Schur's Lemma and Beyond

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Backstory

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Let V be an absolutely irreducible and finite-dimensional representation of a group G over a field k. If there is a nonzero quadratic form q on V that is invariant under G, then by Schur's Lemma q is uniquely determined up to multiplication by an element of $k^\times$.

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Schur's Lemma for Group Representations

$\varphi : V \to V'$ linear

$g.\varphi(v) = \varphi(g.v) \ \forall g \in G, v \in V$

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A. Either φ is the zero map, or φ is an isomorphism of representations.

B. Suppose k is algebraically closed, $V = V'$, and $\rho = \rho'$. Then φ is a scalar multiple of the identity.

$\varphi : V \to V$

$\varphi = \lambda I = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$ for some $\lambda \in k$.

Proof of Schur's Lemma

(a) Either $\varphi \colon \overset{\text{irr.}}{V} \to \overset{\text{irr.}}{V'}$ is the zero map, or φ is an isomorphism of representations.

V irred. $\Rightarrow$ only subreps of V are 0 and V

$\ker \varphi \subseteq V$ is a subrep.

$\cdot\ \varphi(v) = 0 \Rightarrow \varphi(g \cdot v) = g \cdot \varphi(v) = g \cdot 0 = 0.$

$\operatorname{im} \varphi \subseteq V'$ is a subrep.

Proof of Schur's Lemma

(a) Either $\varphi\colon V \to V'$ is the zero map, or φ is an isomorphism of representations.

$\ker \varphi \leq V \Rightarrow \ker \varphi = 0$ or $\cancel{V}$

$\operatorname{im} \varphi \leq V' \Rightarrow \operatorname{im} \varphi = \cancel{0}$ or V'.

Assume $\varphi \neq 0$.

So $\ker \varphi = 0$, $\operatorname{im} \varphi = V' \Rightarrow \varphi$ is an isom.

Proof of Schur's Lemma

(b) Suppose k is algebraically closed, $V = V'$, and $\rho = \rho'$. Then φ is a scalar multiple of the identity. $\varphi : V \to V$

Let λ be an eigenvalue for φ $(k = \bar{k})$

Then $\varphi - \lambda I$ is a hom'm of G-reps:

$$(\varphi - \lambda I)(g \cdot v) = \varphi(g \cdot v) - \lambda(g \cdot v)$$
$$= g \cdot \varphi(v) - g \cdot (\lambda \cdot v) = g \cdot (\varphi - \lambda I)(v).$$

Schur's lemma (a) $\Rightarrow$ $\varphi - \lambda I$ is an ~~iso.~~ or 0

$\Rightarrow$ $\varphi = \lambda I$.

Quadratic Forms

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$g \cdot (v \otimes \alpha) = g \cdot v \otimes \alpha$

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$q(v) = b(v,v)$ $\quad$ $q'(v) = b'(v,v)$

Claim: If $q' \neq 0$ is invariant then $q = \lambda q'$ for some $\lambda \in k^\times$.

- Construct a homomorphism of G-representations $\varphi\colon V \to V$ such that $b(v,-) = b'(\varphi(v),-)$ for all $v \in V$.

$b(v,w) = b'(\varphi(v),w) \quad \forall\, w \in V.$

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- Schur's lemma (b) $\implies$ $\varphi \otimes_k \mathrm{id}_{\overline{k}} = \lambda I$ for some $\lambda \in \overline{k}$.

$$V \otimes_k \overline{k} \to V \otimes_k \overline{k}$$
$$v \otimes \alpha \mapsto \varphi(v) \otimes \alpha$$

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- Then $\lambda \in k$ and $\varphi = \lambda I$.
- So $q = \lambda q'$.

$$q(v) = b(v,v) = b'(\varphi(v), v)$$
$$= b'(\lambda v, v) = \lambda b'(v,v)$$
$$= \lambda q(v).$$

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- In particular, abelian categories

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- $\mathsf{Shv}_{\mathsf{Ab}}(X)$, category of sheaves of abelian groups on X
- $\mathsf{Rep}_{\mathbb{C}}(G)$, category of complex representations of G

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- $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is an abelian group for any two objects A, B (and composition distributes over addition) $(f+f')\circ g = f\circ g + f'\circ g$

  $h\circ(f+f') = h\circ f + h\circ f'$

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Not the full definition!

Kernels in an Abelian Category

$\mathcal{A}$, objects A, B, morphism $A \xrightarrow{f} B$

The kernel of f is $K \xrightarrow{i} A$ satisfying the following univ. prop:

$$\begin{tikzcd}
 & K' \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "i'"'] \arrow[ddr, bend left, "0 \in \mathrm{Hom}_{\mathcal{A}}(K', B)"] & \\
 & K \arrow[dl, "i"'] \arrow[dr, "0"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}$$

Schur's Lemma in an Abelian Category

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V irrep V'

Schur's Lemma. Let A and B be simple objects in an abelian category $\mathcal{A}$. Then any nonzero element $\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ is an isomorphism.

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- Since $\varphi \neq 0 \in \mathrm{Hom}(A, B)$, we have $\ker\varphi = 0$ and $\operatorname{im}\varphi = B$
- Using some more properties of abelian categories, we can conclude that φ is an isomorphism.

Aside: Division Rings

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Div ring D : $\forall d \in D \; \exists d^{-1} \in D$

s.t. $dd^{-1} = d^{-1}d = 1$

Corollary. If A is a simple object in an abelian category, then $\mathrm{End}(A) = \mathrm{Hom}(A, A)$ is a division ring.

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So A determines an element in a Brauer group!

Example: Stable Sheaves

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See Huybrechts and Lehn, *The Geometry of Moduli Spaces of Sheaves*. **(Don't rely on these details!)**

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- E a coherent sheaf on X, $\dim E = \dim X$

$$\dim \underbrace{\mathrm{Supp}(E)}_{\{x \in X \mid E_x \neq 0\}}$$

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- E a coherent sheaf on X, $\dim E = \dim X$
- Can define the reduced Hilbert polynomial $p(E)$
- E is *semi-stable* if E is pure and $p(F) \leq p(E)$ for any proper subsheaf $F \subseteq E$

$x^3+2x+1 \quad \leq x^3+3x$

Example: Stable Sheaves

$X = \operatorname{Spec} A$

$\uparrow$

$(X, \mathcal{O}_X)$

$\mathcal{F}$ sheaf on X

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- Subcategory $\mathcal{C}(p)$ of $\mathrm{Coh}(X)$, of semi-stable sheaves with reduced Hilbert polynomial p, is abelian

E semistable
$p(E) = p$

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- Category $\mathrm{Coh}(X)$ of coherent sheaves on X is abelian
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- Stable sheaves are simple objects in $\mathcal{C}(p)$

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- **Proposition**. If F, G are stable sheaves and $p(F) = p(G)$, then any non-trivial homomorphism $f\colon F \to G$ is an isomorphism.

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- **Proposition**. If F, G are stable sheaves and $p(F) = p(G)$, then any non-trivial homomorphism $f\colon F \to G$ is an isomorphism.

- **Corollary**. If E is a stable sheaf, then $\mathrm{End}(E)$ is a division algebra over k.

References

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