

THE KÄHLER-RICCI FLOW ON KÄHLER SURFACES

Jian Song

Johns Hopkins University
Department of Mathematics
3400 North Charles Street
Baltimore MD 21218

1 Introduction

The problem of finding Kähler-Einstein metrics on a compact Kähler manifold has been the subject of intense study over the last few decades. In his solution to Calabi's conjecture, Yau [Ya1] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with vanishing or negative first Chern class. An alternative proof of Yau's theorem is given by Cao [Ca] using the Kähler-Ricci flow.

As is well-known, Hamilton's Ricci flow has become one of the most powerful tools in geometric analysis with remarkable applications to the study of 3-manifolds [Ha1], [Pe]. In early 90's, Hamilton and Chow used the Ricci flow to give another proof of classical uniformization for Riemann surfaces (see [Ha2], [Ch]). Recently Perelman [Pe] has made a major breakthrough in studying the Ricci flow. The convergence of the Kähler-Ricci flow on Kähler-Einstein manifolds with positive first Chern class was claimed by Perelman and it has been generalized to any Kähler manifolds admitting a Kähler-Ricci soliton by Tian and Zhu [TiZhu]. Previously, in [ChTi], Chen and Tian proved that the Kähler-Ricci flow converges to a Kähler-Einstein metric if the initial metric is of non-negative bisectional curvature.

Most algebraic varieties do not admit Kähler-Einstein metrics, for example, those with indefinite first Chern class, so it is a natural question to ask if there exist any well-defined canonical metrics on these varieties or on their canonical models. Tsuji [Ts] applied the Kähler-Ricci flow to produce a canonical singular Kähler-Einstein metric on non-singular minimal algebraic varieties of general type. In [SoTi], new canonical metrics on the canonical models of projective varieties of positive Kodaira dimension were constructed. We also constructed such canonical metrics by the Kähler-Ricci flow on Kähler surfaces.

In this expository note, we present a metric classification for Kähler surfaces with non-negative Kodaira dimension or positive first Chern class by the Kähler-Ricci flow.

2 Preliminaries

Let X be an n -dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X . In local coordinates $(z_1, \dots, z_n)$, ω can be written in the form of

$$\omega = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $\{g_{i\bar{j}}\}$ is a positive definite hermitian matrix function. The Kähler form ω is a closed $(1,1)$ -form by the Kähler condition. In other words, for $i, j, k = 1, \dots, n$,

$$\frac{\partial g_{i\bar{k}}}{\partial z_j} = \frac{\partial g_{i\bar{k}}}{\partial z_j} \quad \text{and} \quad \frac{\partial g_{k\bar{i}}}{\partial \bar{z}_j} = \frac{\partial g_{k\bar{j}}}{\partial \bar{z}_i}.$$

We also define the space of plurisubharmonic functions with respect to a Kähler form ω by

$$\mathcal{P}(X, \omega) = \{\varphi \in C^\infty(X) \mid \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0\}.$$

The curvature tensor for g is locally given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^n g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}, \quad i, j, k = 1, 2, \dots, n.$$

The bisectional curvature of ω is positive if

$$R_{i\bar{j}k\bar{l}} v^i \bar{v}^j w^k \bar{w}^l \geq 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of X . The Ricci tensor is given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j}, \quad i, j = 1, 2, \dots, n.$$

So its Ricci curvature form is

$$\text{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^n R_{i\bar{j}} dz_i \wedge d\bar{z}_j = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{k\bar{l}}).$$

Definition 2.1 The Kähler metric ω is called a Kähler-Einstein metric on X if

$$\text{Ric}(\omega) = \lambda \omega,$$

for some constant $\lambda \in \mathbf{R}$.

We can always scale the Kähler-Einstein metric ω such that $\lambda = -1, 0$ or 1 . By the Einstein equation, the first Chern class of X has to be definite or vanishing if there exists a Kähler-Einstein metric on X .

The Ricci flow introduced by Hamilton ([Ha1]) on a Riemannian manifold is defined by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}. \quad (2.1)$$

On a Kähler manifold X , the Kähler condition is preserved by the Ricci flow if the initial metric is Kähler, so that the Ricci flow is called the Kähler-Ricci flow. We define the following normalized Kähler-Ricci flow

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} + \lambda g_{i\bar{j}}, \quad (2.2)$$

where $\lambda = -1, 0, 1$.

Let X be a compact complex manifold of complex dimension n and K_X be the canonical line bundle on X . The canonical ring of X is defined by

$$R(X) = \bigoplus_{m=0}^{\infty} H^0(X, K_X^m)$$

with the pairing

$$H^0(X, K_X^{m_1}) \otimes H^0(X, K_X^{m_2}) \rightarrow H^0(X, K_X^{m_1+m_2}).$$

Then we can define the Kodaira dimension $\text{kod}(X)$ of X by

$$\text{kod}(X) = \begin{cases} -\infty & \text{if } R(X) \cong \mathbf{C} \\ \text{tr}(R(X)) - 1 & \text{otherwise.} \end{cases}$$

We always have $\text{kod}(X) \leq \dim(X)$. In fact, if $\text{kod}(X) \geq 0$, we can define the following meromorphic pluricanonical map for sufficiently large m

$$\Phi_m : X \rightarrow \mathbf{CP}^{N_m}$$

by

$$\Phi_m(z) = [S_0^{(m)}(z), S_1^{(m)}(z), \dots, S_{N_m}^{(m)}(z)],$$

where $\{S_0^{(m)}, S_1^{(m)}, \dots, S_{N_m}^{(m)}\}$ is a basis of $H^0(X, K_X^m)$. The Kodaira dimension of X is exactly the complex dimension of the image of X by Φ_m for sufficiently large m .

The compact complex manifolds can be classified according to their Kodaira dimension by $\text{kod}(X) = -\infty, 0, 1, \dots, \dim(X)$.

In the case of smooth compact complex surfaces, the Kodaira dimension must be $-\infty, 0, 1$ or 2 . We have the Enriques-Kodaira classification by Kodaira dimension dividing the minimal surfaces into ten classes. A smooth compact complex surface is called minimal, if it does not contain any (-1) -curve. Any smooth surface has a smooth minimal model by contracting the (-1) -curves.

3 $\text{kod}(X) = 2$

A compact complex surface X of $\text{kod}(X) = 2$ is called a surface of general type. A surface of general type X is minimal if and only if its canonical line bundle K_X is semi-positive, so that $c_1(X) \leq 0$.

If $c_1(X) < 0$, K_X is a positive line bundle so that X must be a minimal algebraic surface. It is proved by Yau [Ya1] and Aubin [Au] independently that there always exists a unique Kähler-Einstein metric on X . Cao [Ca] gave an alternative proof by applying the following Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) - \omega \\ \omega|_{t=0} = \omega_0 \end{cases} \quad (3.1)$$

where ω_0 is a Kähler metric on X .

Theorem 3.1 (Cao) *Let X be a minimal complex surface with $c_1(X) < 0$. Then the Kähler-Ricci flow converges for any initial Kähler metric to the unique Kähler-Einstein metric ω_{KE} with*

$$\text{Ric}(\omega_{KE}) = -\omega_{KE}.$$

In general, a minimal surface of general type might contain a finite number of (-2) -curves and $c_1(X)$ is not negative but semi-negative. Let C be the union of these curves whose connected components are rational curves of $A - D - E$ -type. The canonical model X_{can} of X is then obtained by blowing down C

$$f : X \rightarrow X_{can}.$$

X_{can} is an orbifold surface with singularities of $A - D - E$ -type. We can still apply the normalized Kähler-Ricci flow (3.1) on a minimal surface of general type. Let $Ka(X)$ denote the Kähler cone of X , that is,

$$Ka(X) = \{[\omega] \in H^{1,1}(X, \mathbf{R}) \mid [\omega] > 0\}.$$

The Kähler class will change along the Kähler-Ricci flow by the following ordinary differential equation

$$\begin{cases} \frac{\partial [\omega]}{\partial t} = -2\pi c_1(X) - [\omega] \\ [\omega]|_{t=0} = [\omega_0]. \end{cases} \quad (3.2)$$

It follows

$$[\omega(t, \cdot)] = -2\pi c_1(X) + e^{-t}([\omega_0] + 2\pi c_1(X)).$$

Let $\chi \in -2\pi c_1(X)$ be a closed semi-positive $(1,1)$ -form and Ω be the smooth volume form on X such that

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = \chi.$$

We choose the reference Kähler metric ω_t by

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi)$$

so that

$$\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi.$$

Then the Kähler-Ricci flow can be reduced to a Monge-Ampère flow for the potential φ given by

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} - \varphi \\ \varphi|_{t=0} = 0. \end{cases} \quad (3.3)$$

Tsuji [Ts] proved the following convergence result for the Kähler-Ricci flow.

Theorem 3.2 (Tsuji) *Let X be a minimal complex surface of general type. Then the Kähler-Ricci flow converges to a singular Kähler-Einstein metric ω_∞ smooth outside the (-2) -curves with*

$$\text{Ric}(\omega_{KE}) = -\omega_{KE}$$

for any initial Kähler metric ω_0 satisfying $[\omega_0] > -2\pi c_1(X)$.

The key observation in Tsuji's proof is that the canonical bundle K_X on a minimal surface of general type is big and nef, so that for any sufficiently small $\epsilon > 0$

$$[K_X] - \epsilon[C] > 0.$$

The initial class condition is removed by Tian and Zhang [TiZha] and a stronger uniform C^0 -estimate is obtained.

Theorem 3.3 (Tian-Zhang) *Let X be a minimal complex surface of general type and $f : X \rightarrow X_{\text{can}}$ be the holomorphic map from X to its canonical model contracting the (-2) -curves. Then the Kähler-Ricci flow converges to a singular Kähler-Einstein metric ω_∞ smooth outside the (-2) -curves for any initial Kähler-metric. Furthermore*

$$\omega_\infty = f^*\omega_{KE},$$

where ω_{KE} is the unique smooth Kähler-Einstein orbifold metric on X_{can} . In particular, ω_∞ has local continuous potential, i.e.,

$$\omega_\infty = \chi + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$$

for some $\varphi_\infty \in C^0(X)$.

The critical equation for the Monge-Ampère flow on X is given by

$$\frac{(\chi + \sqrt{-1}\partial\bar{\partial}\varphi_\infty)^2}{\Omega} = e^{\varphi_\infty}.$$

The following corollary is an immediate corollary from Theorem 3.3 by the fact that $\omega_\infty^2 = \Omega e^{\varphi_\infty}$ and φ_∞ is continuous.

Corollary 3.1 *The Kähler-Einstein volume form on X defined by $\Omega_{KE} = \omega_\infty^2$ is a continuous and nonvanishing volume form on X such that*

$$\sqrt{-1}\partial\bar{\partial} \log \Omega_{KE} = \omega_\infty.$$

4 $\text{kod}(X) = 1$

An elliptic fibration of a surface X is a proper, connected holomorphic map $f : X \rightarrow \Sigma$ from X to a curve Σ such that the general fibre is a non-singular elliptic curve. An elliptic surface is a surface admitting an elliptic fibration. Any surface X of $\text{kod}(X) = 1$ must be an elliptic surface. Such an elliptic surface is sometimes called a properly elliptic surface. Since we assume that X is minimal, all fibres are free of (-1) -curves. A very simple example is the product of two curves, one elliptic and the other of genus ≥ 2 .

Let $f : X \rightarrow \Sigma$ be an elliptic surface. The differential df can be viewed as an injection of sheaves $f^*(K_\Sigma) \rightarrow \Omega_X^1$. Its cokernel $\Omega_{X/\Sigma}$ is called the sheaf of relative differentials. In general, $\Omega_{X/\Sigma}$ is far from being locally free. If some fibre has a multiple component, then df vanishes along this component and $\Omega_{X/\Sigma}$ contains a torsion subsheaf with one-dimensional support. Away from the singularities of f we have the following exact sequence

$$0 \rightarrow f^*(K_\Sigma) \rightarrow \Omega_X^1 \rightarrow \Omega_{X/\Sigma} \rightarrow 0$$

including an isomorphism between $\Omega_{X/\Sigma}$ and $K_X \otimes f^*(K_\Sigma^\vee)$. We also call the line bundle $\Omega_{X/\Sigma}$ the dualizing sheaf of f on X . The following theorem is well-known (cf. [BaHuPeVa]).

Theorem 4.1 (Kodaira) *Let $f : X \rightarrow \Sigma$ be a minimal elliptic surface such that its multiple fibres are $X_{s_1} = m_1 F_1, \dots, X_{s_k} = m_k F_k$. Then*

$$K_X = f^*(K_\Sigma \otimes (f_{*1}\mathcal{O}_X)^\vee) \otimes \mathcal{O}_X(\sum (m_i - 1)F_i), \quad (4.1)$$

or

$$K_X = f^*(L \otimes \mathcal{O}_X(\sum (m_i - 1)F_i)),$$

where L is a line bundle of degree $\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_\Sigma)$ on Σ .

Note that $\deg(f_{*1}\mathcal{O}_X)^\vee = \deg(f_*\Omega_{X/\Sigma}) \geq 0$ and the equality holds if and only if f is locally trivial. The following invariant

$$\delta(f) = \chi(\mathcal{O}_X) + \left(2g(\Sigma) - 2 + \sum_{i=1}^k \left(1 - \frac{1}{m_i} \right) \right)$$

determines the Kodaira dimension of X .

Proposition 4.1 (cf. [BaHuPeVa]) *Let $f : X \rightarrow \Sigma$ be a relatively minimal elliptic fibration and X be compact. Then $\text{kod}(X) = 1$ if and only if $\delta(f) > 0$.*

Kodaira classified all possible singular fibres for f . A fibre X_s is stable if

1. X_s is reduced,
2. X_s contains no (-1) -curves,

3. X_s has only node singularities.

The only stable singular fibres are of type I_b for $b > 0$, therefore such singular fibres are particularly interesting. Let $\mathcal{S}_1 = \{z \in \mathbf{C} \mid \text{Im} z > 0\}$ be the upper half plane and $\Gamma_1 = \text{SL}(2, \mathbf{Z}) / \{\pm 1\}$ be the modular group acting by $z \rightarrow \frac{az+b}{cz+d}$. Then $\mathcal{S}_1 / \Gamma_1 \cong \mathbf{C}$ is the period domain or the moduli space of elliptic curves. The j -function gives an isomorphism $\mathcal{S}_1 / \Gamma_1 \rightarrow \mathbf{C}$ with

1. $j(z) = 0$ if $z = e^{\frac{\pi}{3}\sqrt{-1}}$ modular Γ_1 ,
2. $j(z) = 1$ if $z = \sqrt{-1}$ modular Γ_1 .

Let $\Sigma_{reg} = \{s \in \Sigma \mid X_s \text{ is a nonsingular fibre}\}$. Then any elliptic surface $f : X \rightarrow \Sigma$ gives a period map $p : \Sigma_{reg} \rightarrow \mathcal{S}_1 / \Gamma_1$. Set $J : \Sigma_{reg} \rightarrow \mathbf{C}$ by $J(s) = j(p(s))$. For a stable fibre X_s of type I_b , the functional invariant J has a pole of order b at s and the monodromy is given by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

Again we apply the normalized Kähler-Ricci flow (3.1) on such elliptic surfaces of Kodaira dimension one. We choose a semi-positive $(1, 1)$ -form $\chi \in -2\pi c_1(X)$ to be the pullback of a Kähler form on the base Σ and χ might vanish somewhere due to the presence of singular fibres. Let Ω be the smooth volume form on X such that $\sqrt{-1}\partial\bar{\partial}\log \Omega = \chi$. The Kähler class of ω deformed by the Kähler-Ricci flow is given by

$$[\omega] = (1 - e^{-t})[\chi] + e^{-t}[\omega_0],$$

with the initial Kähler metric ω_0 . So we let $\omega_t = (1 - e^{-t})\chi + e^{-t}\omega_0$ be the reference metric and $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$.

As we discussed in the previous section, one can reduce the Kähler-Ricci flow (2.2) to the following Monge-Ampère flow on Kähler potentials.

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{e^{-t}(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} - \varphi \\ \varphi|_{t=0} = 0. \end{cases} \quad (4.2)$$

The simple example is the Kähler-Ricci flow on $X = E \times C$, where E is an elliptic curve and C is a curve of genus greater than one. Let $\pi_1 : X \rightarrow E$ and $\pi_2 : X \rightarrow C$ be the projection maps. Let ω_E be a flat metric on E , ω_C be the hyperbolic metric on C with $\omega_C \in -c_1(C)$ and

$$\omega_0 = \pi_1^* \omega_E + \pi_2^* \omega_C$$

be the initial Kähler metric on X for the Kähler-Ricci flow. Then

$$\omega_t = \pi_2^* \omega_C + e^{-t} \pi_1^* \omega_E$$

solves the Kähler-Ricci flow by

$$\frac{\partial \omega_t}{\partial t} = -e^{-t} \pi_1^* \omega_E = -\text{Ric}(\omega_t) - \omega_t.$$

And

$$\lim_{t \rightarrow \infty} \omega_t = \pi_2^* \omega_C.$$

The following general result is proved in [SoTi]

Theorem 4.2 (Song-Tian) *Let $f : X \rightarrow \Sigma$ be a minimal elliptic surface of $\text{kod}(X) = 1$. Let $\Sigma_{\text{reg}} = \Sigma \setminus \{s_1, \dots, s_k\}$. Then the Kähler-Ricci flow has a global solution $\omega(t, \cdot)$ for any initial Kähler metric satisfying:*

1. $\omega(t, \cdot)$ converges to $f^* \omega_\infty \in -c_1(X)$ as currents for a positive current ω_∞ on Σ ,
2. ω_∞ is smooth on Σ_{reg} and satisfies as currents on Σ

$$\text{Ric}(\omega_\infty) = -\omega_\infty + \omega_{\text{WP}} + \sum_{i=1}^k \frac{m_i - 1}{m_i} [s_i],$$

3. for any compact subset $K \in f^{-1}(\Sigma_{\text{reg}})$, there is a constant C_K such that

$$\|\omega\|_{L^\infty(K)} + \sup_{s \in f(K)} \|e^t \omega|_{f^{-1}(s)}\|_{L^\infty(f^{-1}(s))} + \|S\|_{L^\infty(K)} \leq C_K,$$

where S is the scalar curvature of $\omega(t, \cdot)$.

The Kähler-Ricci flow collapses exponentially fast along the vertical direction and the limit metric satisfies a generalized Kähler-Einstein equation (2) with the correction term as the curvature of the fibration. The generalized Kähler-Einstein equation can also be considered as a local version of Kodaira's adjunction formula (4.1).

Corollary 4.1 *Let $f : X \rightarrow \Sigma$ be an elliptic fibre bundle over a curve Σ of genus greater than one. Then the Kähler-Ricci flow (2.2) has a global solution with any initial Kähler metric. Furthermore, $\omega(t, \cdot)$ converges with uniformly bounded scalar curvature to the pullback of the Kähler-Einstein metric on Σ .*

If $f : X \rightarrow \Sigma$ has only singular fibres of type mI_0 , all the smooth fibres of f are isomorphic so that the period map p is trivial with its image as a point in $\mathcal{M}$. The Kähler-Ricci flow will then converge to a cone hyperbolic metric on Σ and the cone singularities appear from where the multiple fibres sit.

5 $\text{kod}(X) = 0$

If X is a minimal Kähler surface of $\text{kod}(X) = 0$, the canonical line bundle K_X is numerically trivial so that $c_1(X) = 0$. Yau's solution to Calabi's conjecture shows that there always exists a Ricci-flat Kähler metric in any given Kähler class on X . An alternative proof is given by Cao [Ca] by applying the following Kähler-Ricci flow.

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) \\ \omega|_{t=0} = \omega_0 \end{cases} \quad (5.1)$$

where ω_0 is a Kähler metric on X .

Theorem 5.1 *Let X be a minimal complex surface with $c_1(X) = 0$. Then the Kähler-Ricci flow (5.1) converges for any initial Kähler metric ω_0 to the unique Ricci-flat metric $\omega_{KE} \in [\omega_0]$ with*

$$\text{Ric}(\omega_{KE}) = 0.$$

6 Fano surfaces

A compact complex surface X with $c_1(X) > 0$ is called a Fano surface. The anti-canonical bundle K_X^{-1} is thus positive. By the Enriques-Kodaira classification, $\mathbf{CP}^2 \# n \overline{\mathbf{CP}^2}$ for $0 \leq n \leq 8$ and $\mathbf{CP}^1 \times \mathbf{CP}^1$ are the only compact differential four-manifolds on which there is a complex structure with positive first Chern class. Tian [Ti2] proves the following theorem for the existence of a Kähler-Einstein metric on Fano surfaces.

Theorem 6.1 (Tian) *Any compact complex surface X with $c_1(X) > 0$ admits a Kähler-Einstein metric if the Lie algebra of the automorphism group on X is reductive.*

So there always exists a Kähler-Einstein metric on $\mathbf{CP}^2$, $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2 \# n \overline{\mathbf{CP}^2}$ for $3 \leq n \leq 8$.

However, there does not exist any Kähler-Einstein metric on $\mathbf{CP}^2 \# 1 \overline{\mathbf{CP}^2}$ and $\mathbf{CP}^2 \# 2 \overline{\mathbf{CP}^2}$ since the Lie algebra of the automorphism group on these toric Fano surfaces is not reductive. One can also disprove the existence of any Kähler-Einstein metric by showing the Futaki invariant on these two surfaces is non-zero.

Definition 6.1 *Let X be a complex surface with $c_1(X) > 0$. A Kähler metric ω is called a Kähler-Ricci soliton if it satisfies*

$$\text{Ric}(\omega) = \omega + L_V \omega, \quad (6.1)$$

where V is a holomorphic vector field on X and L_V denotes the Lie derivative along V .

A Kähler-Einstein metric is also a special Kähler-Ricci soliton by taking $V = 0$. In [Koi], Koiso constructed a Kähler-Ricci soliton on $\mathbf{CP}^2 \# 1 \overline{\mathbf{CP}^2}$. A general result on the existence of Kähler-Einstein metrics and Kähler-Ricci soliton on toric manifolds is proved in [WaZh].

Theorem 6.2 (Wang-Zhu) *There exists a Kähler-Ricci soliton on a toric Kähler manifold with positive first Chern class.*

Therefore there always exists a Kähler-Einstein metric or a Kähler-Ricci soliton on a complex surface of positive first Chern class.

We apply the the following normalized Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -Ric(\omega) + \omega \\ \omega|_{t=0} = \omega_0 \end{cases} \quad (6.2)$$

with the initial Kähler metric $\omega_0 \in 2\pi c_1(X)$ so that the Kähler class $[\omega]$ will not change.

Perelman proves a gradient estimate on the Kähler-Ricci flow (6.2), which implies the scalar curvature will stay uniformly bounded (see [SeTi]). He has also claimed that the Kähler-Ricci flow converges (6.2) on a Kähler-Einstein manifold X with positive first Chern class to a Kähler-Einstein metric. This is proved by Tian and Zhu [TiZhu] and generalized to the Kähler-Ricci solitons. Suppose X admits a Kähler-Ricci soliton with respect to the holomorphic vector field V . Then the imaginary part of V induces a one-parameter subgroup G_V in the automorphism group of X .

Theorem 6.3 (Tian-Zhu) *Let X be a compact complex surface of $c_1(X) > 0$. Then the Kähler-Ricci flow will converge to a Kähler-Einstein metric or a Kähler-Ricci soliton if the initial Kähler metric is G_V -invariant.*

7 Generalizations

The generalized Kähler-Einstein metric on the canonical model of an elliptic surface with Kodaira dimension one can be generalized and defined on a family of surfaces with a fibration structure.

Mirror symmetry and the SYZ conjecture make predictions for Calabi-Yau manifolds with "large complex structure limit point" (cf. [StYaZa]). It is believed that in the large complex structure limit, the Ricci-flat metrics should converge in the Gromov-Hausdorff sense to a half-dimensional sphere by collapsing a special Lagrangian torus fibration over this sphere. This holds trivially for elliptic curves and is proved by Gross and Wilson (cf. [GrWi]) in the case of $K3$ surfaces. The method of the proof is to find a good approximation for the Ricci-flat metrics near the large complex structure limit. The approximation metric is obtained by gluing together the Oogru-Vafa metrics near the singular fibres and a semi-flat metric on the regular part of the fibration. Such a limit metric of $K3$ surfaces is McLean's metric.

We will apply a deformation for a family of Calabi-Yau metrics and derive Mclean's metric [Mc] without writing down an accurate approximation metric. Such a deformation can be also done in higher dimensions. It will be interesting to have a flow which achieves this limit. The large complex structure limit of a $K3$ surface $\hat{X}$ can be identified as the mirror to the large Kähler limit of X as shown in [GrWi], so we can fix the complex structure on X and deform the Kähler class to infinity. Let $f : X \rightarrow \mathbf{CP}^1$ be an elliptic $K3$ surface. Let $\chi \geq 0$ be the pullback

of a Kähler form on $\mathbf{CP}^1$ and ω_0 be a Kähler form on X . We construct a reference Kähler metric $\omega_t = \chi + e^{-t}\omega_0$ and $[\omega_t]$ tends to $[\chi]$ as $t \rightarrow \infty$. We can always scale ω_0 so that the volume of each fibre of f with respect to ω_t is e^{-t} . Suppose Ω is a Ricci-flat volume form on X with $\partial\bar{\partial}\log\Omega = 0$. Then Yau's proof [Ya1] of Calabi's conjecture yields a unique solution φ_t to the following Monge-Ampère equation for $t \in [0, \infty)$

$$\begin{cases} \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^2}{\Omega} = C_t \\ \int_X \varphi_t \Omega = 0, \end{cases} \quad (7.1)$$

where $C_t = [\omega_t]^2$. Therefore we obtain a family of Ricci-flat metrics $\omega(t, \cdot) = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$. The following theorem is proved in [SoTi].

Theorem 7.1 *Let $f : X \rightarrow \mathbf{CP}^1$ be an elliptically fibred K3 surface with 24 singular fibres of type I_1 . Then the Ricci-flat metrics $\omega(t, \cdot)$ converges to the pullback of a Kähler metric ω_∞ on $\mathbf{CP}^1$ in any compact set of X_{reg} in $C^{1,1}$ as $t \rightarrow \infty$. The Kähler metric ω_∞ on $\mathbf{CP}^1$ satisfies the equation*

$$\text{Ric}(\omega_\infty) = \omega_{WP}. \quad (7.2)$$

This limit metric ω_∞ coincides with McLean's metric as obtained by Gross and Wilson [GrWi]. Their construction is certainly much more delicate and gives an accurate approximation near the singular fibres by the Ooguri-Vafa metrics. Also Mclean's metric is an example of the generalized Kähler-Einstein metric defined as

$$\text{Ric}(\omega) = -\lambda\omega + \omega_{WP}$$

when $\lambda = 0$.

In fact, these canonical metrics belong to a class of Kähler metrics defined in [SoTi], which generalize Calabi's extremal metrics. Let Y be a Kähler manifold of complex dimension n together with a fixed closed (1,1)-form θ . Fix a Kähler class $[\omega]$, denote by $\mathcal{K}_{[\omega]}$ the space of Kähler metrics within the same Kähler class, that is, all Kähler metrics of the form $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. One may consider the following equation

$$\bar{\partial}V_\varphi = 0, \quad (7.3)$$

where V_φ is defined by

$$\omega_\varphi(V_\varphi, \cdot) = \bar{\partial}(S(\omega_\varphi) - \text{tr}_{\omega_\varphi}(\theta)). \quad (7.4)$$

Clearly, when $\theta = 0$, (7.3) is exactly the equation for Calabi's extremal metrics. For this reason, we call a solution of (7.3) a generalized extremal metric. If Y does not admit any nontrivial holomorphic vector fields, then any generalized extremal metric ω_φ satisfies

$$S(\omega_\varphi) - \text{tr}_{\omega_\varphi}(\theta) = \mu, \quad (7.5)$$

where μ is the constant given by

$$\mu = \frac{n(2\pi c_1(Y) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Moreover, if $2\pi c_1(Y) - [\theta] = \lambda[\omega]$, then any such a metric satisfies

$$\text{Ric}(\omega_\varphi) = \lambda\omega_\varphi + \theta,$$

that is, ω_φ is a generalized Kähler-Einstein metric.

Another example of such extremal metrics is constructed by Fine [Fi]. Let $f : X \rightarrow \Sigma$ be a Kähler surface admitting a non-singular holomorphic fibration over Σ , with fibres of genus at least 2. We also assume $c_1(\Sigma) \leq 0$. Let V be the vertical tangent bundle of X and $[\omega_t] = -f^*c_1(\Sigma) - e^{-t}c_1(V)$.

Let χ be a Kähler form in $-c_1(\Sigma)$ and $\bar{\omega} \in -c_1(V)$. Then $\bar{\omega} = \omega_H \oplus \theta\chi$, where ω_H is the hyperbolic Kähler form on each fiber and θ is a smooth function on X . We then set

$$\omega_t = \chi + e^{-t}\bar{\omega}.$$

The following theorem is proved by Fine in [Fi].

Theorem 7.2 *For sufficiently large $t \geq 0$, there exists a constant scalar curvature Kähler metric in $[\omega_t]$. Furthermore, such a family of metrics converge to a Kähler metric ω_∞ on Σ defined by*

$$S(\omega_\infty) = \text{tr}_{\omega_\infty}(\omega_{WP}) + \text{const}, \quad (7.6)$$

where ω_{WP} is the pullback of the Weil-Petersson metric from the moduli spaces of the fibre curves with a certain polarization.

Acknowledgements The author would like to thank Professor Selman Akbulut and the organizers of Gökova Geometry-Topology Conferences for their invitation and hospitality.

References

- [Au] Aubin, T. *Equations du type Monge-Ampère sur les variétés Kähleriennes compactes*, Bull. Sc. Math. 102 (1976), 119-121.
- [BaMu] Bando, S. and Mabuchi, T., *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, 11-40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
- [BaHuPeVa] Barth, W., Hulek, K., Peters, C., and Van De Ven, A., *Compact complex surfaces*, 2003, Springer.
- [Ca] Cao, H., *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2, 359-372.

- [Ch] Chow, B., *The Ricci flow on the 2-sphere*, J. Differential Geom. 33 (1991), no. 2, 325–334.
- [ChTi] Chen, X.X. and Tian, G., *Ricci flow on Kähler-Einstein surfaces*, Invent. Math. 147 (2002), no. 3, 487–544.
- [Do] Donaldson, S. K., *Scalar curvature and projective embeddings, I.*, J. Differential Geom. 59 (2001), no. 3, 479–522.
- [Fi] Fine, J., *Constant scalar curvature Kähler metrics on fibred complex surfaces*, J. Differential Geom. 68 (2004), no. 3, 397–432.
- [GrWi] Gross, M. and Wilson, P. M. H., *Large complex structure limits of K3 surfaces*, J. Differential Geom. 55 (2000), no. 3, 475–546.
- [Ha1] Hamilton, R., *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255–306.
- [Ha2] Hamilton, R., *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.
- [Hö] Hörmander, L., *An introduction to complex analysis in several variables*, Van Nostrand, Princeton 1973.
- [Ko] Kodaira, K., *On compact complex analytic surfaces I*, Ann. of Math. (2) 71 (1960), 111–152, *II*, Ann. of Math. (2) 77 (1963), 563–626, *III*, Ann. of Math. (2) 78 (1963), 1–40.
- [Koi] Koiso, N., *On rotationally symmetric Hamilton’s equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, 327–337, Adv. Stud. Pure Math., 18-I, Academic Press, Boston, MA, 1990.
- [Kol1] Kolodziej, S., *The complex Monge-Ampère equation*, Acta Math. 180 (1998), no. 1, 69–117.
- [KoSo] Kontsevich, M. and Soibelman, Y., *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), 203–263, World Sci. Publishing, River Edge, NJ, 2001.
- [Mi] Miranda, R., *The basic theory of elliptic surfaces*, Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research] ETS Editrice, Pisa, 1989. vi+108 pp.
- [Mc] McLean, R., *Deformations of calibrated submanifolds*, Comm. Anal. Geom. 6 (1998), no. 4, 705–747.
- [OoVa] Ooguri, H. and Vafa, C., *Summing up Dirichlet instantons*, Phys. Rev. Lett. 77 (1996), no. 16, 3296–3298.

- [Pe] Perelman, P., *The entropy formula for the Ricci flow and its geometric applications*, preprint math.DG/0211159.
- [PhSt] Phong, D. H. and Sturm, J., *On stability and the convergence of the Kähler-Ricci flow*, preprint math.DG/0412185.
- [SeTi] Sesum, N. and Tian, G., *Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman)*, lecture note.
- [Si] Siu, Y-T. *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, Birkhäuser Verlag, Basel 1987.
- [SoTi] Song, J. and Tian, G., *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, preprint, math.DG/0602150.
- [StYaZa] Strominger, A., Yau, S.T. and Zaslow, E., *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243–259.
- [Ti1] Tian, G., *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$* , Invent. Math. 89 (1987), no. 2, 225–246.
- [Ti2] Tian, G., *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101, no. 1 (1990), 101–172.
- [TiZha] Tian, G. and Zhang, Z., *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179–192.
- [TiZhu] Tian, G. and Zhu, X., *Convergence of Kähler Ricci flow*, preprint.
- [Ts] Tsuji, H. *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), 123–133.
- [WaZh] Wang, X. and Zhu, X., *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. 188 (2004), no. 1, 87–103.
- [Ya1] Yau, S.T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. 31 (1978), 339–411.
- [Ya2] Yau, S.T., *Open problems in geometry*, Proc. Symposia Pure Math. 54 (1993), 1–28.
- [Zh] Zhang, Z., Thesis at MIT.