Tokuyama's formula
Combinatorics of the deformed Schur polynomials.

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Based on my recent post on Thuses.com, website for math discussions.

1. The Schur polynomial
2. The deformed Schur polynomial and the Tokuyama formula
3. Big picture and Open problems.

① The Schur polynomials.

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, we define polynomials

$s_\lambda(z_1, \dots, z_n)$ as $\dfrac{a_{\lambda+\rho}}{a_\rho}$, where

$a_\mu(z_1, \dots, z_n) = \det\left(z_i^{\mu_j}\right)$, explicitly:

$$a_\mu = \det \begin{pmatrix} z_1^{\mu_1} & z_2^{\mu_1} & \cdots & z_n^{\mu_1} \\ z_1^{\mu_2} & z_2^{\mu_2} & \cdots & z_n^{\mu_n} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{\mu_n} & z_2^{\mu_n} & \cdots & z_n^{\mu_n} \end{pmatrix},$$

and where $\rho = (n-1, n-2, \dots, 1, 0)$.

Ex $\lambda = (2,0)$, $\rho = (1,0)$, $\lambda + \rho = (3,0)$.

$$a_{\lambda+\rho}(z_1, z_2) = \det \begin{pmatrix} z_1^3 & z_2^3 \\ z_1^0 & z_2^0 \end{pmatrix} = (z_1 - z_2)(z_1 + z_1 z_2 + z_2)$$

$$a_\rho(z_1,z_2)=\det\begin{pmatrix} z_1^1 & z_2^1 \\ z_1^0 & z_2^0\end{pmatrix} = z_1-z_2.$$

$$s_\lambda(z_1,z_2)=\frac{(z_1-z_2)(z_1+z_1z_2+z_2)}{z_1-z_2}=z_1^2+z_1z_2+z_2^2.$$

Rem $a_\rho(z_1,\dots,z_n)=\prod_{i<j}(z_i-z_j)$ by Vandermonde.

Rep. Theory s_λ is the character of the highest weight representation V_λ of $GL_n(\mathbb{C})$.

By **Weyl Character Formula**,

$W=S_n$
$\Phi^+=\{(i,j)\}_{i<j}$

$$\chi_\lambda(z_1,\dots,z_n)=\frac{\sum_{w\in W}\varepsilon(w)\,z^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi^+}\left(z^{\alpha/2}-z^{-\alpha/2}\right)}.$$

Note that the denominator is basically a_ρ and the numerator is $a_{\lambda+\rho}$ by expansion.

Branching rules Consider restriction of V_λ of $GL_n(\mathbb{C})$ to $GL_{n-1}(\mathbb{C})$. Then

$$\begin{pmatrix} 1 & \\ & GL_{n-1} \end{pmatrix}$$

$$V_\lambda\big|_{GL_{n-1}(\mathbb{C})} = \bigoplus_\mu V'_\mu, \qquad \begin{array}{ccccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_n \\ & \mu_1 & & \mu_2 & & \mu_3 & \cdots & \mu_{n-1} & \end{array}$$

where $\mu = (\mu_1, \dots, \mu_{n-1})$ interleave λ, that is,

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n,$$

and V'_μ is highest weight rep. of $GL_{n-1}(\mathbb{C})$.

In terms of characters (Schur polynomials),

$$s_\lambda(z_1, \dots, z_n) = \sum_\mu z_1^{|\lambda/\mu|} s_\mu(z_2, \dots, z_n),$$

where $|\lambda/\mu| = (\lambda_1 - \mu_1) + (\lambda_2 - \mu_2) + \dots + (\lambda_{n-1} - \mu_{n-1})$.

Rem Note that the expansion is multiplicity-free, so we can effectively repeat the procedure.

Gelfand's parametrization

Repeat branching rules until you get to one-dimensional rep-ns of $GL_1(\mathbb{C})$.

Gelfand–Tsetlin patterns

$$\left\{ \begin{array}{ccccccc} a_{11} & & a_{12} & & \cdots & & a_{1n} \\ & a_{22} & & \cdots & & a_{2n} & \\ & & \ddots & & \iddots & & \\ & & & a_{nn} & & & \end{array} \right\} \quad GT(a_{11}, \ldots, a_{1n})$$

top row

with $a_{ij} \geqslant a_{i+1,j+1} \geqslant a_{i,j+1}$ „betweenness" cond-n.

Note that rows interleave next ones. Then repeated branching rules give the result:

$$S_\lambda(z_1, \ldots, z_n) = \sum_{T \in GT(\lambda)} z^{m(T)},$$

$\boxed{d_k \text{ - sum of } k\text{-th row.}}$

where $z^{m(T)} = z_1^{|d_1/d_2|} z_2^{|d_2/d_3|} \cdots z_n^{|d_n|}$.

Ex Let $\lambda = (2,0)$. Then

$$\left\{\begin{array}{ccc} 2 & & 0 \\ & 0 & \end{array}\right\} \qquad \left\{\begin{array}{ccc} 2 & & 0 \\ & 1 & \end{array}\right\} \qquad \left\{\begin{array}{ccc} 2 & & 0 \\ & 2 & \end{array}\right\}$$

$$z_1^2 z_2^0 \qquad z_1^1 z_2^1 \qquad z_1^0 z_2^2 .$$

Hence, $S_\lambda(z_1, z_2) = z_1^2 + z_1 z_2 + z_2^2$, as before.

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② Tokuyama's Formula

$$\left\{\begin{array}{ccccc} a_{11} & a_{12} & \cdots & & a_{1n} \\ & a_{22} & \cdots & & a_{2n} \\ & & \ddots & \iddots & \\ & & a_{nn} & & \end{array}\right\}$$

is strict if rows are strictly decreasing.

a_{ij} is left-leaning if $a_{ij} = a_{i-1,j-1}$. $\ell(T) = \#$ of left

right-leaning if $a_{ij} = a_{i-1,j}$.

Otherwise generic. $g(T) = \#$ of generic

Theorem (Tokuyama, '88) Let λ be a partition.

$$\sum_{T \in SGT(\lambda+\rho)}' (t+1)^{g(T)}\, t^{\ell(T)}\, z^{m(T)} = \prod_{i<j} (z_i + t z_j)\, s_\lambda(z_1, \dots, z_n).$$

Rem $\prod_{i<j}(z_i + t z_j)$ is the deformed Weyl denominator. The classical one is $t=-1$:

$\prod_{i<j}(z_i - z_j)$.

$\lambda = (1,0,0)$

$\rho = (2,1,0)$

$\lambda+\rho = (3,1,0)$

let's see Tokuyama Formula in action.

Corollaries.

- Gelfand's parametrization (naturally)
- Weyl's character Formula
- Stanley's Formula on Hall-Littlewood polys.

Gelfand's parametrization

Set $t=0$. Then the only terms that survive in expression

$$\sum_{T\in SGT(\lambda+\rho)} (t+1)^{g(T)} t^{\ell(T)} z^{m(T)} = \prod_{i<j} (z_i + t z_j) S_\lambda(z)$$

are terms with $\ell(T)=0$. That is, no entries are left-leaning. But then

$\lambda+\rho=(5,3,0)$ $\qquad\qquad\qquad\qquad$ $\lambda=(3,2,0)$

$$\left\{\begin{matrix} 5 & & 3 & & 0 \\ & 4 & & 3 & \\ & & 3 & & \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} 3 & & 2 & & 0 \\ & 3 & & 3 & \\ & & 3 & & \end{matrix}\right\}$$

We subtract $\rho_k=(k-1,k-2,\dots,0)$ from k-th row. It gives a bijection with GT-patterns with top row λ. Then the sum is more or less becomes

$$\sum_{T\in GT(\lambda)} z^{m(T)} = \prod_{i=1}^{n} z_i^{n-i} \cdot S_\lambda(z).$$

Weyl's Character Formula

Set $t=-1$. Then the only terms that survive in

expression $\displaystyle\sum_{T\in SGT(\lambda+\rho)} (t+1)^{g(T)}\, t^{\ell(T)}\, z^{m(T)} = \prod_{i<j}(z_i+tz_j)\,S_\lambda(z)$

are terms with $g(T)=0$. That is, no entries are generic, so each entry is either left- or right-leaning.

$$\left\{\begin{array}{ccccc} 7 & & 5 & & 2 \\ & 5 & & 2 & \\ & & 5 & & \end{array}\right\} \longleftrightarrow \text{element of } W \simeq S_n$$

weight $z^{m(T)}$ will be $z^{\pi(\lambda+\rho)}$ for some permutation $\pi\in W\simeq S_n$. Then $\ell(T)$ is the number of inversions, and so the sum becomes

$$\sum_{\pi\in S_n} (-1)^{\varepsilon(\pi)} z^{\pi(\lambda+\rho)} = \prod_{i<j}(z_i-z_j)\,S_\lambda(z), \text{ or}$$

$$S_\lambda(z) = \frac{\sum_{\pi \in S_n} (-1)^{\varepsilon(\pi)} z^{\pi(\lambda+\rho)}}{\prod_{i<j}(z_i - z_j)} = \frac{a_{\lambda+\rho}}{a_\rho}$$

by the expansion of the determinant.

Stanley's Set $t=1$ and get Hall-Littlewood polynomials values in terms of strict GT-patterns with weights $2^{g(T)} z^{m(T)}$, rediscovering combinatorial formula of Stanley.

Why does everything work so well? What are these values $\prod_{i<j}(z_i + t z_j) S_\lambda(z)$ anyway?

③ Representations of p-adic groups

Let $G = GL_n(F)$, where F is a local non-archimedean field (e.g. $\mathbb{Q}_p$) with the uniformizer ϖ and the cardinality of the residue field q.

There is a natural matrix coefficient on a principal series representation V_z, $z \in \mathbb{C}^n$ called the spherical Whittaker function. The value on the diagonal element $\begin{pmatrix} \varpi^{\lambda_1} & & & \\ & \varpi^{\lambda_2} & & \\ & & \ddots & \\ & & & \varpi^{\lambda_n} \end{pmatrix}$ is given by

Casselman-Shalika' 80 by

$$z^{-\rho} \prod_{i<j} (z_i - q^{-1} z_j) S_\lambda(z).$$

It's Tokuyama's expression with $t = -q^{-1}$!

In other words, Tokuyama's formula gives a combinatorial expression of the spherical Whittaker function of p-adic group $GL_n(F)$!

Open Problems

Casselman-Shalika formula works not only for $GL_n(F)$ but for any reductive group as follows.

$$\text{Whittaker function on } \varpi^\lambda = \prod_{\alpha \in \Phi^+} (1 - q^{-1} z^\alpha) \, \chi_\lambda(z_1, \dots, z_n).$$

$$\prod_{i<j} (z_i - q^{-1} z_j) \, s_\lambda(z)$$

For example, for $G = Sp_{2n}(F)$ we get

$$\prod_{i<j} (z_i - q^{-1} z_j) \prod_{i=1}^{n} (1 - q^{-1} z_i^2) \, \chi_\lambda^C(z_1, \dots, z_n).$$

Is there combinatorial evaluation in terms of GT?

Yes. For the symplectic group the corresponding Tokuyama formula was obtained by D. Ivanov'12. The proof uses machinery of solvable lattice models and no „simple" proof is known.

What about the orthogonal group?

No. No expression is found so far on how to express

$$\prod_{i<j}(z_i - q^{-1}z_j)\prod_{i=1}^{n}(1-q^{-1}z_i)\,\chi_\lambda^B(z_1,\dots,z_n)$$

in terms of corresponding GT-patterns.

There are partial results but they are not satisfactory.

About the proofs

1. For $G = GL_n$ there is a simple direct proof using branching rules for the Schur polynomial and Pieri's rules. No proof like this is known for other types. (look my post on Thuses.)

2. For GL_n and Sp_{2n} there are proofs using the solvable lattice models using Yang-Baxter equation. It doesn't use anything about the branching rules or anything like that.
Unfortunately, all attempts to extend proofs for the orthogonal groups failed.

3. For GL_n there is a geometric-combinatorial evaluation of the spherical Whittaker function in terms of crystal bases, Lusztig data, Mirković-Vilonen cycles and such. It's harder but a

seems to work uniformly over all types, thus, giving a hope it could be extended to all types. That's what I'm working on now.

Thank you!

$G_0 = G(F)/G(\mathcal{O}_F)$.

$\uparrow$

subvarieties

MV cycles $\longleftrightarrow$ basis of V_λ rep-n.

$G^\vee$ - dual Langlands group.

$\updownarrow$

parametrize basis of cohomology ring of G_0 (Geometric Satake).