**Connections in Type A Representation Theory**

\[ \text{reps of } \GL_n(C) \rightarrow \text{reps of } S_k \]

\[ \text{Schur functions } \rightarrow \text{polynomials} \]

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**Symmetric Functions**

(See Macdonald, Symmetric Function and Hall Polynomials.)

A polynomial \( f(x_1, \ldots, x_n) \) is symmetric if it is unchanged when you swap \( x_i, x_j \).

Example: \( x_1^2 + x_2^2 + x_3^2 \).

Symmetric functions are "symmetric polynomials in infinitely many variables." Formally, let \( \Lambda_n \) be symmetric polynomials in \( n \) variables. For \( m \geq n \), define a map

\[ \Lambda_m \rightarrow \Lambda_n \]

\[ f(x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_n, 0). \]

We have, for \( r \leq m \leq n \),

\[ \Lambda_r \leq \Lambda_m \leq \Lambda_n. \]

Define the inverse limit \( \Lambda = \varprojlim \Lambda_n \).
to be the space of symmetric functions. Elements are sequences \((f_n)_{n \geq 1}\) with 
\[ f_m(x_1, \ldots, x_n, 0, \ldots, 0) = f_n(x_1, \ldots, x_n). \quad (m \geq n) \]

Example: \(p_r(x) = x_1^r + x_2^r + \cdots\) can be thought of as the sequence \((x_1^r, x_1^r + x_2^r, x_1^r + x_2^r + x_3^r, \ldots)\).

A partition is an infinite sequence \(\lambda = (\lambda_1, \lambda_2, \ldots)\) of nonnegative integers s.t. \(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots\) and all but finitely many \(\lambda_i\) are 0.

The length of \(\lambda\), \(l(\lambda)\), is the number of nonzero parts \(\lambda_i\).

Example: \(\lambda = (3, 3, 0, 0, \ldots) = (3, 1, 0) = (3, 1, 0), \) etc., \(l(\lambda) = 2\). ^ variant notation

Given \(\lambda\) with \(l(\lambda) = k\), define \(x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}\) and let the monomial symmetric function be

\(m_\lambda(x) = \text{all permutations of } x^\lambda\)

Example: \(m_{(3, 1)}(x_1, x_2) = x_1^3 x_2 + x_1 x_2^3\)

\(m_{(3, 1)}(x_1) = m_{(3, 1)}(x_1, 0) = 0.\)

Note that if \(l(\lambda) > n\), then \(m_\lambda(x_1, \ldots, x_n) = 0.\)

Lemma: \(\{m_\lambda : \lambda \text{ a partition}\}\) is a basis for \(\Lambda.\)
Want to define Schur polynomials. First, for \( \lambda \) with \( \ell(\lambda) \leq n \), define

\[
\alpha_{\lambda}(x_1, \ldots, x_n) = \det \left( x_i^{\lambda_j + (n-j)} \right) = \det \begin{pmatrix}
X_1^1 & X_2^{1+1} & \cdots & X_n^{1+n-1} \\
X_1^{2+1} & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
X_1^{n+1} & \cdots & \cdots & X_n^{n+n-1}
\end{pmatrix}
\]

\( \alpha_{\lambda} \) is antisymmetric: \( x_i \leftrightarrow x_j \) is a row swap.
So it is divisible by \( (x_i - x_j) \). Then the Schur polynomial

\[
S_{\lambda}(x_1, \ldots, x_n) = \frac{\alpha_{\lambda}(x_1, \ldots, x_n)}{\prod_{i<j} (x_i - x_j)}
\]

is a polynomial.

Numerator and denominator are antisymmetric, so \( S_{\lambda} \) is symmetric.

Thm: \( S_{\lambda} \) extends to a symmetric function if we define,
for \( n < \ell(\lambda) \), \( S_{\lambda}(x_1, \ldots, x_n, 0) = 0 \).

Proof: Must show \( S_{\lambda}(x_1, \ldots, x_n, 0) = S_{\lambda}(x_1, \ldots, x_n) \).
First let \( \ell(\lambda) \leq n \). Then \( x_{n+1} = 0 \).

\[
\alpha_{\lambda}(x_1, \ldots, x_{n+1}) = \det \begin{pmatrix}
X_1^{1+1} & X_2^{1+2} & \cdots & X_n^{1+n} & X_{n+1}^{1+n} \\
X_1^{2+2} & \ddots & \cdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
X_1^{n+n} & \cdots & \cdots & X_n^{n+n} & X_{n+1}^{n+n} \\
X_1^{n+n} & \cdots & \cdots & \cdots & X_{n+1}^{n+n}
\end{pmatrix}
\]

\[
\alpha_{\lambda}(x_1, \ldots, x_n, 0) = \det \begin{pmatrix}
X_1^{1+1} & X_2^{1+2} & \cdots & X_{n-1}^{1+n} & X_n^{1+n} \\
X_1^{2+2} & \ddots & \cdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
X_1^{n+n} & \cdots & \cdots & X_{n-1}^{n+n} & X_n^{n+n} \\
0 & \cdots & \cdots & 0 & 0
\end{pmatrix}
\]
\[
\det \left( \begin{array}{cccc}
X_1^{\lambda_1(n-1)} & \lambda_2(n-2)^{\lambda_1} & \cdots & \lambda_n^{n-1} \\
\vdots & \ddots & \ddots & \vdots \\
X_n^{\lambda_1(n-1)} & \lambda_2(n-2)^{\lambda_1} & \cdots & \lambda_n^{n-1} 
\end{array} \right) = x_1 \cdots x_n \cdot a_\lambda(x_1, \ldots, x_n).
\]

Similarly, \( \prod_{1 \leq i < j \leq n+1} (x_i - x_j) \bigg|_{x_{n+1} = 0} = x_1 \cdots x_n \cdot \prod_{1 \leq i < j \leq n} (x_i - x_j) \).
So \( s_\lambda(x_1, \ldots, x_n, 0) = s_\lambda(x_1, \ldots, x_n) \).

If instead \( \lambda(1) > n \), then \( \lambda_1, \ldots, \lambda_n > 0 \) and setting \( x_{n+1} = 0 \) gives you a zero row. \( \square \)

Nice Theorems:
1) \( \exists s_\lambda \text{ s.t. } \lambda \text{ a partition} \) is an orthonormal basis for \( \Lambda \).

   In fact, 
   \[ s_\lambda = m_\lambda + \sum_{\mu \neq \lambda} d_{\lambda \mu} \cdot m_\mu \]
   where \( < \) is dominance ordering.

2) \( s_\lambda = \Xi x^T \) where \( T \) runs over all semi-standard Young tableaux of shape \( \lambda \).

3) Littlewood–Richardson Rule: Define \( c_{\lambda\mu}^\nu \) by 
   \[ s_\lambda s_\mu = \sum c_{\lambda\mu}^\nu s_\nu. \]
   There is a nice combinatorial formula for \( c_{\lambda\mu}^\nu \).

   (See Macdonald.)
Representations of $\text{GL}_n$

All vector spaces will be finite-dimensional and complex.

$\text{GL}(V) =$ invertible linear transformations of $V$.

$\text{GL}_n = \text{GL}_n(\mathbb{C}) = GL(\mathbb{C}^n) =$ $n \times n$ matrices.

A representation (rep) of a group $G$ is a pair $(\rho, V)$: a vector space $V$ and a homomorphism

$\rho : G \to \text{GL}(V)$.

Ex: $G = \text{GL}_n$, $V = \mathbb{C}^n$, $\rho = \text{id}$.
Matrix multiplication.

Ex: $V = \mathbb{C}$, $\rho : G = \text{GL}_n \to \mathbb{C}^n = \text{GL}(\mathbb{C})$ by

$\rho(A) = |\det A|^\text{c}$.

We will only work with polynomial reps: we assume

$\rho(A) = (f_{ij}(A))$, where each $f_{ij}$ is a polynomial in the entries of $A$.

Ex: $V = \mathbb{C}^n$, $\rho = \text{id}$.

Ex: $V = \mathbb{C}$, $\rho(A) = |\det A|^k$, $k \in \mathbb{Z} \geq 0$. 
Given a rep $\rho$, its character is the function $X_\rho: G \rightarrow \mathbb{C}$ defined by $X_\rho(g) = \text{Tr}(\rho(g))$.

**Ex.** For $V = \mathbb{C}^n$, $\rho = \text{id}$, $X_\rho(A) = \text{Tr}A$.

**Thm.** Reps of $\text{GL}_n$ are uniquely determined by their characters.

**Note:** $X_\rho$ is conjugation-invariant, since $\text{Tr}$ is.

$$X_\rho(BAB^{-1}) = \text{Tr}(\rho(BAB^{-1})) = \text{Tr}(\rho(B)\rho(A)\rho(B)^{-1}) = \text{Tr}\rho(A) = X_\rho(A).$$

**Note 2:** Since $X_\rho$ is a polynomial, it is determined by its values on the dense set of diagonalizable matrices.

By previous note, $X_\rho$ is determined by its values on diagonal matrices. Write

$$X_\rho(x_1, \ldots, x_n) = X_\rho\left(\begin{array}{c} x_1 \\ \vdots \\ 0 \\ \vdots \\ x_n \end{array}\right).$$

**Ex.** $V = \mathbb{C}^n$, $\rho = \text{id}$.

$$X_\rho(x_1, \ldots, x_n) = \text{Tr}\left(\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array}\right) = x_1 + \ldots + x_n.$$
Note: $S_n$ permutes $x_1, ..., x_n$ by conjugation by permutation matrices.

By conjugation-invariance, $X \rho$ is a symmetric polynomial, (called an *irrep*)

A rep $(\rho, V)$ is irreducible if there does not exist a subspace $0 \subset W \subset V$ s.t.

$\forall g \in G, \rho(g)W \subset W.$

**Thm:**

$(\text{polynomial}) \leftrightarrow (\text{irrep of } GL_n) \leftrightarrow (\text{partition } \lambda : \ell(\lambda) \leq n).

Call the irrep corresponding to $\lambda$ $(\rho_\lambda, V(\lambda))$.

**Thm:** $X^{\rho_\lambda} = S_\lambda(x_1, ..., x_n)$.

$(\text{polynomial}) \leftrightarrow (\text{irrep of } GL_n) \leftrightarrow \Lambda_n.$
Corollary: Recall \( x_{\rho_1, \rho_2} = x_{\rho_1} + x_{\rho_2} \), \( x_{\rho_1, \rho_2} = x_{\rho_1} x_{\rho_2} \). Then

\[
S_1 S_n = \sum C_{d_1 \cdots d_n} S_{d_1} \quad \text{implies}
\]

\[
\rho_1 \otimes \rho_n = \bigoplus \rho_{d_1} \otimes \cdots \otimes \rho_{d_n}.
\]

Reps of \( S_k \)

(Warning: \( n \) and \( k \) will be different numbers. I always mean \( S_k \), even if I say \( S_n \).)

Theorem: \((\text{irreps of } S_k) \leftrightarrow (\text{partitions } \lambda : \ell(\lambda) = 3 \ell(\lambda) = k)\).

We have the characteristic map

\[
\chi: (\text{reps of } S_k) \rightarrow \Lambda \quad \text{by}
\]

\[
\chi_\lambda: 1 \rightarrow S_1
\]

\( S_k \)-rep \hspace{2cm} \text{schur function}

Preserves inner products and a nice multiplication operation.
Fix $n, k$. Let $L_{n,k} = \{ \lambda : \ell(\lambda) \leq n, |\lambda| = nk \}$. We have nice bijections:

\[
\begin{align*}
( \text{irreps } \nu_{\lambda} & \quad \text{of } S_k : \lambda \in L_{n,k} ) \xrightarrow{\text{Ch}} \left( \sigma_{\lambda} \in \Lambda : \lambda \in L_{n,k} \right) \\
( \text{irreps } V(\lambda) & \quad \text{of } \text{GL}_n : \lambda \in L_{n,k} ) \xleftarrow{\text{character}} \left( \sigma_{\lambda}(x, \ldots, x_n) \in \Lambda_n : \lambda \in L_{n,k} \right)
\end{align*}
\]

Composing, we find a correspondence between reps of $S_k$ and $\text{GL}_n$!

Can construct this map explicitly. Let

\[ W = (\mathbb{C}^n)^{\otimes k} = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n. \]

Get a $\text{GL}_n$ rep, $\rho(A) = A \otimes \cdots \otimes A$.

Also an $S_k$ rep, permuting the tensor factors.

These actions commute, we have
Proposition:
\[ V(\lambda) = \text{Hom}_{S_k}(V_\lambda, W). \]

Follow from

\underline{Schur-Weyl Duality:}

\[ \text{End}_{S_k}(W) \text{ is generated by } \rho(\text{GL}_n) \]
and

\[ \text{End}_{\text{GL}_n}(W) \text{ is generated by the } S_k\text{-action}. \]

Get the decomposition

\[ W = \bigoplus_{\lambda: \ell(\lambda) \leq n, \ell(\lambda) = k} V_\lambda \otimes V(\lambda). \]

Sketch of Proposition:

Consider \( W_\lambda = \text{Hom}_{S_k}(V_\lambda, W) \).

This is a rep \((\Psi, W_\lambda)\) of \( \text{GL}_n \) with

\[ (\Psi(A) f)(v) = \rho(A) f(v) \quad (f \in W_\lambda, v \in V_\lambda). \]

Lemma: \( W_\lambda \) is an irrep for \( \text{End}_{S_k}(W) \).

\( \text{SW Duality} \Rightarrow \text{Lemma} \Rightarrow W_\lambda \) is an irrep for \( \text{GL}_n \), since \( \rho(\text{GL}_n) \) generates \( \text{End}_{S_k}(W) \).
Proof of Lemma: Fix nonzero $u \in V_\lambda$. Let $E = \text{End}_{S_k}(W)$. V_\lambda \text{ irrep } \Rightarrow F \circ E \text{ is determined by } F(u) = w \in W.

Given arbitrary $w' \in W$ and nonzero $F$, show that we can take $F$ to an endomorphism with $u_1 \mapsto w'$.

By Maschke’s Thm,

$W = Ew \otimes Y$ as $S_k$-modules.

Define $T$ on $W$ by

$T(gw) = gw' \ (g \in E), \ T|_Y = \text{id}.$

$(T \circ F)(u) = T(F(u)) = w'$. \hfill \square$