The Brauer Group through the Lens of Crossed Product Algebras

March 24, 2021
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What is a division algebra over a field $F$?

**Definition**

Given a field $F$, a **division algebra over $F$** is an associative, unital algebra over $F$ (i.e., vector space over $F$ with multiplication such that $1_A F = Z(A)$) and such that all nonzero elements are invertible.

**Example:**

1) $F$ is a division algebra over $F$.
2) $\mathbb{H}$ is a division algebra over $\mathbb{R}$.

Let $\mathbb{F} = \mathbb{R}$. Then $\mathbb{F}^2 \cong \mathbb{R}, \mathbb{F}^3 \cong \mathbb{H}$.

- $i^2 = j^2 = k^2 = -1$
- $k = ij = -ji$
What are all of the division algebras over \( \mathbb{C} \)?

Suppose \( \mathcal{D} \) division over \( \mathbb{C} \).

\[ \dim_{\mathbb{C}} \mathcal{D} = n. \]

\( d \in \mathcal{D} \setminus \mathbb{C} \)

\( \mathcal{C}(d) \) field

\( \text{linearly dependent} \)

\( 1, d, d^2, d^3, \ldots, d^n \)

\( d \) is algebraic over \( \mathbb{C} \)

\[ \mathcal{C}(d) = \mathbb{C} \] b/c algebraically closed.
What are all of the division algebras over $\mathbb{R}$?

$\mathbb{R}$, $\mathbb{H}$

what happens when you take tensor product of two division algebras over $\mathbb{F}^2$?

Not necessarily division.

$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$

CSA that’s not division
Central Simple Algebras over a field $F$

**Definition**

Given a field $F$, a **central simple algebra (CSA) over $F$** is an associative, unital finite dimensional algebra, $A$, over $F$ such that:

1. $A$ is central ($Z(A) = 1_A F$)
2. $A$ is simple (i.e. $A$ has no proper, nontrivial, two-sided ideals.)

CSA’s

1. Tensor of two CSA's is a CSA (over the same field).
2. Every division alg is a CSA over its center.
3. Every CSA/$F$ is of the form $M_n(CD)$ for some division alg $D/F$ (Wedderburn's Thm.).
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What is the Brauer group?

Definition

The Brauer Group of a field $F$, denoted $Br(F)$, is the set of all CSA’s over $F$ modulo the equivalence relation $\sim$, where

$$A \sim B \iff A \otimes_F M_{n_1}(F) \cong B \otimes_F M_{n_2}(F)$$

for some $n_1, n_2 \in \mathbb{Z}_{>0}$. 
Equivalence Relation?

We have $A \sim B \iff A \otimes_F M_{n_1}(F) \cong B \otimes_F M_{n_2}(F)$ for some $n_1, n_2 \in \mathbb{Z}_{>0}$.

$A \cong M_{n}(D) \cong D \otimes M_{n}(F)$

$[AT] = [D]$. 

$A \otimes_F M_{n}(F) \cong B \otimes_F M_{n}(F)$

$B \otimes_F M_{n}(F) \cong C \otimes_F M_{n}(F)$

$A \otimes_F M_{n_1}(F) \cong A \otimes_F (M_{n}(F) \otimes M_{n_1}(F))$

$B \otimes_F M_{n_1}(F) \cong (B \otimes_F M_{n}(F)) \otimes M_{n_1}(F)$

$A \sim C \otimes M_{n}(F)$
How is this a group?

\[ \text{CSA's over } F^{3}/ \sim , \times \]

1. Associative \( \checkmark \)
2. What's the identity elt. \( \mathbb{F} = [M_n(F)] \).
3. What is the inverse of \( [AJ] \)?
   \[ A^oP = \frac{1}{2} a^o \mid a \in A \]
   \[ a^o b^o = (ba)^o \]
The Opposite Algebra as the Inverse

\[ [A] \otimes [A^\text{op}] = [\text{End}_F(A)] \cong \text{GL}_n(F) \]

\[ \psi : A \otimes F A^\text{op} \rightarrow \text{End}_F(A) \quad \text{sandwich} \]

\[ \psi((a \otimes b^\text{op}) (c \otimes d^\text{op})) = \psi(a \otimes b^\text{op}) \psi(c \otimes d^\text{op}) \]

\[ \begin{align*}
\psi(a \otimes b^\text{op}) & = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\
\psi(c \otimes d^\text{op}) & = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}
\end{align*} \]
\( A \otimes A^\circ \) is CSA/F

\begin{itemize}
  \item \( \ker \mathcal{U} \neq A \otimes A^\circ \implies \ker \mathcal{U} = 0 \)
  
  \text{so } \mathcal{U} \text{ is injective.}
\end{itemize}

\begin{itemize}
  \item \( \dim_F A = n \)
  \item \( \dim_F A^\circ = n^2 = \dim \text{End}_F A. \)
\end{itemize}

\( Br(\mathbb{F}) \) is an abelian gp.

\( Br(\mathbb{C}) \) is trivial.
Some Remarks about the Brauer Group

\[ H_1 \otimes_R H_1 \cong M_4(\mathbb{C}) \]

\( H_1 \cong H_1^\text{op} \).

\[ [H_1]^2 = [R]^J \]

\( \text{ord} \ [H_1] = 2 \) in \( \text{Br}(\mathbb{R}) \).

\[ \text{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z} \]
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Let’s build some CSA’s

\[ E \text{ a finite Galois extension of } F. \]

\[ A = \bigoplus_{\sigma \in G} \mathcal{U}_\sigma E \]

Need to define multi:

\[ \left( \sum_{\sigma \in G} \mathcal{U}_\sigma \gamma \right) \left( \sum_{\tau \in G} \mathcal{U}_\tau \delta \right) = \sum_{\sigma,\tau \in G} \mathcal{U}_{\sigma \tau} \overline{\varphi(\sigma,\tau)} C_{\sigma,\tau}^\tau \delta \]

\[ \varphi: G \times G \rightarrow E^x \]
What do we need $\Phi : G \times G \to E^\times$ to satisfy?

\[
\text{Want } A \text{ to be associative!}
\]

\[
(U_\sigma U_\tau) U_\nu = (U_{\sigma \tau} \dashv (\sigma, \tau)) U_\nu
\]

\[
= U_{\sigma \tau \nu} \dashv (\sigma \tau, \nu)
\]

should be equal to

\[
U_\sigma (U_\tau U_\nu) = U_\sigma (U_{\tau \nu} \dashv (\tau, \nu))
\]

\[
= U_{\sigma \tau \nu} \dashv (\sigma, \tau \nu)
\]

$\dashv$ should satisfy

\[
\dashv (\sigma, \tau, \nu) \dashv (\sigma, \tau, \nu) = \dashv (\sigma, \tau \nu) \dashv (\tau, \nu)
\]
\[ 1_A = U_{id} \Phi(id_G, id_G)^{-1} \]

Normalize: \[ \Phi(id_G, id_G) = 1 \]
Why is $A$ central?

$$x = \sum_{\sigma \in G} u_{\sigma} c_\sigma \in \mathbb{Z}(A)$$

$\forall d \in E$

$$(1_A d) x = x (1_A d)$$

$\Rightarrow 0 = (1_A d) x - x (1_A d) = \sum_{\sigma \in G} u_{\sigma} d^\sigma c_\sigma - \sum_{\sigma \in G} u_{\sigma} c_\sigma$

$= \sum_{\sigma \in G} u_d (d^\sigma - d) c_\sigma$

either $\Rightarrow d = d^\sigma$ $\forall \sigma \in G$.

$$\Rightarrow x = U_{id_E} c = 1_A c^1 \quad \cdots$$

$\Rightarrow c' \in F$
A = (E, G, \Phi) \cong (E, G, \Psi)

\(
\Phi(\sigma, T) = \theta(T) \Theta(T) \otimes \Theta(T)
\)

\(\theta : G \to E^x\)

\[
\left[ \mathcal{C} (E, G, \Phi) \right] \rightarrow H^2 (G, E^x).
\]

\(\text{Br}(E/F) = \text{CSA's over } F \text{ s.t. } [\Lambda \otimes_F E] = [E].\)
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Main Connection

**Theorem**

If $E/F$ is a finite Galois extension with $G = \text{Gal}(E/F)$, then the mapping

$$\Theta_{E/F} : [\Phi] \rightarrow [(E, G, \Phi)]$$

is an isomorphism of $H^2(G, E^\times)$ to $Br(E/F)$. 

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