Schur's Lemma and Beyond

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Backstory

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Let V be an absolutely irreducible and finite-dimensional representation of a group G over a field k. If there is a nonzero quadratic form q on V that is invariant under G, then by Schur's Lemma q is uniquely determined up to multiplication by an element of $k^\times$.

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- B Suppose k is algebraically closed, $V = V'$, and $\rho = \rho'$. Then φ is a scalar multiple of the identity.

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Quadratic Forms

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- So $q = \lambda q'$.

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- In particular, abelian categories

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- $\mathsf{Rep}_{\mathbb{C}}(G)$, category of complex representations of G

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Not the full definition!

Kernels in an Abelian Category

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- Since $\varphi \neq 0 \in \mathrm{Hom}(A, B)$, we have $\ker\varphi = 0$ and $\operatorname{im}\varphi = B$
- Using some more properties of abelian categories, we can conclude that φ is an isomorphism.

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So A determines an element in a Brauer group!

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- Stable sheaves are simple objects in $\mathcal{C}(p)$

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- **Corollary**. If E is a stable sheaf, then $\mathrm{End}(E)$ is a division algebra over k.

References

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