$\boxed{\text{PLEASE UNMASK}}$

$\exists$ handout
A-S
familiar ?

$\exists$ prob's ... not too many
grades — show up

- uses ←
- tools
- random structures (e.g. $G_{n,p}$)

?

Lin. of $\mathbb{E}$

$X_1, \dots$ r.v.'s $(\in \mathbb{R})$

$$\mathbb{E}\left[\sum X_i\right] = \sum \mathbb{E} X_i$$

$$\left(\& \quad \mathbb{E} \sum \underset{\uparrow \mathbb{R}}{c_i} X_i = \sum c_i \mathbb{E} X_i \right)$$

e.g. not for var; e.g. X, Y "sym Ber":

$$\mathbb{P}(X=1) = \mathbb{P}(X=-1) = 1/2$$

$$\mathbb{E}X = 0, \quad \sigma_X^2 = 1$$

$$\mathrm{var}(\underbrace{X+Y}_{Z}) = ?$$

e.g. X, Y ind $\rightarrow$ $\sigma_Z^2 = 2$

$X = Y$ $\rightarrow$ $\sigma_Z^2 = 4$

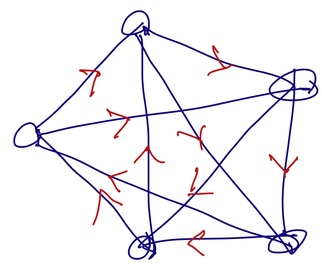
$X = -Y$ $\quad \sigma_Z^2 = 0$

E.g. Szele

tournament on $[n] = \{1, \dots, n\}$:

T: or. of K_n

set of "arcs"



$P(T) = \#$ of Ham. paths

σ: $x_1, \dots, x_n$ ordering of $[n]$.

$x_i x_{i+1} \in T \quad i \in [n-1]$

$P(n) := \max\{P(T) : T \text{ on } [n]\}$

Q: min? 1 ($\leftrightarrow$ trans. T)

Thm (Redei 34): $\exists$

Ham cycle? no e.g.

Szele 43: $P(n) \geq 2^{-(n-1)} n! = g(n)$

persp: $n! \begin{cases} \approx (n/e)^n \\ \asymp \sqrt{n} \, (n/e)^n \end{cases}$ (i.e. $\Theta(\sqrt{n}\,(n/e)^n)$)

Alon 90 $P(n) = O(n^{3/2} g(n))$

Q: $O(g(n))$?

Q: $\min\{P(T) : T \text{ regular}\}$?

guess: $\Omega(g(n))$??

Pf $\quad T$ random tourn $\quad$ (T poss. value)

$$\mathbb{P}(xy \in T) = \tfrac{1}{2} = \mathbb{P}(yx \in T) \quad \forall x,y$$

σ : ordering of $[n]$

$$X_\sigma = \mathbb{1}_{\{\sigma \text{ H.p. of } T\}}$$

$$\mathbb{E}X_\sigma = \mathbb{P}(X_\sigma = 1) = 2^{-(n-1)}$$

A event

$$\mathbb{1}_A = \begin{cases} 1 & A \text{ occurs} \\ 0 & \end{cases}$$

$$X = P(T) = \sum X_\sigma$$

$$\to \mathbb{E}X = \mathbb{E}\sum X_\sigma = \sum \mathbb{E}X_\sigma = n!\, 2^{-(n-1)} \qquad \blacksquare$$

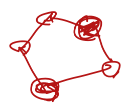
$$= \sum_k k\, \mathbb{P}(X = k)$$

E.g. (Caro-Wei) $\quad$ AS pp 100-101

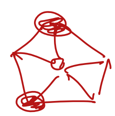
$G = (V, E)$, $\quad \alpha(G)$ = ind. #

[ind set : $W \subseteq V$ w/ no edges

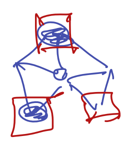
$\alpha(G) = \max |W|$ $\quad$ e.g.



$\alpha = 2$



$\alpha = 2$



$\alpha = 3$

d_v = degree

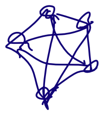
$$\bar{d} = n^{-1} \sum_v d_v = \frac{2|E(G)|}{n}$$

avg deg

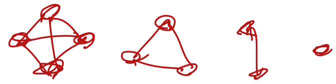
Thm $\quad \alpha(G) \geq \sum_v \frac{1}{d_v + 1} \geq \frac{n}{\bar{d} + 1}$

Jenssen

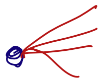
⊛ sharp iff G = disjt $\cup$ of cliques



K_5 $\quad \sum \frac{1}{d_v+1} = 5 \cdot \frac{1}{4+1} = 1 = \alpha(G)$

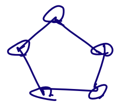


e.g. G d-reg $\xrightarrow{\text{triv}}$ $\alpha(G) \geq \frac{n}{d+1}$



$\}\, d$

I $\qquad$ $\overline{I}$

e.g. G =  $\alpha(G) = 2 > \frac{5}{3}$

$= \sum \frac{1}{d_v+1}$

▷ Cor Turán's thm (41) $\qquad |V(G)| = n$ default

one ver: $\alpha(G) \leq r \implies$

$|E(G)| \geq |E(\underbrace{\bullet \ \cdots \ \bullet}_{r;\ \text{"equipart'n"}})|$ $\to T_r(n)$ "Turán graph"

sketch: want: $|E(G)| < |E(T_r(n))| \implies \alpha(G) > r$

show: $|E(G)| = |E(T_r(n))| \implies$

$\alpha(G) \geq r$ and eq. $\iff$ $G \cong T_r(n)$ (⊛)

(Pf) $\alpha(G) \underset{\text{C-W}}{\geq} \underbrace{\sum \frac{1}{d_v+1}}_{\text{in } G} \geq \underbrace{\sum \frac{1}{d_v+1}}_{\text{in } T_r(n)} = r$

C-W

eq. $\iff$ disjt $\cup$ of cliques

$\sum d_v = 2|E|$

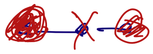
Pf of CW $\quad \underbrace{x_1, \dots, x_n}_{\sigma}$ unif ord. of $[n]$

I : "random greedy" :

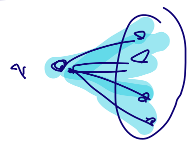
$$x_k \in I \iff \left[\left\{ \begin{matrix} j < k \\ x_j \sim x_k \end{matrix} \right\} \Rightarrow x_j \notin I \right]$$

$$\Rightarrow \quad \underset{=?}{\mathbb{P}(x_k \in I) \geq} \mathbb{P}(x_k <_\sigma \underbrace{N(x_k)}_{\text{nbhd.}}) = \frac{1}{d_v + 1}$$

e.g.



$\sigma \rightarrow$



$N(v)$

$$\mathbb{E}|I| = \sum_v \mathbb{P}(v \in I) \geq \sum \frac{1}{d_v + 1}$$

QED

(eq iff …:) $G \neq$ d. u. of c's

$\iff G \supseteq$  etc

Pf?

E.g. sum-free sets

Thm Erdős 65 A–S Thm 1.4.1

$A=\{a_1,\dots,a_n\}\subseteq \mathbb{Z}\setminus\{0\}$ (multiset okay)

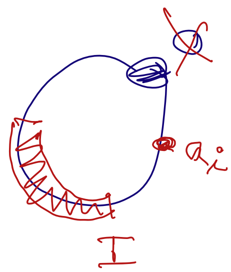
$\Longrightarrow \exists\, B\subseteq A$ $\begin{cases} \text{sum-free} \hookrightarrow \nexists\, x+y=z & (x=y \text{ allowed}) \\ |B|>|A|/3 \end{cases}$

Pf $p=3k+2$, $(p,a_i)=1\ \forall i$

ET prove in $\mathbb{Z}_p$ ($a_i \to$ res. mod p)

expt: x unif $\in \mathbb{Z}_p^\times$ $(=\mathbb{Z}_p\setminus\{0\})$

$I=\{k+1,\dots,2k+1\}\subseteq \mathbb{Z}_p$



① sum-free $\left(\begin{array}{l} 2(k+1)=2k+2 \\ 2(2k+1)=4k+2=k \end{array} \right.$

② $|I|=k+1>\frac{p-1}{3}$

xa_i unif $\in \mathbb{Z}_p^\times$

$b_i=xa_i \longrightarrow B=\{a_i : b_i\in I\}$

$\Rightarrow$ B is sum-free

$\mathbb{E}\,|B|>|A|/3$ ■

$\uparrow$

$\mathbb{P}(a_i\in B)=\frac{|I|}{p-1}>1/3$

Eberhard-Green-Manner '14: $1/3$ is best possible (!)

e.g. $\exists$? sum-free B

$|B| > |A|/3 + 100$? (A large)

how about u.b $|A|/3 + O(1)$?