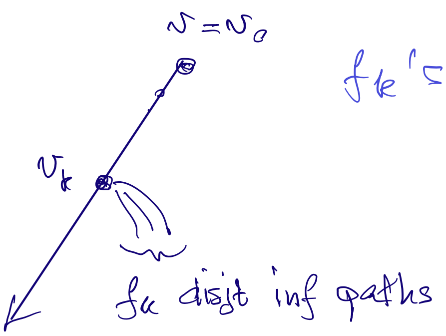
back to gen'l T (notation as above)

ess. saw: $p_c \geq \overline{\lim} |V_n|^{-1/n}$

$(\theta(p) \leq \mathbb{E}_p |C_v \cap V_n| \leq |V_n| p^n \ \ldots)$

▷ Example where strict?



f_k's arb. $\longrightarrow$

e.g. can arrange $\overline{\lim} = \infty$, $\boxed{p_c = 1}$

(why?)

gen'l G (for a bit; v = "root" = distinguished vert)

Def cutset: $\Pi \subseteq V \setminus \{v\}$ "sep'g v from ∞"

$:=$ cpt of v in $G - \Pi$ finite

$\overset{\text{remk}}{\Longleftrightarrow}$ every inf. path from v meets Π

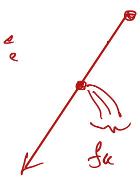
[Q: why? A: König's Lemma]

(as before:) $\forall\, \Pi, p$

$$\theta(p) \leq \mu_p(C_v \cap \Pi \neq \emptyset) \leq \mathbb{E}_p |C_v \cap \Pi|$$

$$\longrightarrow \boxed{\theta(p) \leq \inf_{\Pi} \mathbb{E}_p |C_v \cap \Pi|} \quad (\text{nondecr. in } p)$$

$\circledast$

[Ex:  $\longrightarrow \inf = 0 \;\; \forall p < 1$]

$\longrightarrow$ another "crit" prob:

$$p_{cut} = p_{cut}(G) = \sup\{p : \inf_{\Pi} \mathbb{E}_p |C_v \cap \Pi| = 0\} \overset{\circledast}{\leq} p_c$$

$$(= \inf\{\text{——}\})$$

[sup of p's for which 1st moment ($\leftrightarrow$ $\cup$-bd) gives $\theta(p) = 0$]

$$\mathbb{E}|C_v \cap \Pi| \to 0 \;\Rightarrow\; \mathbb{P}(v \leftrightarrow \Pi) \to 0$$

Thm (R. Lyons '89) $\forall\, T$ $\quad p_c(T) = p_{cut}(T)$

Pf (show: $q > p_{cut} \Rightarrow \theta(q) > 0$)

ETS: $\boxed{q > p_{cut} \Rightarrow \mathbb{P}(v \leftrightarrow V_n) > \varepsilon_q > 0 \quad \forall n}$

Fix n $\to$ $\mathbb{E}_q = \mathbb{E}$, $V_n = W$

not $\mu_q = \mu$ — μ TBA

PLAN ("weighted 2^{nd} m.m."):

$\alpha : W \to \mathbb{R}^+$ TBA

$X = \sum_{y \in W} \boxed{\alpha_y \mathbb{1}_{\{v \leftrightarrow y\}}} =: X_y$ if desired (we won't)

$\mu_q(v \leftrightarrow W) \geq \mathbb{P}(X \neq 0) \geq \mathbb{E}^2 X / \mathbb{E} X^2 \overset{?}{>} \varepsilon_q$ (TBA)

$\to$ NEED $\boxed{\mathbb{E} X^2 < K_q\, \mathbb{E}^2 X}$

μ TBA prob. meas. on W

$\to$ unimp. but convenient

you will p'ly wander for a while where this is going

$\alpha_y := q^{-n} \mu(y)$ $\to$

$\mathbb{E} X = \sum_{y \in W} \alpha_y q^n = \sum_{y \in W} \mu(y) = 1$

$\mathbb{E} X^2 = \sum_{y \in W} \sum_{z \in W} \alpha_y \alpha_z\, q^{2n - |y \wedge z|}$

$= \sum \sum \mu(y) \mu(z)\, q^{-|y \wedge z|}$

$\overset{?}{\leq} K_q$ (= ?)

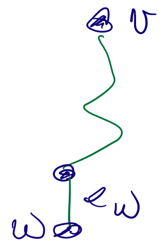
▶ ROUGH: want μ spread: $|y \wedge z|$'s tend to be small

▶ where will we use $q > p_{cut}$?

minor tech addition (unnec. but moves disc. to more familiar territory):

$w \in V \setminus \{v\} \longrightarrow e_w$ as shown

$|e_w| := |w|$



bijection: $\underset{\text{vert cut}}{\Pi} \longrightarrow \underset{\text{edge cut}}{\Pi'} = \{e_w : w \in \Pi\}$

- $\{v \longleftrightarrow w\} = \{e_w \in C_v\}$
- $\mu_p(e_w \in C_v) = \mu_p(v \longleftrightarrow w) = p^{|e_w|}$
- $\mathbb{E}_p |C_v \cap \Pi| = \mathbb{E}_p |C_v \cap \Pi'|$ (since $|\cdots| = |\cdots|$)

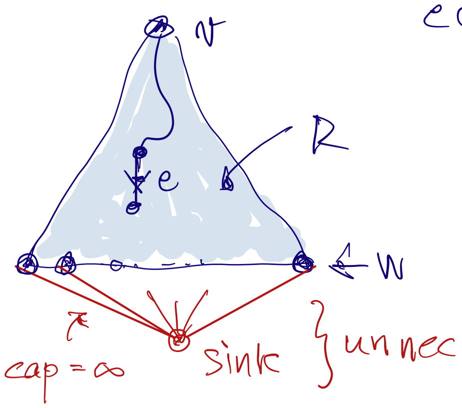
Fix $p = p(q) \in (p_{cut}, q)$ & define C by

$$\inf_{\Pi'} \underbrace{\mathbb{E}_p |C_v \cap \Pi'|}_{\sum_{e \in \Pi'} p^{|e|}} = \inf_{\Pi} \underbrace{\mathbb{E}_p |C_v \cap \Pi'|}_{> 0 \ \because \ p > p_{cut}} = \frac{1}{C}$$

fn of p, not n

$\rightarrow \forall \Pi' \quad \sum_{e \in \Pi'} C p^{|e|} \geq 1 \quad (*)$

network:



$e \in R \mapsto cap(e) = c\, q^{|e|}$

[ref: Diestel or any book on GT]

$\xrightarrow{\circledast\ +\ \text{mfmc}^{*}} \exists\ (v, W)\text{-flow } \varphi$ s.t.

(i) $\varphi(e) \leq c\, p^{|e|}$

(ii) $\boxed{\text{strength}} = 1$

a.k.a. "value" $= \sum_{e \ni v} \varphi(e) = \sum_{y \in W} \varphi(e_y)$

* max-flow min-cut

Now μ: