Connections in Type A Representation Theory

$$\begin{tikzcd}
\text{reps of } GL_n(\mathbb{C}) \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{reps of } S_k \arrow[dl, leftrightarrow] \\
& \text{Schur functions/polynomials} &
\end{tikzcd}$$

Symmetric Functions (See Macdonald, Symmetric functions and Hall polynomials.)

A polynomial $f(x_1, \ldots, x_n)$ is symmetric if it is unchanged when you swap x_i, x_j.

Example: $x_1^2 + x_2^2 + x_3^2$.

Symmetric functions are "symmetric polynomials in infinitely many variables." Formally, let

Λ_n = symmetric polynomials in n variables.

For $m \geq n$, define a map

$$\begin{aligned} \Lambda_m &\longrightarrow \Lambda_n \\ f(x_1, \ldots, x_m) &\longmapsto f(x_1, \ldots, x_n, 0). \end{aligned}$$

We have, for $r \geq m \geq n$,

$$\begin{tikzcd}
\Lambda_r \arrow[d] \arrow[dr] & \\
\Lambda_m \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \Lambda_n
\end{tikzcd}$$

. Define the inverse limit $\Lambda = \varprojlim \Lambda_n$

to be the space of symmetric functions.

Elements are sequences $(f_n)_{n\geq 1}$ with

$f_m(x_1,\dots,x_n,0,\dots,0) = f_n(x_1,\dots,x_n)$. $(m \geq n)$

<u>Ex</u>: $p_r(x) = x_1^r + x_2^r + \dots$ can be thought of as the sequence $(x_1^r,\ x_1^r + x_2^r,\ x_1^r + x_2^r + x_3^r, \dots)$.

A <u>partition</u> is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers s.t. $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ and all but finitely many λ_i are 0.

The <u>length</u> of λ, $\ell(\lambda)$, is the number of nonzero parts λ_i.

<u>Ex</u>: $\lambda = (3,1,0,0,\dots) = (3,1) = (3,1,0)$, etc. $\ell(\lambda) = 2$.

variant notation

Given λ with $\ell(\lambda) = k$, define $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ and let the <u>monomial symmetric function</u> be

$m_\lambda(x) =$ all permutations of x^λ

<u>Ex</u>: $m_{(3,1)}(x_1, x_2) = x_1^3 x_2 + x_1 x_2^3$

$m_{(3,1)}(x_1) = m_{(3,1)}(x_1, 0) = 0.$

Note that if $\ell(\lambda) > n$, then $m_\lambda(x_1,\dots,x_n) = 0$.

<u>Lemma</u>: $\{m_\lambda : \lambda \text{ a partition}\}$ is a basis for Λ.

Want to define Schur polynomials. First, for λ with $\ell(\lambda) \le n$, define

$$a_\lambda(x_1,\dots,x_n) = \det\left(x_i^{\lambda_j+(n-j)}\right) = \det\begin{pmatrix} x_1^{\lambda_1+(n-1)} & x_1^{\lambda_2+(n-2)} & \cdots & x_n^{\lambda_n+0} \\ \vdots & & & \\ x_n^{\lambda_1+(n-1)} & x_n^{\lambda_2+(n-2)} & \cdots & x_n^{\lambda_n} \end{pmatrix}.$$

a_λ is antisymmetric: $x_i \longleftrightarrow x_j$ is a row swap.

So it is divisible by $(x_i - x_j)$. Then the <u>Schur polynomial</u>

$$s_\lambda(x_1,\dots,x_n) = \frac{a_\lambda(x_1,\dots,x_n)}{\prod_{i<j}(x_i - x_j)} \quad \text{is a polynomial.}$$

Numerator and denominator are antisymmetric, so s_λ is symmetric.

<u>Thm</u>: s_λ extends to a symmetric function if we define, for $n < \ell(\lambda)$, $s_\lambda(x_1,\dots,x_n) = 0$.

<u>Proof</u>: Must show $s_\lambda(x_1,\dots,x_n,0) = s_\lambda(x_1,\dots,x_n)$.

First let $\ell(\lambda) \le n$. Then $\lambda_{n+1} = 0$.

$x_1^{\lambda_{n+1}} = x_1^0 = 1$

$$a_\lambda(x_1,\dots,x_{n+1}) = \det\begin{pmatrix} x_1^{\lambda_1+(n-1)+1} & x_1^{\lambda_2+(n-2)+1} & \cdots & x_1^{\lambda_n+1} & 1 \\ \vdots & & & & \vdots \\ x_n^{\lambda_1+(n-1)+1} & x_n^{\lambda_2+(n-2)+1} & \cdots & x_n^{\lambda_n+1} & \vdots \\ x_{n+1}^{\lambda_1+(n-1)+1} & x_{n+1}^{\lambda_2+(n-2)+1} & \cdots & x_{n+1}^{\lambda_n+1} & 1 \end{pmatrix}$$

$$a_\lambda(x_1,\dots,x_n,0) = \det\begin{pmatrix} x_1^{\lambda_1+(n-1)+1} & x_1^{\lambda_2+(n-2)+1} & \cdots & x_1^{\lambda_n+1} & 1 \\ \vdots & & & & \vdots \\ x_n^{\lambda_1+(n-1)+1} & x_n^{\lambda_2+(n-2)+1} & \cdots & x_n^{\lambda_n+1} & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} x_1^{\lambda_1+(n-1)+1} & x_1^{\lambda_2+(n-2)+1} & \cdots & x_1^{\lambda_n+1} \\ \vdots & & & \\ x_n^{\lambda_1+(n-1)+1} & x_n^{\lambda_2+(n-2)+1} & \cdots & x_n^{\lambda_n+1} \end{pmatrix} = x_1 \cdots x_n \cdot a_\lambda(x_1, \dots, x_n).$$

Similarly $\prod_{1 \le i < j \le n+1} (x_i - x_j) \Big|_{x_{n+1}=0} = x_1 \cdots x_n \cdot \prod_{1 \le i < j \le n} (x_i - x_j)$.

So $s_\lambda(x_1, \dots, x_n, 0) = s_\lambda(x_1, \dots, x_n)$.

If instead $\ell(\lambda) > n$, then $\lambda_1, \dots, \lambda_n > 0$ and setting $x_{n+1} = 0$ gives you a zero row. $\square$

Nice Theorems:

1) $\{ s_\lambda : \lambda \text{ a partition} \}$ is an orthonormal basis for Λ.
   In fact,
   $$s_\lambda = m_\lambda + \sum_{\mu < \lambda} d_\lambda^\mu m_\mu$$
   where $<$ is dominance ordering.
2) $s_\lambda = \sum_T x^T$, where T runs over all semistandard Young tableaux of shape λ.
3) Littlewood-Richardson Rule: Define $c_{\lambda\mu}^\nu$ by
   $$s_\lambda s_\mu = \sum c_{\lambda\mu}^\nu s_\nu .$$
   There is a nice combinatorial formula for $c_{\lambda\mu}^\nu$.

(See Macdonald.)

Representations of GL_n

All vector spaces will be finite-dimensional and complex.

$GL(V)$ = invertible linear transformations of V.

$GL_n = GL_n(\mathbb{C}) = GL(\mathbb{C}^n)$ = $n \times n$ matrices.

A representation (rep) of a group G is a pair (ρ, V): a vector space V and a homomorphism

$$\rho: G \to GL(V).$$

Ex: $G = GL_n$, $V = \mathbb{C}^n$, $\rho = id$.
Matrix multiplication.

Ex: $V = \mathbb{C}$, $r \in \mathbb{R}$, define $\rho: GL_n \to \mathbb{C}^* = GL(\mathbb{C})$ by
$\rho(A) = |\det A|^r$.

We will only work with polynomial reps: we assume

$$\rho(A) = \big(f_{ij}(A)\big),$$ where each f_{ij} is a polynomial in the entries of A.

Ex: $V = \mathbb{C}^n$, $\rho = id$.

Ex: $V = \mathbb{C}$, $\rho(A) = (\det A)^k$, $k \in \mathbb{Z}_{\geq 0}$.

Given a rep ρ, its character is the function

$$\chi_\rho : G \longrightarrow \mathbb{C} \quad \text{defined by}$$

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)).$$

Ex: For $V = \mathbb{C}^n$, $\rho = id$, $\chi_\rho(A) = \mathrm{Tr}\, A$.

Thm: Reps of GL_n are uniquely determined by their characters.

Note: χ_ρ is conjugation-invariant, since Tr is.

$$\chi_\rho(BAB^{-1}) = \mathrm{Tr}\left(\rho(BAB^{-1})\right) = \mathrm{Tr}\left(\rho(B)\rho(A)\rho(B)^{-1}\right)$$
$$= \mathrm{Tr}\, \rho(A) = \chi_\rho(A).$$

Note 2: Since χ_ρ is a polynomial, it is determined by its values on the dense set of diagonalizable matrices.

By previous note, χ_ρ is determined by its values on diagonal matrices. Write

$$\chi_\rho(x_1, \dots, x_n) = \chi_\rho \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}.$$

Ex: $V = \mathbb{C}^n$, $\rho = id$.

$$\chi_{id}(x_1, \dots, x_n) = \mathrm{Tr}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = x_1 + \dots + x_n.$$

Note: S_n permutes $x_1, \dots, x_n$ by conjugation by permutation matrices.

By conjugation-invariance, χ_ρ is a symmetric polynomial.

A rep (ρ, V) is <u>irreducible</u> (called an <u>irrep</u>) if there does not exist a subspace $0 \subset W \subset V$ s.t. $\forall g \in G$, $\rho(g) W \subseteq W$.

Thm:

$$\begin{pmatrix} \text{polynomial} \\ \text{irreps of } GL_n \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{partitions } \lambda: \\ \ell(\lambda) \leq n \end{pmatrix}.$$

Call the irrep corresponding to λ $(\rho_\lambda, V(\lambda))$.

Thm: $\chi_{\rho_\lambda} = s_\lambda(x_1, \dots, x_n)$.

$$\begin{pmatrix} \text{polynomial} \\ \text{reps of } GL_n \end{pmatrix} \longleftrightarrow \Lambda_n$$

Corollary: Recall $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$, $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}$. Then

$$s_\lambda s_\mu = \sum c_{\lambda\mu}^{\nu} s_\nu \quad \text{implies}$$

$$\rho_\lambda \otimes \rho_\mu = \bigoplus \rho_\nu^{\oplus c_{\lambda\mu}^{\nu}}.$$

Reps of S_k

(Warning: n and k will be different numbers. I always mean S_k, even if I say S_n.)

Thm: $\begin{pmatrix} \text{irreps of} \\ S_k \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{partitions } \lambda: \\ |\lambda| = \sum \lambda_i = k \end{pmatrix}$.

We have the characteristic map

$$ch: \begin{pmatrix} \text{reps of} \\ S_k \end{pmatrix} \longrightarrow \Lambda \quad \text{by}$$

$$\underset{S_k\text{-rep}}{V_\lambda} \longmapsto \underset{\text{schur function}}{s_\lambda}$$

Preserves inner products and a nice multiplication operation.

Cool Consequence

Fix n, k. Let $L_{n,k} = \{\lambda : \ell(\lambda) \leq n,\ |\lambda| = \sum \lambda_i = k\}$.

We have nice bijections

$$\begin{tikzcd}
\left(\begin{matrix} \text{irreps } V_\lambda \\ \text{of } S_k : \lambda \in L_{n,k} \end{matrix}\right) \arrow[r, "\mathrm{ch}"] & \left(\begin{matrix} s_\lambda \in \Lambda : \\ \lambda \in L_{n,k} \end{matrix}\right) \arrow[d, "f \mapsto f(x_1, \dots, x_n, 0, 0, \dots)"] \\
\left(\begin{matrix} \text{irreps } V(\lambda) \\ \text{of } GL_n : \lambda \in L_{n,k} \end{matrix}\right) & \left(\begin{matrix} s_\lambda(x_1, \dots, x_n) \in \Lambda_n : \\ \lambda \in L_{n,k} \end{matrix}\right) \arrow[l, "\text{character}"']
\end{tikzcd}$$

Composing, we find a correspondence between reps of S_k and GL_n!

Can construct this map explicitly. Let

$$W = (\mathbb{C}^n)^{\otimes k} = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n.$$

Get a GL_n rep, $\rho(A) = A \otimes \cdots \otimes A$.

Also an S_k rep, permuting the tensor factors.

These actions commute. We have

Proposition:

$$V(\lambda) = \mathrm{Hom}_{S_k}(V_\lambda, W).$$

GL_n irrep $\qquad\qquad$ S_k irrep

Follows from

Schur-Weyl Duality:

$\mathrm{End}_{S_k}(W)$ is generated by $\rho(GL_n)$

and

$\mathrm{End}_{GL_n}(W)$ is generated by the S_k-action.

Get the decomposition

$$W = \bigoplus_{\substack{\lambda : \ell(\lambda) \leq n, \\ |\lambda| = k}} V_\lambda \otimes V(\lambda).$$

Sketch of Proposition:

Consider $W_\lambda = \mathrm{Hom}_{S_k}(V_\lambda, W)$.

This is a rep (φ, W_λ) of GL_n with

$$(\varphi(A)f)(v) = \rho(A) f(v) \qquad (f \in W_\lambda,\ v \in V_\lambda).$$

Lemma: W_λ is an irrep for $\mathrm{End}_{S_k}(W)$.

SW Duality + Lemma $\Rightarrow$ W_λ is an irrep for GL_n, since $\rho(GL_n)$ generates $\mathrm{End}_{S_k}(W)$. $\square$

<u>Proof of Lemma:</u> Fix nonzero $u \in V_\lambda$. Let $E = \mathrm{End}_{S_k}(W)$.

V_λ irrep $\Rightarrow$ $f \in E$ is determined by $f(u) = w \in W$.

Given arbitrary $w' \in W$ and nonzero f, show that we can take f to an endomorphism with $u \mapsto w'$.

By Maschke's Thm,

$$W = Ew \oplus Y \text{ as } S_k\text{-modules.}$$

Define T on W by

$$T(gw) = gw' \quad (g \in E), \quad T|_Y = \mathrm{id}.$$

$$(T \circ f)(u) = T(w) = w'. \qquad \square$$