E.g. $X := \#$ of Δ's in $G_{n,p}$ $\longrightarrow$

$$\mu = \binom{n}{3} p^3 \sim n^3 p^3/6$$

$$\Delta = \binom{n}{3} 3(n-3) p^5 \sim n^4 p^5/2$$

- $p \ll n^{-4/5}$: Φ $(:= \mathbb{P}(X=0)) \sim e^{-\mu}$
  - cf. Brun (A-S, §8.3)
  - EX: $\forall k$ $\mathbb{P}(X=k) \sim e^{-\mu} \mu^k/k!$
- $p \ll n^{-1/2}$ $(\Leftrightarrow \Delta \ll \mu)$: $\Phi = e^{-\mu + o(\mu)}$
- $p = n^{-1/2}$ (e.g.): ?? (wait)

Pf of BJI (B-S; Janson: lower tails, J{\L}R T 2.14)

$\boxed{\text{L 23:}}$

MP: $\underbrace{\mathbb{P}\left(\bar{A}_i \,\middle|\, \bigcap_{j<i} \bar{A}_j\right)}_{\geq 1-p_i} \leq \underbrace{\mathbb{P}(\bar{A}_i)}_{1-p_i} + \sum_{i\sim j<i} \mathbb{P}(A_i A_j)$

then done: $\mathbb{P}(\cap \bar{A}_i) = \prod \mathbb{P}\left(\bar{A}_i \,\middle|\, \bigcap_{j<i} \bar{A}_j\right)$

$$\leq \prod \exp\left[-p_i + \sum_{i\sim j<i} \mathbb{P}(A_i A_j)\right]$$

$$= \exp\left[-\mu + \Delta/2\right] \quad \blacksquare$$

<u>equiv:</u>

$$\mathbb{P}\Big(A_i \,\Big|\, \bigcap_{j<i} \bar{A}_j\Big) \geq \mathbb{P}(\bar{A}_i) - \sum_{i\sim j<i} \mathbb{P}(A_iA_j) \qquad (*)$$

<u>Pf</u> w.l.o.g. $i \begin{cases} \sim 1,\dots,d \\ \not\sim d+1,\dots,i-1 \end{cases}$

$$\text{l.h.s. of } (*) = \frac{\mathbb{P}(A_i\bar{A}_1\cdots\bar{A}_d \mid \bar{A}_{d+1}\cdots\bar{A}_{i-1})}{\mathbb{P}(\bar{A}_1\cdots\bar{A}_d \mid \boxed{\bar{A}_{d+1}\cdots\bar{A}_{i-1}})} \quad B$$

$$\mathbb{P}(X|YZ) = \frac{\mathbb{P}(XY|Z)}{\mathbb{P}(Y|Z)}$$

$$\geq \text{NUMERATOR} \quad (!)$$

$$= \underbrace{\mathbb{P}(A_i|B)}_{=} - \mathbb{P}\Big(A_i \cap \bigcup_{j=1}^{d} A_j \,\Big|\, B\Big)$$

$$\geq \overbrace{\mathbb{P}(A_i)} - \sum_{j=1}^{d} \mathbb{P}(A_iA_j|B)$$

$$\longrightarrow ?$$

[■]

<u>(!) maybe not so bad?:</u>

$$\text{DENOM} \geq \mathbb{P}\Big(\bigcap_{j\sim i} \bar{A}_j\Big) = 1 - \mathbb{P}\Big(\bigcup_{j\sim i} A_j\Big)$$

OTOH $\quad \mathbb{P}\Big(\bigcup_{j\sim i} A_j\Big) < \mathbb{P}\Big(\bigcup_{j\sim i} A_j \,\Big|\, A_i\Big) \leq \sum_{j\sim i} \mathbb{P}(A_j|A_i)$

But $\quad \Delta = \sum_i \mathbb{P}(A_i) \sum_{j\sim i} \mathbb{P}(A_j|A_i) \longrightarrow$

<u>if</u> $\Delta \ll \mu = \sum \mathbb{P}(A_i)$, then <u>inner $\sum$ typ. small</u>

BJI gives nothing if $\Delta \geq 2\mu$ $\rightsquigarrow$

▶ "EJI": if $\Delta \geq \mu$ then

$$\Phi \leq \exp\left[-\frac{\mu^2}{2\Delta}\right] \begin{cases} = e^{-\mu+\Delta/2} & \text{if } \Delta = \mu \\ < e^{-\mu} \ (\approx \text{l.b.}) & \text{if } \Delta < \mu/2 \end{cases}$$

E.g. (again): $X = \#$ of $\triangle$'s in $G_{n,p}$

$$\mu \sim n^3 p^3/6, \quad \Delta \sim n^4 p^5/2$$

$\boxed{\text{BJI:}}$

$$\mathbb{P}(X=0) \begin{cases} \sim e^{-\mu} & p \ll n^{-4/5} \\ = e^{-\mu+o(\mu)} & p \ll n^{-1/2} \\ ?? & p = n^{-1/2} \end{cases}$$

Extended Janson Ineq:

$$p \gtrsim (3n)^{-1/2} \longrightarrow \mathbb{P}(X=0) < \exp[-\Omega(n^2 p)]$$

- right order in expt (?); vs. $e^{-\mu}$ (soon) too small $\longleftrightarrow$

too many A_i's $\longrightarrow$ lose independence

poss fix: use only some A_i's – but which?

Pf of EJI: BJI $\longrightarrow$ $\forall\, I \subseteq [r]$

$$\Phi \leq \mathbb{P}\Big(\bigcap_{i\in I} \overline{A_i}\Big) < \exp\underbrace{\Big[-\sum_{i\in I} \mathbb{P}(A_i) + \frac{1}{2}\sum_{\substack{i\sim j \\ i,j\in I}}\sum \mathbb{P}(A_i A_j)\Big]}_{L(I)}$$

▷ I random: $I = [r]_\alpha$ $\quad$ α TBA

$\longrightarrow$ $\mathbb{E}L(I) = -\alpha\mu + \alpha^2\Delta/2$

min $= -\mu^2/2\Delta$ $\quad$ at $\alpha = \mu/\Delta$

▷ where did we use $\Delta \geq \mu$?

<u>back to Bollobás</u>

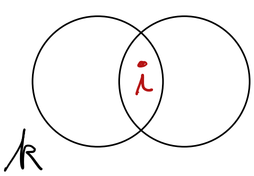
MP: $\mathbb{P}\big(\alpha(\underbrace{G_{m,1/2}}_{G}) < \underbrace{k_0(m) - 3}_{k}\big) < \exp\left[-\Omega\left(\frac{m^2}{\log^4 m}\right)\right]$

X = # of ind. k-sets in G

EJI: $\mathbb{P} \leq \exp\left[-\frac{\mu^2}{2\Delta}\right]$ $\quad$ (if $\Delta \geq \mu$)

$$\mu = \binom{m}{k} 2^{-\binom{k}{2}}$$

$$\Delta = \binom{m}{k}\left[\binom{k}{2}\binom{m-k}{k-2} 2^{-2\binom{k}{2}+1} + \cdots \right.$$
$$\left. \cdots + \binom{k}{k-1}\binom{m-k}{1} 2^{-2\binom{k}{2}+\binom{k-1}{2}}\right]$$



$$\longrightarrow \binom{m}{k}\binom{k}{i}\binom{m-k}{k-i} 2^{-2\binom{k}{2}+\binom{i}{2}}$$

$$\frac{\Delta}{\mu^2} = \frac{\binom{k}{2}\binom{m-k}{k-2} 2^{\binom{2}{2}} + \cdots + \binom{k}{k-1}\binom{m-k}{1} 2^{\binom{k-1}{2}}}{\binom{m}{k}}$$

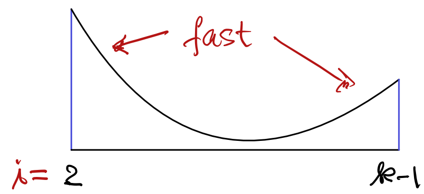
$$\underline{\text{1st term}} \sim \frac{16 \log^4 m}{m^2}$$

$$\underline{\text{last term}} \sim \frac{k\, m^{\binom{k}{2}-(k-1)}}{\binom{m}{k}} = \frac{km}{2^{k-1}\mu(k)}$$

$$= O\left(\frac{k^5}{m^3}\right) \ll \frac{\log^4 m}{m^2}$$

$$\left[\mu(k) \gtrsim \left(\frac{m}{k}\right)^2, \quad 2^{k-1} \asymp \left(\frac{m}{k}\right)^2 \right]$$

check ratios:



$$\rightarrow \quad \frac{\Delta}{\mu^2} \sim \frac{16 \log^4 m}{m^2}$$

check $\Delta \geq \mu$: $\dfrac{\Delta}{\mu} \sim \dfrac{16 \log^4 m}{m^2} \cdot \mu = \Omega(\log^2 m)$ ∎

(ess. same calcs for 2nd m.m.)

Arithmetic progressions

$\boxed{\text{Ref: GRS}}$

- van der Waerden's Thm ('27):

  $\forall k$ & $f: \mathbb{N} \longrightarrow \{R, B\}$ $\exists$ monochr. k-term A.P.

  (EX: maybe not an inf. AP)

- 2nd ver (crucial for pf):

  $\forall k, r \ \exists N$ s.t. $\forall f: [N] \longrightarrow [r]$

  $\exists$ monochr. k-term A.P.

- van der Waerden number $W(k,r)$: least such N

- $W(k) := W(k,2) \approx$?
  - best l.b.'s like 2^k
  - Shelah '88: primitive recursive ($500)
  - Gowers '98: u.b. $2^{2^{2^{2^{2^{k+9}}}}}$ ($1000)
  - Graham: $1000 for $W(k) \overset{?}{<} 2^{k^2}$

Szemerédi's Thm & density thms

- stronger than vdW (if true): density version:

  $\forall\ k, r\ \exists\ N_0$ s.t. $\forall\ N > N_0$

  $A \subseteq [N],\ |A| > N/r \implies A \supseteq k$-t. AP

- in hypergraph language*:

  $\mathcal{H}_N = \{k\text{-t. AP's in } [N]\}$ $\quad (V(\mathcal{H}_N) = [N])$

  vdW: $\chi(\mathcal{H}_N) \longrightarrow \infty$

  density: $\alpha(\mathcal{H}_N) = o(N)$ $\quad$ (?)

  * (α: ind. #; ind: $\not\supseteq$ edge)

- analogue for Ramsey: $V(\mathcal{H}_n) = \binom{[n]}{2}$, $\mathcal{H}_n = \{\text{triangles}\}$ (say) $\longrightarrow$

  Ramsey: $\chi(\mathcal{H}_n) \longrightarrow \infty$

  density: $\alpha(\mathcal{H}_n) = o(n^2)$ $\quad$ (?)

  — nonsense: $\alpha(\mathcal{H}_n) = \lceil n^2/4 \rceil$

... But density vdW is true