[RECAP]

Proof of RL (sketch)

arb. equipart'n $V_1 \cup \cdots \cup V_{t_0}$ ($t \geq t_0$) and refine

"potential" of $V_1 \cup \cdots \cup V_q$:

$$P(\{V_i\}) = q^{-2} \sum \sum d^2(V_i, V_j) \quad (\in [0,1])$$

$[d(V_i, V_i) := 0 \quad (\text{irrelevant})]$

$$\boxed{\begin{array}{l}\text{MAIN:}\\ \text{if \# irreg pairs} > \varepsilon\binom{q}{2}\text{, can refine so}\\ \quad (a)\ q \leftarrow q\,4^q\\ \quad (b)\ P \text{ increases by} > \varepsilon^5\end{array}}$$

$\longrightarrow$ # steps $< \varepsilon^{-5}$

$\longrightarrow$ ε-reg. part'n w/ $t \lesssim \underbrace{4^{4^{\cdot^{\cdot^{\cdot^{4^{t_0}}}}}}}_{T}$ $\Big\}\ \varepsilon^{-5}$

ridiculous! (?) ... NO (!)

- Gowers '97: l.b. $\approx 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\}\ \varepsilon^{-1/16}$
- Conlon-Fox '12: ε^{-1}
- Fox-Lovász '17: $\Theta(\varepsilon^{-2})$ is the truth

$\boxed{L\,28}$ (and a bit more)

"Pf" of MAIN

I. Gen'l $\quad V_i = \dot{\bigcup}_{k=1}^{s} V_{ik}$, $\quad V_j = \dot{\bigcup}_{\ell=1}^{s} V_{j\ell}$ equipart'ns

k, ℓ unif $\in [s]$ (indept), $Z = d(V_{ik}, V_{j\ell})$

① $\mathbb{E} Z = s^{-2} \sum_{k} \sum_{\ell} d(V_{ik}, V_{j\ell}) = \underbrace{d(V_i, V_j)}_{d}$

[law of t.P: u unif $\in V_i$, v unif $\in V_j$ } (indept) $\longrightarrow$

$d = \mathbb{P}(u \sim v) = \sum \sum \underbrace{\mathbb{P}(k, \ell)}_{s^{-2}} \underbrace{\mathbb{P}(u \sim v \mid k, \ell)}_{d(V_{ik}, V_{j\ell})}$ ☒]

② $\mathbb{E} Z^2 = s^{-2} \sum \sum d^2(V_{ik}, V_{j\ell}) \geq d^2(V_i, V_j)$

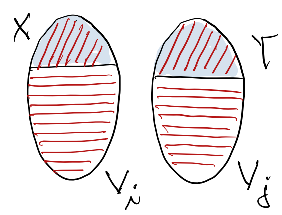
$\boxed{\text{refinement doesn't decr. } \mathbb{P}}$

II. (V_i, V_j) irreg w "witness" (X, Y)

[$X \subseteq V_i$, $Y \subseteq V_j$; $|X|, |Y| > \varepsilon |V_i|$, $d(X, Y) \neq d \pm \varepsilon$]

$\{V_{ik}\}, \{V_{j\ell}\}$ as above $\boxed{\text{refining } (X, V_i \setminus X), (Y, V_j \setminus Y)}$

s TBA (will be large)



Claim $s^{-2}\sum\sum d^2(V_{ik},V_{j\ell}) \geq d^2(V_i,V_j) + \varepsilon^4$

i.e. $\boxed{\operatorname{Var} Z \geq \varepsilon^4}$

Pf $\operatorname{Var} Z = \mathbb{E}(Z-d)^2$

$$= \mathbb{P}(X,Y)\,\mathbb{E}[(Z-d)^2 \mid X,Y] + \underbrace{(1-\mathbb{P}(k,\ell))\,\mathbb{E}[\cdots]}_{\geq 0}$$

$V_{ik} \subseteq X$, $V_{j\ell} \subseteq Y$

$$\geq \underbrace{\mathbb{P}(X,Y)}_{\geq \varepsilon^2}\ \underbrace{\mathbb{E}^2[|Z-d| \mid X,Y]}_{\geq |\mathbb{E}[Z-d \mid X,Y]|^2 = |d(X,Y)-d|^2 > \varepsilon^2}$$

III. $\forall i \neq j$: (V_i,V_j) irreg $\longrightarrow$ (X_{ij}, X_{ji}) "witness"

(V_i,V_j) reg. $\longrightarrow$ $X_{ij} = V_i$

$\forall i$: $\{X_{ij} : i \neq j \in [q]\} \longrightarrow$ B.A. $\overline{\omega} \leq 2^q$ $\boxed{\text{atoms}}$

$\bigcap_{j\in I} X_{ij} \cap \bigcap_{j\notin I} (V_i \setminus X_{ij})$

$\longrightarrow$ $s = 4^q$ (say); $\{V_{ik}\}_{k=1}^{s}$ equipart'n of V_i

refining $(X_{ij}, V_i \setminus X_{ij})$ $\forall j \neq i$ (more or less)

$\longrightarrow$ potential gain:

$$q^{-2}\sum_i\sum_j \left[s^{-2}\sum\sum d^2(V_{ik},V_{j\ell}) - d^2(V_i,V_j)\right]$$

$$[\cdots] \geq \begin{cases} \varepsilon^4 & \text{if } (V_i,V_j) \text{ irreg} \\ 0 & \text{———} \end{cases}$$

[1st appl:] $|V(G)| = n$, $\tau(G) = \#$ of Δ's in G

Triangle removal Lemma (Ruzsa-Szemerédi '78)

$$\tau(G) = o(n^3) \Rightarrow G \text{ } o(n^2)\text{-close to } \Delta\text{-free}$$

i.e.

[can be made Δ-free by removing $o(n^2)$ edges

$\equiv$ can cover Δ's w/ $o(n^2)$ edges]

equiv: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$\tau(G) < \delta n^3 \Rightarrow G \ (\varepsilon n^2)\text{-close to } \Delta\text{-free}$$

This is not obvious; e.g.:

① why not: $\Omega(n^2)$ edges, each in δn Δ's $\rightarrow$

$\Omega(\delta n^3)$ Δ's $\rightarrow$ $\Omega(n^2)$ edges to cover ?

② how large can we make δ $(= \delta_\varepsilon)$?

- orig. pf (via RL): $\delta^{-1} \approx \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} \varepsilon^{-c}$

- Fox '11: $\delta^{-1} \approx \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} \log \frac{1}{\varepsilon}$ (Ann. Math.)

... vs. upper bd: Behrend '46: $\delta < \varepsilon^{c \log(1/\varepsilon)}$ (!)

$r_3(N)$ (mentioned earlier)

[also: Graph RL (Erdős-Frankl-Rödl '86)

ref: Conlon-Fox: Gph removal lemmas]

Proof (ε given)

- δ TBA; assume $\tau(G) < \delta n^3$
- RL $\to$ $V_1 \cup \cdots \cup V_t$ α-reg. part'n (α :: ε is taken)

  α TBA; $t_0 = \alpha^{-1}$ (unimp., convenient)

- define regularity graph H: $V(H) = [t]$

  $i \sim_H j \iff (V_i, V_j)$ reg, density $\geq d$ TBA

MP: H is Δ-free

why? [think first, then see below]

[MP: H is Δ-free (why?)]

$\boxed{\text{PLAN:}}$

▷ Counting Lemma $\longrightarrow$

H Δ-free (or too many Δ's in G)

$\longrightarrow$ can just delete edges in $\left.\begin{cases} V_i\text{'s} \\ (V_i, V_j)\text{'s } ij \notin H \end{cases}\right]$ ✱

$\boxed{\text{DETAILS:}}$ $d = \varepsilon \boxed{\gg \alpha}$ ← minor

$$\alpha^{-1} = t_0 \leq t \leq T = T(\alpha, t_0)$$

$$\delta n^3 \overset{\text{given}}{>} \tau(G) \overset{CL}{\gtrsim} \tau(H) \cdot d^3 (n/T)^3$$

$$= \tau(H) \cdot (d/T)^3 n^3 \longrightarrow$$

Now choosing $\delta = \frac{1}{2}(d/T)^3$ (tiny ∵ T from RL)

makes $\underline{H\ \Delta\text{-free}}$ $\longrightarrow$

$$\#\text{ of deletions (at ✱)} < \underbrace{\tfrac{1}{t}\tbinom{n}{2}}_{\text{edges in } V_i\text{'s}} + \underbrace{\alpha\tbinom{n}{2}}_{\text{irreg. pairs}} + \underbrace{\varepsilon\tbinom{n}{2}}_{\text{low density pairs}} < \varepsilon n^2$$

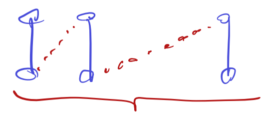
∎

applications of TRL (still R-Sz) $\boxed{\text{ref: Tao-Vu}}$

[orig. motivation for TRL was hypergraphic
("(6,3) problem" of Brown-Erdős-Sós), BUT:]

Cor 1 G (on n) $\cup$ of $O(n)$ induced matchings

$\Longrightarrow e_G = o(n^2)$



no other edges

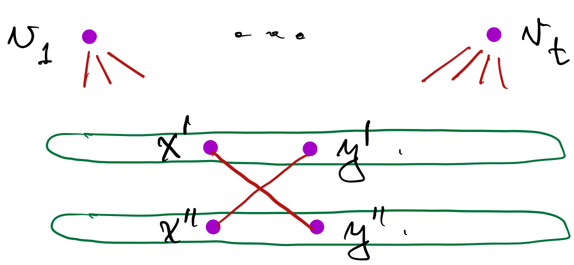
Cor 2 $r_3(n) = o(n)$ (! & unexpected by R-Sz)

$\rho(G) :=$ # of edges to cover Δ's

Pf of Cor 1:

$G = M_1 \cup \dots \cup M_t$ $\begin{cases} M_i\text{'s disjt I.M.'s} & \text{WMA} \\ t = O(n) & \end{cases}$

G^*:



bip. double cover of G

$v_i \sim$ copies of vertices used in M_i

$\{\Delta\text{'s of } G^*\} = \{ v_i x' y'' : xy \in M_i \}$

▷ Δ's are edge-disjoint

$\longrightarrow \tau(G^*) = 2e_G < n^2 \ll n^3$

$\xRightarrow{\text{TRL}} \rho(G^*) \ll n^2 \quad \Rightarrow \qquad \square$

$[\Delta\text{'s e-d} \Rightarrow \rho(G^*) = \tau(G^*) \ (= 2e_G)]$

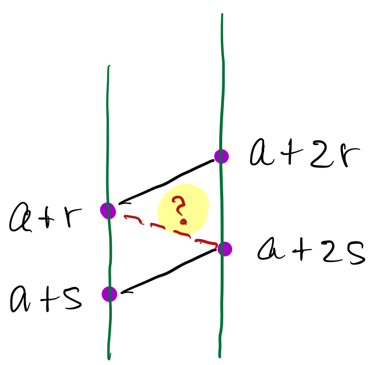
(where did we use $t = O(n)$?)

Pf of Cor 2: $A \subseteq [n]$ no 3-t. AP

bigraph G on $[2n] \cup [3n]$:

$$a + r \sim a + 2r \quad \forall\, a \in [n],\ r \in A$$

▷ OBS: $\{(a+r, a+2r) : r \in A\}$ I.M. $(\forall\, a \in [n])$



Pf Else $\exists$

$$\left.\begin{aligned} a + r &= c + t \\ a + 2s &= c + 2t \end{aligned}\right\} \quad \begin{aligned} r, s, t &\in A \\ c &\in [n] \end{aligned}$$

$\longrightarrow r + t = 2s$ (A.P.) $\square$

$\xrightarrow{\text{Cor 1}} e_G = o(n^2)$

(but $e_G = n|A|$) $\square$