L12

<u>Cultural</u>: most studied: $p_c(d) = p_c(\mathbb{Z}^d)$

- $p_c(1) = 1$, $p_c(d+1) \le p_c(d)$ (why?)
- $p_c(2) = 1/2$ (Kesten '80; $\ge \frac{1}{2}$: Harris '60 → H's Ineq)

  ← duality

- $d \geq 3$: don't expect nice val, but

  Menshikov '86, Aizenmann-Barsky '87:

  gen'l thm ⇒'g $p_c(2) = 1/2$ (wait)

- EX: ass'g $p_c(2) < 1$ (not obv.), show

$$p_c(d+1) < p_c(d)$$

- <u>Conj</u> ("no perc. at crit. pt.")

$$\forall d \qquad \mu_{p_c(d)}(\exists \text{ i.o.c.}) = 0$$

(known for $d = 2$ & $d \geq 11$)

Prop. $\forall v \quad p_c(G) = \sup\{p : \mu_p(|C_v| = \infty) = 0\} =: p_c(v)$ temp.

- pf $\leq$ needs G conn
- "$\leq$" triv. (why?)

notation: $A_v = \{|C_v| = \infty\}$

$\theta_p(v) = \mu_p(A_v)$ [remk $\theta_p(v) < 1$ if $p \neq 1$]

Claim $\forall v, w \quad p_c(v) = p_c(w)$

- N.B. not $\theta_v(p) = \theta_w(p)$ — why?
- this is enough: $p < p_c(v) \Rightarrow \mu_p(A_w) = 0 \ \forall w$

$\Rightarrow \mu_p(\underbrace{\cup A_w}_{\{\exists \text{ i.o.c.}\}}) = 0 \quad (\Rightarrow p \leq p_c(G))$ ∎

Pf of Claim ETS $\theta_v(p) > 0 \Rightarrow \theta_w(p) > 0$

fix (v,w)-path $P \rightarrow$

$\theta_w(p) \geq \mu_p(\{P \text{ open}\} \cap A_v)$

$= \underbrace{\mu_p(P \text{ open})}_{>0} \mu_p(A_v \mid P \text{ open})$

$\rightarrow$ ETS

$\mu_p(A_v \mid P \text{ open}) \geq \mu_p(A_v)$

$\uparrow \subseteq$ Harris but easier (coupling):

$\omega \sim \mu_p, \quad \omega' = \omega \vee \mathbb{1}_P \quad$ (i.e. $\omega'_e = \begin{cases} 1 & e \in P \\ \omega_e & \end{cases}$)

$\rightarrow \omega' \sim \mu_p(\cdot \mid P \text{ open}) \rightarrow$ ∎

⊣

Quick cases: $p_T := \sup\{p : \mathbb{E}_p |C_v| < \infty\} \leq p_c$ ← why?

- p_T ind. of v — why? (same arg)
- $p > p_c \; (= p_c(v)) \Rightarrow \Theta_v(p) > 0 \Rightarrow \mathbb{E}_p |C_v| = \infty \longrightarrow p \geq p_T$

Menshikov, A-B: $p_T(d) = p_c(d)$

e.g. $\Rightarrow$ Kesten (not trivially)

- false for gen'l G (even T) — why?

Int'ed in $p_c(G)$ for $G = T =$ tree

not. $p_c := p_c(T)$; fix root v & use:

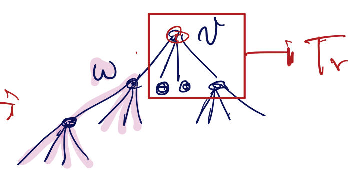
$\Theta(p) = \Theta_v(p)$, $V_n = \{w : \text{dist}(v, w) = n\}$ "level n"

$X_w = \mathbb{1}_{\{v \leftrightarrow w\}} \longrightarrow \mathbb{E}_p X_w = p^{|w|}$ ↳ ...

Warmup $T = T_r :=$ r-branching tree $\longrightarrow p_c$?

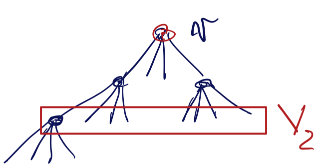
⌈ cf: $T' = (r+1)$-reg. tree

extra branch →



EX: same p_c

(e.g. $\Theta_v(p) \leq \Theta'_v(p) \leq \boxed{2\Theta_v(p)}$ — or $(1+p)\Theta_v(p)$ ⌋



given n: $X = X^{(n)} = |C_v \cap V_n| = \sum_{w \in V_n} X_w$

$$\mathbb{E}_p X = r^n p^n \longrightarrow \begin{cases} 0 & \text{if } p < 1/r \\ \infty & \text{if } p > 1/r \end{cases}$$

$\longrightarrow$ $p_c \geq 1/r$ (as us.: $p < 1/r \Rightarrow$

$$\Theta(p) \leq \mu_p(X^{(n)} \neq 0) \leq \mathbb{E}_p(X^{(n)}) \longrightarrow 0$$

$p_c \leq 1/r$ (i.e. $=$)?

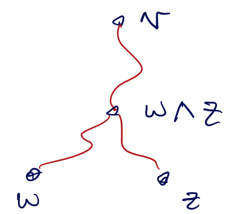
<u>sketch</u> [more gen'l soon ; diff arg. below]

▷ ETS : $p > 1/r \Rightarrow \mu_p(\underbrace{X^{(n)} \neq 0}_{B_n}) > \varepsilon = \underline{\underline{\varepsilon_p}} > 0$ ⊛

Then $B_n \downarrow A_v \Rightarrow \mu_p(A_v) = \lim \mu_p(B_n) \geq \varepsilon$ ▨

(<u>finite</u> prob but need a <u>unif.</u> bd)

<u>Pf</u> of ⊛ : Ex using



(a) $w, z \in V_n \Rightarrow \mathbb{E}\, X_w X_z = p^{2n - |w \wedge z|}$

(b) "improved" 2nd m.m.

(→ ETS $\mathbb{E} X^2 < K_p\, \mathbb{E}^2 X$)

<u>Excursion</u> Galton-Watson (or "br'g") proc :

L : $\mathbb{N}$-val'd r.v.

$Z_0 \equiv 1, \quad Z_1 = Z_1^1 \sim L$

$Z_n = Z_1^n + \dots + Z_{Z_{n-1}}^n$

} Z_i^n's ind copies of L

[start w root & iterate : each node does #children ~ L (ind'ly)]

$\mathcal{E} := \{\exists\, n \;\; Z_n = 0\}$ "extinction"

→ $\boxed{\mathbb{P}(\mathcal{E}) < 1 \,?}$

▷ note $\mathbb{P}(\mathcal{E}) > 0$ if (& only if) $\underbrace{\mathbb{P}(L=0) > 0}_{\text{we assume}}$

▷ E.g. C_v for perc. at p on T_r ⟷ G-W w $L \sim \mathrm{Bin}(r, p)$