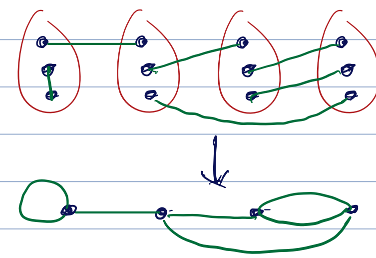
RECAP

▶ $G_{n,d}$: unif $\in \mathcal{G}_{n,d}$

▶ how to work w/ it?

▶ pairing (a.k.a. config) model

$$W = \bigcup_{i=1}^{n} W_i \qquad |W_i| = d$$



$$\mathcal{P}^*_{n,d} := \{\text{pairings of } W\}$$

$\downarrow \pi$ (nat.)

$$\mathcal{G}^*_{n,d} := \{(\text{lab.})\ d\text{-reg multigraphs on } [n]\}$$

loops contrib 2

$$\supseteq \mathcal{G}_{n,d}$$

$\mathbb{P}$: unif $\in \mathcal{P}^*_{n,d}$

$\downarrow \pi$

$G \in \mathcal{G}^*_{n,d}$ (not unif but :)

▶ G simple $\Rightarrow |\pi^{-1}(G)| = (d!)^n$

a.k.a. $G \in \mathcal{G}_{n,d}$

$\mathcal{P}_{n,d} := \pi^{-1}(\mathcal{G}_{n,d})$ ("simple pairings")

$$\to \quad |\mathcal{G}_{n,d}| = (d!)^{-n}|\mathcal{P}_{n,d}| = (d!)^{-n}\underbrace{|\mathcal{P}^*_{n,d}|}_{(nd-1)!!}\ \boxed{\frac{|\mathcal{P}_{n,d}|}{|\mathcal{P}^*_{n,d}|}} \quad \mathbb{P}(\text{simple})$$

$\to$ want $\boxed{\mathbb{P}(\text{simple}) \to e^{-(d^2-1)/4}}$

L10

⊛ $X_\ell \; (= X_\ell^{(n)}) = \#\ \ell\text{-cycles in } \mathbb{G}$ $(\ell \geq 1)$

$= \#\ \underbrace{\text{pre-}\ell\text{-cycles}}_{\text{obv. def.}} \text{ in } \mathbb{P}$

$$\boxed{\mathbb{P} \in \mathcal{G}_{n,d} \iff X_1 = X_2 = 0}$$

⊛ Obs fix s: $\forall S$ = partial pairing w $|S| = s$

$$\mathbb{P}(\mathbb{P} \supseteq S) = \frac{1}{(nd-1)(nd-3)\cdots(nd-2s+1)} =: p_s \sim (nd)^{-s}$$

$$\to \mathbb{E} X_\ell \overset{⊛}{=} \frac{(n)_\ell}{2\ell}(d(d-1))^\ell p_\ell \sim \frac{(d-1)^\ell}{2\ell} =: \mu_\ell \; (= \Theta(1))$$

⊛ $\ell = 1$: formula correct but $\because \; = n\binom{d}{2}$

expect $\longrightarrow$

Thm $\mathbb{P}(X_1 = X_2 = 0) \to \exp\left[-\frac{d-1}{2} - \frac{(d-1)^2}{4}\right] = e^{-(d^2-1)/4}$

(▯)

Remk $\mathcal{G}_{n,d}$ somewhat "manageable" via:

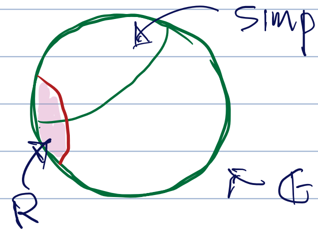
$\mathcal{G}^*_{n,d}$ "manageable" $+$ $\mathbb{P}(\text{simple}) = \Omega(1)$

▷ e.g. $\mathbb{G}^* \models Q$ a.s. $\Rightarrow$ same for $\mathbb{G}_{n,d}$ (why?)

[$\overset{?}{\rightsquigarrow}$ $\mathbb{G}_{n,3}$ Ham (NO — why not?) → OR: not immediately]

N.B. n.g. for large d

simple ($\leftrightarrow G_{n,d}$) $\qquad R = \overline{Q}$



$$\mathbb{P}(G_{n,d} \models R) = \frac{\mathbb{P}(\mathbb{G} \text{ simple}, \models R)}{\mathbb{P}(\mathbb{G} \text{ simple})}$$

"Thm" $\leq$ Thm 2 (Boll. '80)

$\forall \ell$ asymptotically, $X_1, \ldots, X_\ell$ are indept Poissons (w means μ_i)] $\circledast$

i.e. $\mathbb{P}(X_i = k_i \ \forall i \in [\ell]) \to e^{-\sum \mu_i} \prod \mu_i^{k_i}/k_i!$

— follows from

Thm 3 ("multi-Brun") $\circledast$ holds if:

$$\forall r_1, \ldots, r_\ell \quad \mathbb{E} \prod (X_i)_{r_i} \to \prod \mu_i^{r_i} \quad \circledast$$

T3 $\Rightarrow$ T2: need $\circledast$

$$\mathbb{E} \prod (X_i)_{r_i} = \sum \cdots \sum \mathbb{E} \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \mathbb{1}_{C_{i,j}}$$

[$C_{i,1}, \ldots, C_{i,r_i}$ distinct i-cycles of K_n

$$= \sum_{\text{disjt}}{}' + \sum_{\text{not (all) disjt}}{}''$$ and:

$\sum' \to \prod \mu_i^{r_i}$ (easy); $\boxed{\sum'' \overset{\circledast}{=} O(1/n)}$

$\circledast$: combine:

Obs (EX): $\{C_{ij}\} \models \sum'' \Rightarrow$

$$|E(\bigcup_i \bigcup_j C_{ij})| > |V(\bigcup_i \bigcup_j C_{ij})|$$

+ [s fixed, S = partial pairing, $|S| = s$ $\Rightarrow$ $\mathbb{P}(P \supseteq S) \sim (nd)^{-s}$]

EX to finish

[**Thm 3** If $\forall r_1,\dots,r_\ell$ $\mathbb{E}\prod (X_i)_{r_i} \to \prod \mu_i^{r_i}$

then $\forall k_1,\dots,k_\ell$ $\mathbb{P}(X_i = k_i\ \forall i) \to e^{-\sum \mu_i} \prod \mu_i^{k_i}/k_i!$]

Pf (Wormald) just $\ell = 2$, then EX (ind. on ℓ is same arg.)

notation: $X_1 = X$, $X_2 = Y$, $\mu_1 = \mu$, $\mu_2 = \nu$ $\to$

hyp: $\forall r, s$ $\mathbb{E}(X)_r (Y)_s \to \mu^r \nu^s$

① $\left.\begin{array}{l} X \xrightarrow{d} Po(\mu) \\ Y \xrightarrow{d} Po(\nu) \end{array}\right\}$ why?

WMA (w.l.o.g.) $\nu \neq 0$ [if $\mu = \nu = 0$ then $\mathbb{P}(X = Y = 0) \to 1$ ☒]

② $\mathbb{P}(X = k, Y = \ell) = \underset{\text{okay}}{\mathbb{P}(X = k)} \boxed{\mathbb{P}(Y = \ell \mid X = k)}$

② $\to e^{-\nu}\nu^\ell/\ell!$ (= WISB)

fix s and $\boxed{\text{reweight by } (Y)_s}$

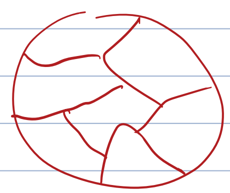
int'g part & reason we're doing

gen'l (discrete) $(\Omega, \mathbb{P})$, ω: elt. of Ω

$Z: \Omega \to \mathbb{R}^+$ (r.v.) $\longrightarrow$

reweight by Z: $\mathbb{P}'(\omega) = \dfrac{\mathbb{P}(\omega) Z(\omega)}{\boxed{\sum \mathbb{P}(\omega') Z(\omega')} = \mathbb{E}Z}$

[a.k.a. $\mathbb{P}'(\omega) \propto \mathbb{P}(\omega) Z(\omega)$]



$\leftarrow$ part'n acc. Z & rewt blocks $\longrightarrow$

$\Rightarrow$ no effect on W if W indep't of Z

① $W: \Omega \to \mathbb{R}$: $\mathbb{E}'W = \dfrac{\sum \mathbb{P}(\omega) Z(\omega) W(\omega)}{\mathbb{E}Z} = \dfrac{\mathbb{E}ZW}{\mathbb{E}Z}$

② $R \subseteq \Omega$:

$$\mathbb{P}'(R) = \mathbb{E}'\mathbb{1}_R = \frac{\mathbb{E}Z\mathbb{1}_R}{\mathbb{E}Z} = \frac{\mathbb{P}(R)\,\mathbb{E}[Z \mid R]}{\mathbb{E}Z}$$

why?

$$\mathbb{P}(X=k, Y=\ell) = \underset{\substack{\uparrow \\ \text{okay}}}{\mathbb{P}(X=k)}\ \boxed{\mathbb{P}(Y=\ell \mid X=k)}$$

② $\longrightarrow e^{-\nu}\nu^{\ell}/\ell!$ (= WISB)

① $W : \Omega \to \mathbb{R}$: $\quad \mathbb{E}'W = \dfrac{\mathbb{E}ZW}{\mathbb{E}Z}$

② $R \subseteq \Omega$: $\quad \mathbb{P}'(R) = \dfrac{\mathbb{P}(R)\,\mathbb{E}[Z \mid R]}{\mathbb{E}Z}$

Back to reweight by $(Y)_s$ (s fixed; $\mathbb{P}'$, $\mathbb{E}'$ as above)

$$\forall r \quad \mathbb{E}'(X)_r \overset{①}{=} \frac{\mathbb{E}(X)_r (Y)_s}{\mathbb{E}(Y)_s} \longrightarrow \mu^r \qquad (\text{using } \nu \neq 0)$$

$$\overset{\text{Brun}}{\Longrightarrow} \quad \forall k \quad \mathbb{P}'(X=k) \longrightarrow e^{-\mu}\mu^k/k!$$

$$\overset{②}{\|}$$

$$\boxed{\mathbb{P}(X=k)}\ \frac{\mathbb{E}[(Y)_s \mid X=k]}{\boxed{\mathbb{E}(Y)_s}}$$

$\mathbb{P}(X=k) \longrightarrow e^{-\mu}\mu^k/k!$

$\mathbb{E}(Y)_s \longrightarrow \nu^s$

$$\longrightarrow \quad (\forall s)\ \mathbb{E}[(Y)_s \mid X=k] \longrightarrow \nu^s$$

SA ?