Finite and Affine Reflection Groups

Terence Coelho

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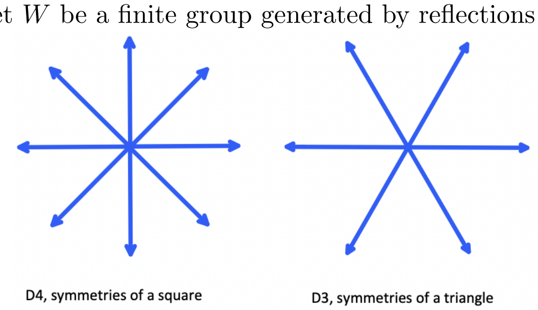
1 Finite Reflection Groups

Let W be a finite group generated by reflections in $\mathbb{R}^n$ through the origin:

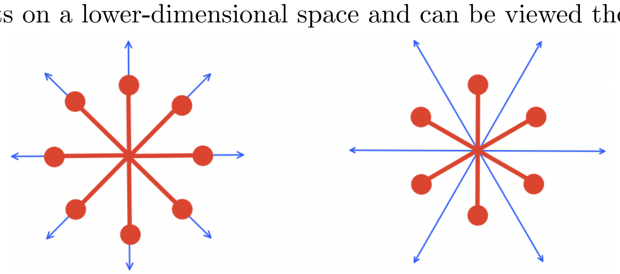


Let $\mathcal{H}$ be the set of hyperplanes reflected over in W. Let $\mathcal{L}$ be the set of lines perpendicular to these hyperplanes.

$\mathcal{H}$ (and equivalently $\mathcal{L}$) are fixed by the action of W. For if $H \in \mathcal{H}, w \in W$ and $s \in W$ is reflection over H, then wsw^{-1} is a reflection over wH.

A choice of opposite vectors on each line in $\mathcal{L}$, Φ is called a *root system* if it is closed under W.

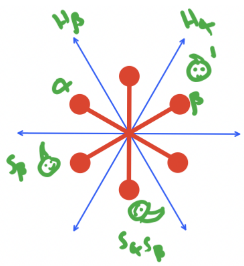
For instance, the set of unit vectors on line in $\mathcal{L}$ forms a root system. We assume Φ spans $\mathbb{R}^n$; if not, W acts on a lower-dimensional space and can be viewed there instead.



For $\alpha \in \Phi$, we denote the corresponding perpendicular hyperplane H_α and reflection across this hyperplane s_α. Note that

$$s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$$

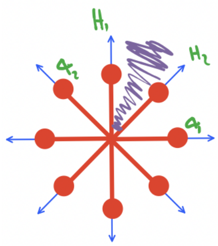
Next, observe that if the angle between α, β is θ, $s_\alpha s_\beta$ will be rotation by 2θ in the plane spanned by α, β (and will leave fixed the space orthogonal to this plane). Thus if $m(\alpha,\beta)$ is the least integer such that $2\theta m(\alpha,\beta) \equiv 0$, then $m(\alpha,\beta)$ is the order of $s_\alpha s_\beta$. Note that twice the angle between H_α, H_β restricted to this plane is the same angle.

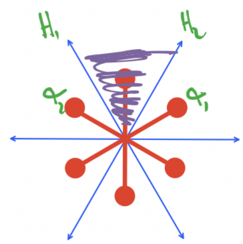


We say the *chambers* of W are the connected components of $\mathbb{R}^n$ with all hyperplanes removed.

Note that since W preserves hyperplanes, it will act on the set of chambers. We will momentarily see that this action is transitive and thus all chambers are congruent.

Let $\{H_1, H_2, \dots H_n\}$ be the walls of some chamber C_0. Let s_i be reflection over H_i and let α_i be the root perpendicular to H_i on the same side as C_0. Call these simple hyperplanes, simple reflections, and simple roots respectively.

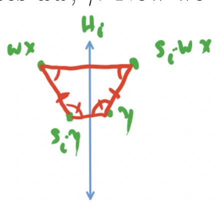




We now show that the simple reflections generate W. Let W' be the subgroup generated by the s_i.

We first note that W' acts transitively on chambers. Suppose not - then there is some chamber C such that there is no $w \in W'$ with $wC = C_0$. Pick some $x \in C$ and choose $w \in W'$ such that $|wx - \gamma|$ is minimal.

Then since wx, γ are in different chambers and the walls around γ are the H_i, there is some H_i that separates wx, γ. Now we can show by a geometric argument that $|s_iwx - \gamma| < |wx - \gamma|$ giving the contradiction.



Now given any H_β, we can find some w, i such that $wH_\beta = H_i$ by considering some chamber C bounded by H_β, taking $w \in W'$ such that $wC = C_0$, then picking H_i based on which wall w sends H_β to.

Finally, recognize that if $wH_\beta = H_i$, then $w^{-1}s_iw = s_\beta$. Thus all reflections are in W' so $W = W'$.

Thus the s_i generate W. Actually something mmuch more fascinating is true (see appendix): W acts *simply transitively* on the set of chambers (so there is a natural bijection between the chambers and W). In addition, we show in the appendix that W has a presentation as a *Coxeter Group*:

$$W = \{s_i, s_i^2 = 1, (s_is_j)^{m(\alpha_i,\alpha_j)} = 1\}$$

We now state some important observations that can be made by looking at the plane spanned by the two roots:

1. Any pair of reflections s_α, s_β generate the Dihedral group of symmetries of an $m(i,j)$-gon
2. The angle between H_i and H_j in the plane spanned by α_i, α_j is $\frac{\pi}{m(\alpha_i,\alpha_j)}$, as it is the smallest angle between two reflections in the corresponding dihedral group.
3. Accordingly, the angle between α_i, α_j is $\pi - \frac{\pi}{m(\alpha_i,\alpha_j)}$. In particular, simple roots are pairwise non-acute

2 Coxeter Groups

A (possibly infinite) group G is a *Coxeter group* if it has a presentation of the form:

$$G = \{s_i, s_i^2 = 1, (s_i s_j)^{n_{i,j}} = 1$$

where the $n_{i,j}$ may be infinite.

To each Coxeter group we associate a Coxeter matrix defined by

$$C_{i,j} = cos(\pi - \frac{\pi}{n_{i,j}}) = -cos(\frac{\pi}{n_{i,j}})$$

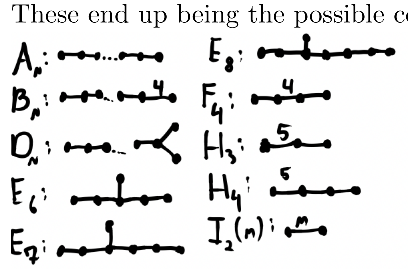
noting that $n_{i,i} = 1$ and this is well-defined even if $n_{i,j} = \infty$.

If G was a finite reflection group, this would be the matrix of inner-products of the normalized simple roots.

Thus to classify all possible reflection groups, it suffices to classify all possible positive-definite Coxeter matrices, which is not too hard to do with a little linear algebra (proof in Humphreys' Reflection Groups). The matrices are a bit unaesthetic to describe, so we will instead draw the associated *Coxeter diagrams.*

The Coxeter diagram associated to a Coxetre group is a graph with a node corresponding to each generator s_i in the Coxeter group, with no edge between nodes i, j if $n_{i,j} = 2$, one edge if $n_{i,j} = 3$, and an edge labeled with $n_{i,j}$ if larger than 3.

These end up being the possible connected diagrams (subscripts denote the number of nodes).



To give some examples of matrices, we have:

$$A_3 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} B_3 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix} I_2(6) = \begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 1 \end{bmatrix}$$

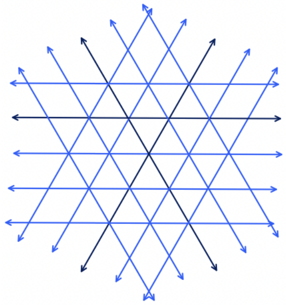
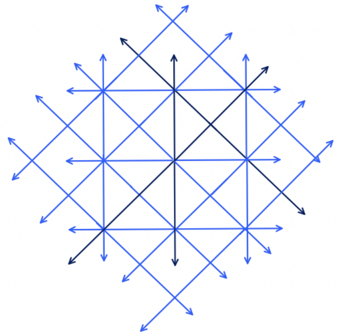
While these represent the possible Coxeter matrices (and thus all possible finite reflection groups), we don't know that all of these really are finite reflection groups. They are, and can be described to prove such a statement. At the moment, I cannot think of nor find a deep reason why they all must be reflection group.

If a diagram is disconnected, the simple hyperplanes can be partitioned into two mutually orthogonal sets $\mathcal{H}_1$, $\mathcal{H}_2$. Let W_1, W_2 be the groups generated by the simple reflections in each; by orthogonality W_1 and W_2 commute so $W = W_1 \times W_2$. Thus what we have is a classification of the indecomposable reflection groups and the remaining reflection groups are simply a direct product of indecomposable reflection groups.

Note a diagram is disconnected precisely when the corresponding Coxeter matrix is decomposable.

3 Affine Reflection Groups

Before, we required all our hyperplanes to go through the origin. Now, we will relax that condition, but just require that there is some ball in $\mathbb{R}^n$ that isn't cut by a hyperplane in the generated group.

Let $\mathcal{H}$ be the set of hyperplanes flipped over by some element of W. Call the connected components of $\mathbb{R}^n$ with the hyperplanes removed *alcoves*. Since $\mathcal{H}$ is preserved by W, W will act on the set of alcoves and thus all alcoves in the same W-orbit will be congruent.

Pick one alcove and call it A_0. Call the hyperplanes that form the walls $\{H_i\}$ and call the corresponding reflections $\{s_i\}$. We will refer to these as simple hyperplanes and reflections respectively.

By the exact same argument as the finite case, we can show that the s_i generate W and W acts transitively on the alcoves. Thus all alcoves are congruent and will tesselate $\mathbb{R}^n$.

In fact we can show in the same way that W will act simply transitively on the alcoves and W will be a Coxeter group (the appendix does this and the finite case simulatneously).

Now to each simple H_i, let α_i be the unit vector orthogonal to it on the same side as A_0. Note if α_i and α_j are non-proportional, then $s_i s_j$ will be a rotation by twice the angle between α_i and α_j in the plane spanned by α_i and α_j. Furthermore, note that in this plane, these reflections will generate the dihedral group of an $m(i,j)$-gon (where $m(i,j)$ is the order of $s_i s_j$). So for these to be walls of an alcove, they must restrict to adjacent lines of reflection in this dihedral group. Thus the angle between H_i and H_j is $\frac{\pi}{m(i,j)}$ (in this plane) and the angle between α_i and α_j is $\pi - \frac{\pi}{m(i,j)}$.

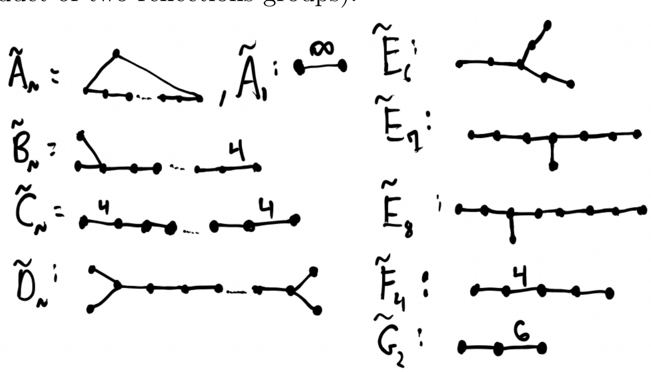
If the α_i, α_j are proportional, they must be in opposite directions to be walls of an alcove. Furthermore, in this case $s_i s_j$ will be a translation and thus have infinite order. Since they are in opposite directions, the angle between them is $\pi = \pi - \frac{\pi}{\infty}$.

This in particular shows that the α_i are all mutually non-acute, so there are at most $2n$ of them in $\mathbb{R}^n$

4 Affine Coxeter Groups

Since these affine reflection groups are Coxeter groups, we can write their associated Coxeter matrices. As before, these Coxeter matrices will be the inner product matrix of the simple roots. But now the simple roots must be linearly dependent, so the resulting Coxeter matrix will be positive semi-definite.

Thus, it suffices to classify the determinant 0 positive semi-definite Coxeter matrices. Here are the associated connected Coxeter diagrams (as before, a disconnected Coxeter diagram corresponds to a direct product of two reflections groups):

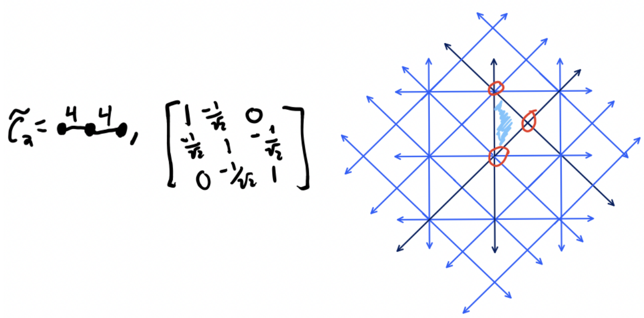


For reasons that will be explained later (along with the choice of names) , the n here is chosen to be one less than the number of nodes in the diagram.

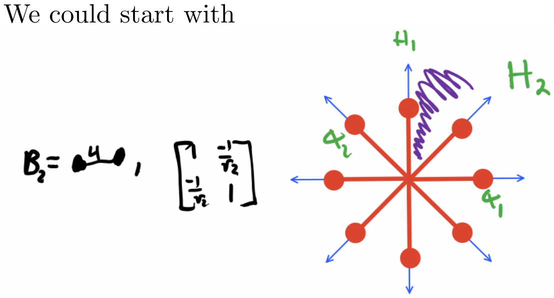
Note that in all these diagrams, removal of any node would make the resulting diagram finite. There are some interesting consequences of this:

1. Indecomposable Coxeter matrices are corank 1
2. The subgroup generated by all but 1 simple reflections will be a finite reflection group. If there are n nodes, the finite reflection group lies in $\mathbb{R}^{n-1}$ and since the remaining simple root is a combination of the rest, the affine reflection group acts on $\mathbb{R}^{n-1}$.
3. The alcoves are bounded by n walls and thus they are $n-1$-dimensional simplices.
4. Each of the n vertices of this simplex is a point where all but 1 simple hyperplane intersect. Thus W can be realized by starting with a finite reflection group corresponding to its diagram with any one point missing, then adding an off-center hyperplane (that would complete the diagram) and determining the group these reflections generate.

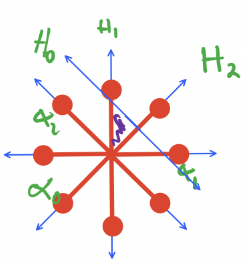
For example, consider:



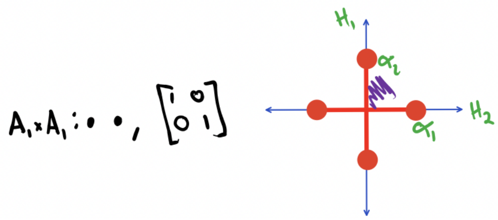
We could start with



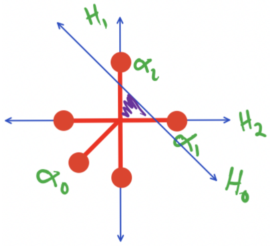
And now we identify a unit vector α_0 with $(\alpha_0, \alpha_1) = 0$, $(\alpha_0, \alpha_2) = -\frac{1}{\sqrt{2}}$ and append an off-center hyperplane orthogonal to this vector that cuts through the fundamental chamber. It happens to be a root already but didn't have to be:



Alternatively, we could have begun with



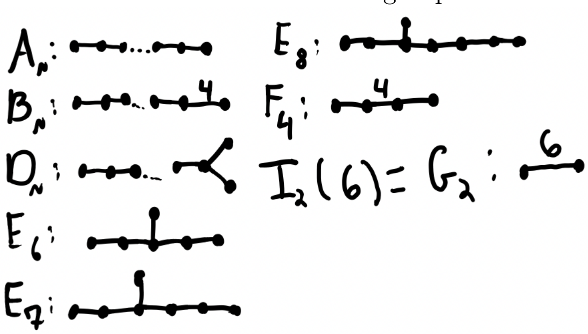
And now we identify a unit vector α_0 with $(\alpha_0, \alpha_1) = -\frac{1}{\sqrt{2}}$, $(\alpha_0, \alpha_2) = -\frac{1}{\sqrt{2}}$ and append an off-center hyperplane orthogonal to this vector that cuts through the fundamental chamber.



Now observe that every finite Coxeter diagram can be found by removing a node from some affine Coxeter diagram. Since we know that an affine Coxeter diagram with n nodes can be realized as a reflection group in $\mathbb{R}^{n-1}$ nodes and removal of any node from a Coxeter diagram gives a finite reflection group at one of the vertices of the alcove, we know that any finite Coxeter diagram with k nodes is a reflection group on $\mathbb{R}^k$.

Now notice that for all affine Coxeter groups, we have all $n_{i,j} \in \{2, 3, 4, 6, \infty\}$. This means for all $i, j, 4C_{i,j}C_{j,i} = 4\cos^2(\frac{\pi}{n_{i,j}})$ are integral.

Furthermore, the finite reflection groups with this property are precisely the ones that occur as subdiagrams of the affine Coxeter diagrams. We call reflection groups with this property *crystallographic*, as they are either affine reflection groups (which tesselate Euclidean space) or a finite reflection group at the vertex of an alcove one such affine reflection group.



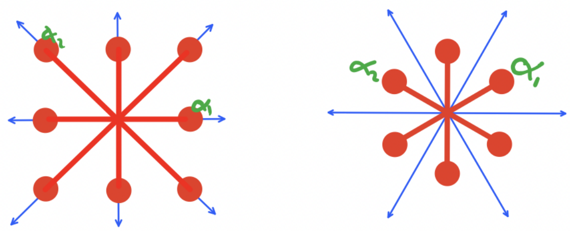
5 Classification of Crystallographic Root systems

Let W be a crystallographic finite reflection group. Let $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$ (so we can write $s_\alpha : \beta \to \beta - \langle \alpha, \beta \rangle \alpha$).

We have $\langle \alpha, \beta \rangle = \frac{2|\alpha|}{|\beta|}\cos(\theta)$. Thus, if we define a matrix $A_{i,j} = \langle \alpha_i, \alpha_j \rangle$, we will have

$$A_{i,j}A_{j,i} = 4\cos^2(\theta_{i,j}) = 4C_{i,j}C_{j,i} \in \mathbb{Z}$$

A root system Φ for a finite reflection group W is said to be *crystallographic* if for all $\alpha, \beta \in \Phi$, $\langle \alpha, \beta \rangle \in \mathbb{Z}$. This naming choice will be apparent later when we see that $\mathbb{Z}[\Phi]$ is a lattice.



For crystallographic root systems, the matrix $A_{i,j} = \langle \alpha_i, \alpha_j \rangle$ is called its *Cartan matrix* and determines the root length ratios (and thus the whole root system, as we can recover the Coxeter matrix). Note this matrix will have integral entries by assumption.

We can determine all possible Cartan matrices associated to a reflection group W with Coxeter matrix C by the relation $A_{i,j}A_{j,i} = 4C_{i,j}C_{j,i}$ and noting $A_{i,j}$ is integral and non-positive off the diagonal (since all simple roots are non-acute), 2 on the diagonal and of course 0s off the diagonal must be symmetric.

Ex: $Cox(B_3)$: $\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}$

$Car_1(B_3)$: $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$: $|\alpha_1| = |\alpha_2| = \sqrt{2}\,|\alpha_3|$

$Car_2(B_3)$: $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$: $|\alpha_1| = |\alpha_2| = \frac{1}{\sqrt{2}}|\alpha_3|$

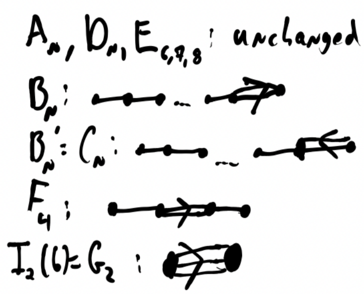
For the crystallographic reflection groups except B_n, F_4, G_2, all off-diagonal entries in the Coxeter matrix will be $-1/2$ or 0. This means any associated Cartan matrix must have $A_{(i,j)}A_{(j,i)} = 0$ or 1 (if $C_{(i,j)}$ is 0, $-1/2$ respectively for $i \neq j$). Thus there is only one choice of Cartan matrix for these - the one that gives all simple roots equal norm.

 $\to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

As with Coxeter matrices, we use diagrams to show all possible Cartan matrices. These diagrams are called Dynkin diagrams, and we define them in a way that makes them resemble the Coxeter diagram for its reflection group:

1. If $A_{i,j} = A_{j,i} = 0$, no edge between nodes i, j.
2. If $A_{i,j} = A_{j,i} = -1$, one edge between nodes i, j.
3. If $A_{i,j} = -1$, $A_{j,i} = -k$, k edges with arrow from i to j.

Here are the connected Dynkin Diagrams, labeled according to the reflection groups they come from. There are two that come from B_n:



One can show (e.g. by construction) that there is really a crystallographic root system associated to each of these Cartan matrices and thus we have a classification of the crystallographic root systems.

6 Properties of Crystallographic Root systems

Crystallographic root systems have a lot of neat properties.

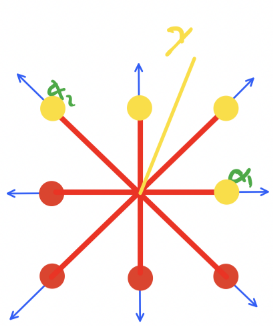
Since $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\cos^2(\theta) \in \mathbb{Z}$: if $\alpha \neq -\beta$ and $(\alpha,\beta) < 0$, we have $\langle\alpha,\beta\rangle = -1$ or $\langle\beta,\alpha\rangle = -1$ since the product is 1, 2, or 3.

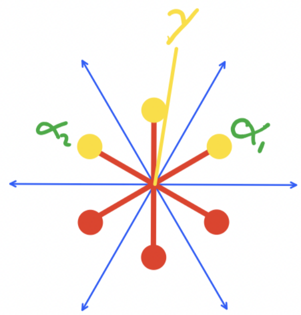
In other words, either $s_\alpha(\beta) = \beta + \alpha$ or $s_\beta(\alpha) = \alpha + \beta$ (recall $s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$)

So since Φ is preserved by W, if $(\alpha,\beta) < 0$ and $\alpha \neq -\beta$ then $\alpha + \beta \in \Phi$. Likewise if $(\alpha,\beta) < 0$ and $\alpha \neq \beta$, then $\alpha - \beta, \beta - \alpha \in \Phi$.

Next, we can show that every root in Φ is a non-negative or non-positive integral combination of the simple roots.

Let γ be a vector in the C_0 (the chamber whose walls are used to determine the simple roots). Let $\Pi \subset \Phi$ be the roots with $(\alpha,\gamma) > 0$ and call these positive roots. Call the remaining roots negative roots.





Let $\Delta = \{\beta_i\}$ be a set of minimal size such that all roots in Π are a non-negative integral combination of the β_i. We aim to show that Δ are precisely the simple roots.

It turns out that Δ is pairwise non-acute. Suppose we have $(\beta_i, \beta_j) < 0$. Then we know $\beta_i - \beta_j \in \Phi$ and thus $\beta_j - \beta_i \in \Phi$. One of these is positive so without loss of generality, say $\beta_i - \beta_j \in \Pi$.

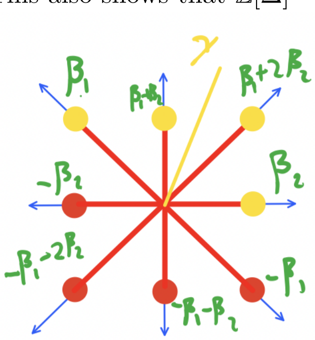
So $\beta_i = \beta_j + \sum b_k\beta_k$ with $b_k \in \mathbb{Z}_{\geq 0}$. If the coefficient on the RHS is positive, we could subtract β_i from both sides and have a non-negative combination of the β_k equals 0, which is absurd since they are all acute with γ.

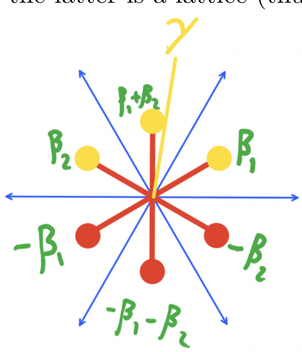
Thus this expression of β_i is a non-negative integral combination of $\Delta \setminus \beta_i$, which contradicts minimality.

Furthermore, one can show that any set of pairwise non-acute vectors in the same half-space of $\mathbb{R}^n$ must be linearly independent.

Thus Δ is a basis and by construction every root can be expressed as a strictly non-negative or strictly non-positive integral combination of the simple roots.

This also shows that $\mathbb{Z}[\Delta] = \mathbb{Z}[\Phi]$, so the latter is a lattice (thus inspiring the name "crystallographic")





We now show that Δ is precisely the simple roots.

Note that $C_0 = \{x \in \mathbb{R}^n, (x,\beta) > 0, \beta_i \in \Pi\}$ since this set contains γ, no hyperplanes, and every point on its boundary is a hyperplane (so it is a chamber, and the chamber with γ).

But since any $\alpha \in \Pi$ is a non-negative linear combination of the β_i, this can equivalently be written

$$C_0 = \{x \in \mathbb{R}^n, (x,\beta_i) > 0, \beta_i \in \Delta\}$$

This shows that all possible walls of the C_0 are the H_{β_i} and there are n of these since Δ is a basis. But the walls of C_0 are the $H_i = H_{\alpha_i}$ (which have the same cardinality n).

So $\{H_{\alpha_i}\} = \{H_{\beta_i}\}$ and since the β_i and α_i are all acute with γ, we have $\{\alpha_i\} = \{\beta_i\}$, so Δ is the set of simple roots.

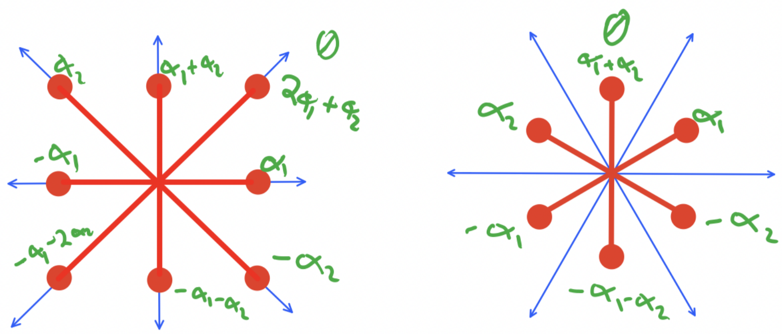
We can introduce a partial order on Φ where $\alpha \geq \beta$ if $\alpha - \beta$ is a non-negative combination of the simple roots

Any crystallographic root system will have a highest root θ that is not only maximal with respect to this ordering, but $\theta \geq \Phi$ (even though this is just a partial order).

To show this, take θ maximal. We want to show $\theta \geq \alpha$ for all $\alpha \in \Phi$. It suffices to assume $\alpha \in \Pi$ and $\alpha + \alpha_i \notin \Phi$ for all i.

Thus we have $(\theta, \alpha_i) \geq 0$, $(\alpha, \alpha_i) \geq 0$ for all i. Now if $(\theta, \alpha) > 0$, the set $\{\alpha_i, -\theta, \alpha\}$ is a set of $n+2$ pairwise non-acute vectors in Rn which is impossible (any $n+1$ of them will be in the same half-space, so we'd have $n+1$ linearly independent vectors)

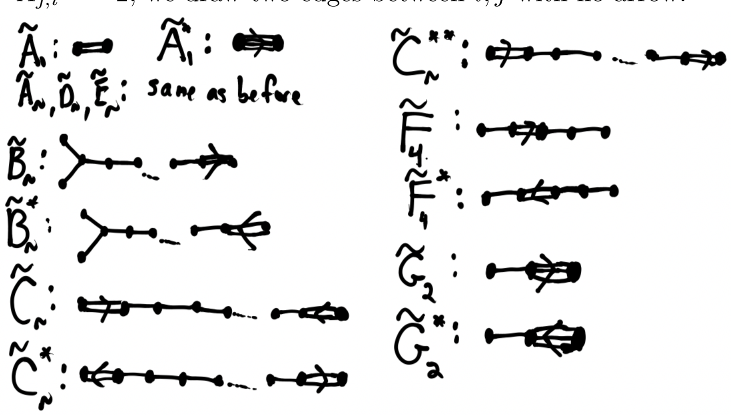
Thus $(\theta, \alpha) > 0$, so if $\theta \neq \alpha, \theta - \alpha \in \Phi$. If the difference is in Π, $\theta > \alpha$ and otherwise $\alpha > \theta$ which is impossible by maximality.



Now, even though there isn't a root system associated each of the affine reflection groups, we can still ask what affine Cartan matrices would correspond to these Coxeter matrices.

In other words, we want matrices A with integral entries and the usual conditions such that $A_{i,j}A_{j,i} = 4C_{i,j}C_{j,i}$.

To each of these matrices, we associate a Dynkin diagram as before. There is one additional case: if $A_{i,j} = A_{j,i} = -2$, we draw two edges between i, j with no arrow.

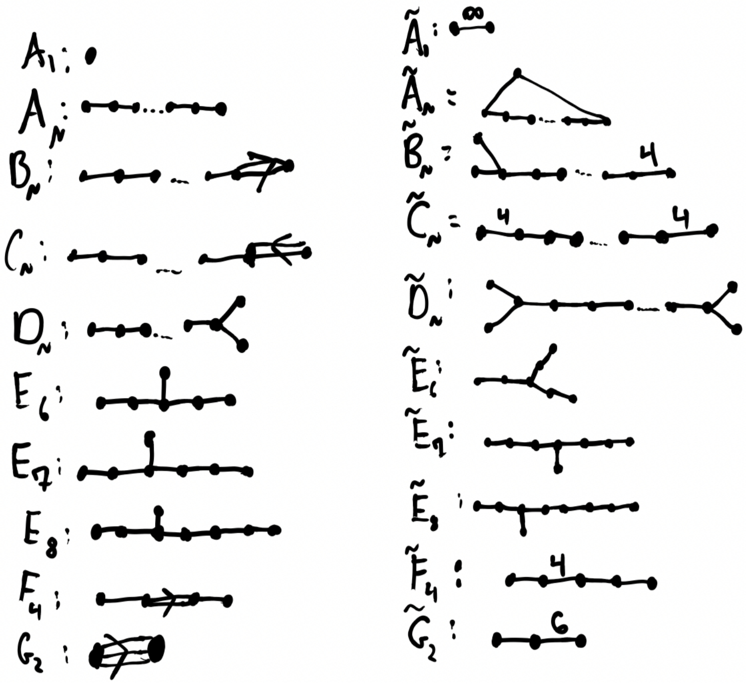


We will discuss the purpose of these at the very end.

7 Standard Construction of Affine Reflection Groups

Earlier we showed how we can construct an affine reflection group from an affine Coxeter diagram by starting with any finite reflection group formed by removing a node.

However, there is actually a nice bijection between crystallographic root systems and the affine reflection groups that give rise to an alternate construction that can be easier to work with (this bijection was built into our notation)



We now describe this correspondence. Take Φ to be a crystallographic root system with (finite, crystallographic) reflection group W.

For $\alpha \in \Phi, k \in \mathbb{Z}$, let

$$H_{\alpha,k} = \{x \in \mathbb{R}^n, (x, \alpha) = k\}$$

and let $s_{\alpha,k}$ be the reflection across this hyperplane. We have

$$s_{\alpha,k}(x) = x - \frac{2((\alpha, x) - k)}{(\alpha, \alpha)}\alpha = s_\alpha(x) + \frac{2k\alpha}{(\alpha, \beta)}$$

For $\alpha \in \Phi$, define the coroot of α to be $\alpha^v = \frac{2\alpha}{(\alpha,\alpha)}$.

One can quickly show that the set of coroots Φ^v is also a crystallographic root system (for the same W)

Define t_x to be translation by the vector x.

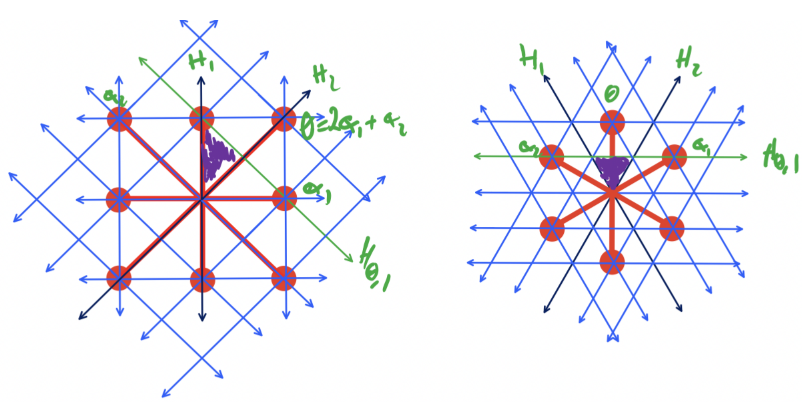
So $s_{\alpha,k} = t_{k\alpha^v} s_\alpha$.

Define the *affine reflection group* W_a to be the group generated by the $s_{\alpha,k}$.

It is clear that W_a contains the group of translations by $\mathbb{Z}[\Phi^v]$ since $s_{\alpha,k}s_\alpha = t_{k\alpha^v}$

Furthermore, since $wt_xw^{-1} = t_{w(x)}$ for $w \in W$ and $\mathbb{Z}[\Phi^v]$ is closed under W, we have:

$$W_a = W \ltimes T_{\mathbb{Z}[\Phi^v]}$$



Pick a chamber C_0 for (Φ, W) with associated positive roots Π and let γ be some element of C_0. Note the following defines an alcove of W_a:

$$A_0 = \{x \in \mathbb{R}^n, 0 < (x, \alpha) < 1 \forall \alpha \in \Pi\}$$

Indeed it is non-empty (contains a small multiple of γ), contains no hyperplane, and any vector on its boundary is a hyperplane.

Now this alcove can be equivalently written:

$$A_0 = \{x \in \mathbb{R}^n, (x, \alpha_i) > 0, (x, \theta) < 1\}$$

by using the fact $\theta \geq \Phi$.

So all possible walls of A_0 are the H_i and $H_{\theta^v,1}$. And since at least $n+1$ walls are needed to bound an alcove, these are precisely the walls.

Thus W_a is generated by $\{s_i, s_{\theta,1}\}$.

So unit vectors in the corresponding Coxeter matrix are the normalized simple roots and $-\theta/|\theta|$.

8 Lie algebras

We say $\mathfrak{g}$ is a Lie algebra if it satisfies:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$
2. $[a, [b, c]] - [b, [a, c]] = [[a, b], c] = 0$ for all $a, b, c \in \mathfrak{g}$

A standard example is $\mathfrak{g} = g\ell(V)$, $[A, B] = AB - BA$.

An ideal $I \subset \mathfrak{g}$ is a subspace that satisfies $[I, \mathfrak{g}] \subset I$.

We say a Lie algebra is *simple* if it has no ideals except 0 and itself. We exclude the trivial 0 and 1 dimensional Lie algebras from this definition.

The set of finite-dimensional simple Lie algebras (over $\mathbb{C}$) turns out to be in natural bijection with the set of crystallographic root systems.

To any finite-dimensional simple Lie algebra, there is a crystallographic root system Φ such that we have the following root space decomposition:

$$\mathfrak{g} = H \oplus \sum_{\alpha \in \Phi} X_\alpha$$

where:

1. H is viewed as the span of Φ^v
2. Each X_α is 1-dimensional
3. $[H, H] = 0$
4. $[h, x_\alpha] = (\alpha, h)x_\alpha$
5. $[X_\alpha, X_{-\alpha}] = \mathbb{C}\alpha^v$
6. $[X_\alpha, X_\beta] \subset X_{\alpha+\beta}$
7. There is a natural form on $\mathfrak{g}$ satisfying $(x, [y, z]) = ([x, y], z)$ which agrees with the euclidean inner product on Φ^v when restricted to H.
8. W acts on $\mathfrak{g}$ in a way that restricts appropriately to $\Phi^v \subset H$.
9. W acts on modules of $\mathfrak{g}$ in a compatible way- $\alpha^v \cdot w(v) = w(\alpha^v) \cdot v$

We can also construct simple Lie algebras $\mathfrak{g}$ directly from an $n \times n$ Cartan matrix A:

1. Construct an n-dimensional vector space H and identify vectors $\{\alpha_i\}$ and $\{\alpha_i^v\}$ such that $(\alpha_j, \alpha_i^v) = A_{i,j}$.

2. Define symbols $e_1 \dots e_n,\ f_1, \dots f_n$. We will view $e_i \in X_{\alpha_i}, f_i \in X_{-\alpha_i}$.
3. Construct the free Lie algebra on these symbols and the vectors in H
4. Mod out the following relations:
   (a) Linear dependence relations in H
   (b) $[h, e_i] = (\alpha_i, h)e_i$
   (c) $[h, f_i] = (-\alpha_i, h)f_i$
   (d) $[e_i, f_i] = \alpha_i^v$
   (e) $[H, H] = 0$
5. Finally, mod out by the largest ideal that does not intersect H (equivalent to modding out by the Serre relations as is done in some texts)

In this way, we have a bijection that maps crystallographic root systems to the finite-dimensional simple Lie algebras (the fact this is a surjection is remarkable).

Now we can apply this algorithm to the affine Cartan matrices we constructed. The only change is the first step, which must be altered slightly to deal with the co-rank 1 nature of affine Cartan matrices:

1. Construct an $n+1$ dimensional vector space H and identify a set of linearly independent vectors $\{\alpha_i\}$, and linearly independent vectors $\{\alpha_i^v\}$ with $(\alpha_j, \alpha_i^v) = A_{i,j}$

We call the resulting Lie algebras *Affine Lie algebras*. They will also have a decomposition:

$$\mathfrak{g} = H \oplus \sum_{\alpha \in \Phi} X_\alpha$$

where Φ is just defined to be the set of non-zero eigenvalues (viewed as functionals on H) for the action of H on $\mathfrak{g}$ (before we didn't have a notion of a root system for affine reflection groups).

This will have very similar properties to the finite case, and perhaps most importantly the affine reflection group still act on $\mathfrak{g}$ in a way that resembles its action on $\mathbb{R}^{n-1}$ (when restricted to any hyperplane in H parallel (but not equal to) the hyperplane spanned by the α_i^v, modulo the dependence relation of the simple roots).

It will also act in a compatible way on the representations of $\mathfrak{g}$ that satisfy some reasonable conditions. This gives rise to the rich representation theory of Affine Lie algebras.

9 Appendix: The Coxeter Presentation

This proof is intended to prove the Coxeter presentation for both the finite and affine case. We will use the term "alcove" to describe chambers in the finite case for the sake of uniform notation.

We have the Euclidean space $\mathbb{R}^n$ separated by a discrete set hyperplanes $\mathcal{H}$ such that $\mathcal{H}$ is closed under reflection over any hyperplane in $\mathcal{H}$. Let W be the group generated by these reflections. We call the connected components of $\mathbb{R}^n$ with the hyperplanes remove alcoves.

Identify one alcove, A_0, and call its walls $H_1 \dots H_n$. Let $s_1 \dots s_n$ be the corresponding reflections. We have shown the group generated by these simple reflections acts transitively on the alcoves and thus generates all of W.

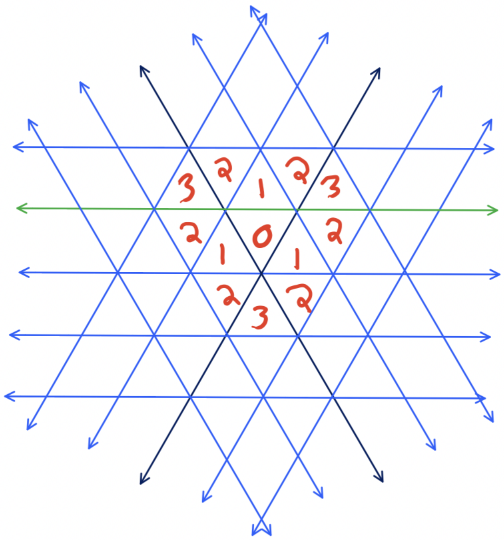
10 Appendex Contd: The Length Function

We have shown that we can write any $w \in W$ as a product of simple reflections s_i. Call an expression of minimum length a *reduced* expression for w.

For $w \in W$, let the length $\ell(w)$ be the length of a reduced expression for w. Let

$$\mathfrak{L}(w) = \{H \in \mathcal{H}, H \text{ separates } A_0 \text{ and } w^{-1}A_0\}$$

Set $n(w) = |\mathfrak{L}(w)$. We will show $\ell(w) = n(w)$. This in particular shows that $wA_0 = A_0 \iff w = 1$, meaning that W acts *simply transitively* on these alcoves



We haven't shown that the Weyl group can be identified with the set of alcoves yet, but consider an identification of each alcove with some shortest w such that $w^{-1}A_0$ is that alcove. One can quickly verify that these numbers will be the corresponding $\ell(w)$ and $n(w)$

First note that for $\ell(w) \leq 1$, we will have $\ell(w) = n(w)$. This can be verified by observing the edge from any vector in s_iA_0 to any vector in A_0 will only cross the hyperplane H_i.

Furthermore, $n(w) = n(w^{-1})$ since H separates A_0 and $w^{-1}A_0$ if and only if wH separates wA_0 and A_0.

We will now show that $\ell(w) \geq n(w)$.

First note that for any $w \in W$, H_i is in exactly one of $\mathfrak{L}(w)$, $\mathfrak{L}(ws_i)$. This is immediate geometrically – $w^{-1}A_0$ and $s_iw^{-1}A_0$ are on opposite sides of H_i, so exactly one of these is on the same side of the hyperplane as A_0.

Next we show that for $H \neq H_i$, if $H \in \mathfrak{L}(w)$, then $s_iH \in \mathfrak{L}(ws_i)$. Since H separates $w^{-1}A_0$ and A_0, we know s_iH separates $s_iw^{-1}A_0$ and s_iA_0. And since s_iA_0 and A_0 are only separated by H_i, we cannot have $s_iw^{-1}A_0$ and A_0 on the same side of s_iH. Thus $s_iH \in \mathfrak{L}(ws_i)$.

By applying s_i again, we see the reverse is also true. In other words:

$$s_i(\mathfrak{L}(w) \setminus \{H_i\}) = \mathfrak{L}(ws_i) \setminus \{H_i\}$$

And since exactly one of $\mathfrak{L}(w), \mathfrak{L}(ws_i)$ contains H_i, we can see that if $H_i \notin \mathfrak{L}(w), n(ws_i) = n(w)+1$, otherwise $n(ws_i) = n(w) - 1$.

This immediately shows inductively that $n(w) \leq \ell(w)$.

We now show that $n(w) = \ell(w)$. Suppose $w = s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced expression for w.

For notational convenience, let $f_j = s_{i_j}$ and $P_j = H_{i_j}$. So $w = f_1f_2\dots f_k$.

We show that

$$\mathfrak{L}(w) = \{P_k, f_kP_{k-1}, f_kf_{k-1}P_{k-2}, \dots f_kf_{k-1}\dots f_2P_1\}$$

and that all these hyperplanes are distinct. Thus we would have $\ell(w) = n(w)$.

We first show distinctness. Suppose for some $p > q$:

$$f_kf_{k-1}f_{k-2}\dots f_{q+1}P_q = f_kf_{k-1}f_{k-2}\dots f_{p+1}P_p$$

Then $f_pf_{p-1}\dots f_{q+1}P_q = P_p$ and recall that $wH_i = H_j \iff ws_iw^{-1} = s_j$.

Thus we have

$$f_pf_{p-1}\dots f_{q+1}f_qf_pf_qf_{q+1}\dots f_{p-1}f_p = f_q$$

rearranging gives

$$f_pf_{p-1}\dots f_{q+1}f_q = f_{p-1}f_{p-2}\dots f_{q+2}f_{q+1}$$

and inverting both sides gives

$$f_q f_{q+1} \dots f_{p-1} f_p = f_{q+1} f_{q+2} \dots f_{p-2} f_{p-1}$$

which gives us a way to shorten the expression for w, which is impossible since this expressiong for w is reduced.

Now to show that these are really the hyperplanes in $\mathfrak{L}(w)$, we proceed by induction, noting it is clear for length 1. Suppose it were true for length less than k.

Let $w' = f_1 f_2 \dots f_{k-1}$. We have by inductive hypothesis:

$$\mathfrak{L}(w') = \{P_{k-1}, f_{k-1}P_{k-2}, \dots f_{k-1}f_{k-2} \dots f_2 P_1\}$$

with all hyperplanes.

We know $P_k \notin f_k \mathfrak{L}(w')$, as the set of hyperplanes

$$\{P_k, f_k P_{k-1}, f_k f_{k-1} P_{k-2}, \dots f_k f_{k-1} \dots f_2 P_1\}$$

are all distinct (since $f_1 \dots f_k$ is a reduced espression).

Thus since $f_k P_k = P_k$, we have $P_k \notin \mathfrak{L}(w')$

We also know that

$$\mathfrak{L}(\mathfrak{w}'\mathfrak{f}_\mathfrak{k}) \setminus \{P_k\} = f_k \mathfrak{L}(w') \setminus \{P_k\}$$

and exactly one of these contains P_k. Thus we can conclude

$$\mathfrak{L}(w' f_k) = f_k \mathfrak{L}(w') \cup \{P_k\} = \{P_k, f_k P_{k-1}, f_k f_{k-1} P_{k-2}, \dots f_k f_{k-1} \dots f_2 P_1\}$$

completing the induction.

We can thus conclude that W acts simply transitively on the alcoves so these alcoves can be identified with elements of W.

11 Appendix Contd: The Deletion Condition

Suppose $w = f_1 f_2 \dots f_k$ is not a reduced expression for w.

We want to show there exist p, q such omitting the reflections and index p and q in this expression leave w unchanged. In other words

$$f_1 f_2 \dots f_{p-1} f_{p+1} \dots f_k = f_1 \dots f_k$$

Since w is not reduced, there must be some p for which

$$n(f_p f_{p+1} \dots f_k) = n(f_{p+1} f_{p+2} \dots f_k) - 1 - 1$$

otherwise we would have $n(w) = k > \ell(w)$.

This means

$$P_p \in \mathfrak{L}(f_{p+1} \dots f_k) = \{P_k, f_k P_{k-1}, \dots f_k f_k \dots f_{p+2} P_{p+1}\}$$

So for some $q > p$,

$$P_p = f_k f_{k-1} \dots f_{q+1} P_q$$

and as before we conclude from this

$$f_q f_{q+1} \dots f_{p-1} f_p = f_{q+1} f_{q+2} \dots f_{p-2} f_{p-1}$$

from which the deletion condition follows

12 Appendix End: Proof of the Coxeter Presentation

Let $m(i,j)$ be the order of $s_i s_j$ in W (possibly infinite).

We know that the s_i generate W. We are now finally ready to show that W is fully described by the Coxeter relations

$$s_i^2 = 1, (s_i s_j)^{m(i,j)} = 1$$

Suppose note. Let $f_1 f_2 \dots f_k = 1$ be the shortest relation in W that cannot be explained as a consequence of the above Coxeter relations.

Note this implies that this relation cannot even be shortened using the Coxeter relations, for if so, the shortened version would equal 1 by the Coxeter relations (by the "shortest" assumption) so the original relation could be fully explained with the Coxeter relations.

We know that k is even by parity of reflections and trivially $k > 2$.

Note that if $w_1 w_2 = 1$ then $w_2 w_1 = 1$. Thus we can cyclically order the $\{f_i\}$ in the above expression (or any expression for 1 in general).

Let $r = \frac{k}{2}$. We can rewrite the relation as

$$f_1 f_2 \dots f_{r+1} = f_l f_{k-1} \dots f_{r+2}$$

The left side is not reduced since it has a shorter expression on the right. Thus we can find some $1 \leq q < p \leq r+1$ such that

$$f_q f_{q+1} \dots f_{p-1} f_p = f_{q+1} f_{q+2} \dots f_{p-2} f_{p-1}$$

by the deletion condition, i.e.

$$f_q f_{q+1} \dots f_{p-1} f_p f_{p-1} f_{p-2} \dots f_{q+1} = 1$$

If this has length less than k, it would be a consequence of the Coxeter relations (and thus we would have a way to shorten the expression $f_1 f_2 \dots f_k = 1$ with Coxeter relation), which is impossible. Thus the length must be k, which is only possible if $q = 1$, $p = r+1$. Thus we must have

$$f_1 f_2 \dots f_r f_{r+1} f_r f_{r-1} \dots f_3 f_2 = 1$$

and this relation must not be a consequence of (nor shortenable by) the Coxeter relations. This shows the following:

Lemma 12.1. *Deletion Rewrite Lemma*

Suppose $f_1 f_2 \dots f_k = 1$ and cannot be shortened by the Coxeter relations.

Then $f_1 f_2 \dots f_r f_{r+1} f_r f_{r-1} \dots f_3 f_2 = 1$ and also cannot be shortened by the Coxeter relations.

Now we cyclically permute our initial relation to conclude $f_2 f_2 \dots f_k f_1 = 1$. Applying the Deletion Rewrite Lemma to this shows that

$$f_2 f_3 \dots f_{r+1} f_{r+2} f_{r+1} \dots f_4 f_3 = 1$$

cannot be shortened, which we can cyclicly permute to conclude

$$f_3 f_2 f_3 f_4 \dots f_{r+1} f_{r+2} f_{r+1} \dots f_4 = 1$$

cannot be shortened.

Now we apply the Deletion Rewrite Lemma to this to conclude

$$f_3 f_2 f_3 \dots f_r f_{r+1} f_r \dots f_3 f_2 = 1$$

But we know from our proof of the Deletion Rewrite Lemma that

$$f_1 f_2 f_3 \dots f_r f_{r+1} f_r \dots f_3 f_2 = 1$$

Thus $f_1 = f_3$.

This gives us:

Lemma 12.2. *Suppose $f_1 f_2 \dots f_k$ cannot be shortened by the Coxeter relations.*
Then $f_1 = f_3$

By applying lemma to each $f_i f_{i+1} \dots f_k f_1 f_2 \dots f_{i-1} = 1$, we conclude that $f_i = f_{i+2}$ for all i.
Thus this relation is really just

$$(f_1 f_2)^k = 1$$

which is of course a consequence of the Coxeter relations. Thus all relations are consequences of the Coxeter relations, so W is a Coxeter group.