

COMPLEX NUMBERS

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Real solutions x of the equation

$$x^2 + 1 = 0 \tag{1}$$

do not exist, because there is no real number x such that $x^2 = -1$.

If we assume the existence of a new number i (obviously not a real one), called the **imaginary unit**, which is defined by the property

$$i \cdot i = i^2 = -1 \tag{2}$$

we can state that the equation (1) is solved by $\pm i$.

1 Arithmetic of Complex Numbers

1.1 Definition

*Definition 1 - A **complex number** z is defined as*

$$z = x + iy \tag{3}$$

*where x and y are real numbers and i is the imaginary unit. The real numbers x and y are called respectively the **real part** and the **imaginary part** of z , and are denoted by the symbols*

$$Re(z) = x \quad \text{and} \quad Im(z) = y \tag{4}$$

*The expression (3) is called **rectangular form** of the complex number z .*

Any complex number is hence expressed as $z = Re(z) + iIm(z)$. The set of real numbers is denoted by \mathbb{R} , and the set of complex numbers by \mathbb{C} .

From (3) it follows that real numbers are complex numbers with a zero imaginary part, namely $x = x + 0i$. In particular 0 is the complex number having both real and imaginary parts equal to zero.

Throughout these notes, we will use the notations $x + iy$ and $x + yi$ interchangeably.

Example 1 - Find the real and imaginary parts of (a) $-\frac{3}{4}i$, (b) $-2 + 3i$, and (c) 5.

Solution - (a) $Re(-\frac{3}{4}i) = 0$ and $Im(-\frac{3}{4}i) = -\frac{3}{4}$; (b) $Re(-2 + 3i) = -2$ and $Im(-2 + 3i) = 3$; (c) $Re(5) = 5$ and $Im(5) = 0$. \square

1.1 EXERCISES

In Exercises 1-6, Find (a) $Re(z)$ and (b) $Im(z)$ for the given complex numbers

- | | |
|-----------------|---------------------------|
| 1. $z = 2 - 2i$ | 2. $z = -\frac{1}{2} + i$ |
| 3. $z = 1$ | 4. $z = i$ |
| 5. $z = -i$ | 6. $z = 0$ |

In Exercises 7-12, write the rectangular form of the complex number z with the given $Re(z)$ and $Im(z)$

- | | |
|--|--|
| 7. $Re(z) = 2, Im(z) = 1$ | 8. $Re(z) = \frac{1}{2}, Im(z) = -1$ |
| 9. $Re(z) = \frac{3}{2}, Im(z) = 0$ | 10. $Re(z) = -\sqrt{2}, Im(z) = 1$ |
| 11. $Re(z) = -\frac{5}{3}, Im(z) = -\frac{2}{3}$ | 12. $Re(z) = 0, Im(z) = -\frac{\sqrt{3}}{2}$ |

1.2 Addition, Subtraction, and Scalar Multiplication

In this section we show how to add and subtract complex numbers, and how to multiply a complex number by a scalar (i.e. a real number) using the common operations of addition, subtraction, and multiplication already in use for real numbers, along with their commutative, associative, and distributive (aka *foil rule*) properties.

In particular, we will use the distributive property of the imaginary unit, according to which for any two real numbers r and s , $i(r + s) = ir + is$.

The **addition** of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is completed by adding their real parts and imaginary parts separately, and by using the distributive property of the imaginary unit i :

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= x_1 + iy_1 + x_2 + iy_2 \\ &= x_1 + x_2 + iy_1 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)\end{aligned}\tag{5}$$

Similarly, for the **subtraction**

$$\begin{aligned}(x_1 + iy_1) - (x_2 + iy_2) &= x_1 + iy_1 - x_2 - iy_2 \\ &= x_1 - x_2 + iy_1 - iy_2 = (x_1 - x_2) + i(y_1 - y_2)\end{aligned}\quad (6)$$

Moreover, the multiplication of a complex number z by a real number r is called **scalar multiplication**, and it is defined by the following rule

$$r \cdot z = r(x + iy) = rx + iry$$

In conclusion

$$Re(z_1 \pm z_2) = Re(z_1) \pm Re(z_2) \quad Im(z_1 \pm z_2) = Im(z_1) \pm Im(z_2) \quad (7)$$

$$Re(r \cdot z) = r \cdot Re(z) \quad Im(r \cdot z) = r \cdot Im(z) \quad (8)$$

Example 2 - Given the complex numbers $z_1 = -2 + i$, $z_2 = 3 - 2i$, and $z_3 = i$, find the rectangular form of

$$(a) 2z_1 + z_2 \quad (b) 3z_2 + 4z_3 \quad (c) 3z_1 + 2z_2 \quad (d) 3z_1 + 2z_2 + z_3$$

Solution -

(a) From (8), since $Re(z_1) = -2$, $Im(z_1) = 1$, then $Re(2z_1) = -4$, $Im(2z_1) = 2$. Because $Re(z_2) = 3$ and $Im(z_2) = -2$, therefore, from (7) we derive that

$Re(2z_1 + z_2) = -4 + 3 = -1$, and $Im(2z_1 + z_2) = 2 - 2 = 0$, so $2z_1 + z_2 = -1$. But the same result can be found directly and in an easier way with the usual properties of addition, subtraction and scalar multiplication, as follows:

$$\begin{aligned}(a) \quad 2z_1 + z_2 &= 2(-2 + i) + (3 - 2i) = (-4 + 2i) + (3 - 2i) = -4 + 2i + 3 - 2i \\ &= (-4 + 3) + (2 - 2)i = -1.\end{aligned}$$

We proceed in the same direct way to solve the remaining problems in this Example

$$\begin{aligned}(b) \quad 3z_2 + 4z_3 &= 3(3 - 2i) + 4i = (9 - 6i) + 4i = 9 - 6i + 4i \\ &= 9 + (-6 + 4)i = 9 - 2i\end{aligned}$$

$$\begin{aligned}(c) \quad 3z_1 + 2z_2 &= 3(-2 + i) + 2(3 - 2i) = (-6 + 3i) + (6 - 4i) = -6 + 3i + 6 - 4i \\ &= (-6 + 6) + (3 - 4)i = -i\end{aligned}$$

$$\begin{aligned}(d) \quad 3z_1 + 2z_2 + z_3 &= 3(-2 + i) + 2(3 - 2i) + i = (-6 + 3i) + (6 - 4i) + i \\ &= -6 + 3i + 6 - 4i + i = (-6 + 6) + (3 - 4 + 1)i = 0 \quad \square\end{aligned}$$

1.2 EXERCISES

In Exercises 1-6, given z_1 and z_2 , find the rectangular form of (a) $z_1 + z_2$ and (b) $z_1 - z_2$

- | | |
|---|---|
| 1. $z_1 = 2 - 2i, z_2 = 3 + 2i$ | 2. $z_1 = 1 - \sqrt{3}i, z_2 = -1 + \sqrt{3}i$ |
| 3. $z_1 = \sqrt{2}, z_2 = -3\sqrt{2} + i$ | 4. $z_1 = \frac{3}{2} + \frac{1}{2}i, z_2 = -\frac{3}{2}$ |
| 5. $z_1 = 2 - 5i, z_2 = -3 + i$ | 6. $z_1 = 2 + 2i, z_2 = 3 - 3i$ |

In Exercises 7-12, given the complex numbers $z_1 = 2 - 4i$ and $z_2 = 2 + i$, find

- | | |
|---------------------------------|-----------------------------|
| 7. $3z_1$ | 8. $-\frac{2}{3}z_2$ |
| 9. $z_1 + 2z_2$ | 10. $\frac{1}{2}z_1 + 3z_2$ |
| 11. $\frac{3}{2}z_1 - z_2 + 2i$ | 12. $1 - \pi z_2$ |
-

1.3 Multiplication of Complex Numbers

The **multiplication** of two complex numbers is performed using all properties (commutative, associative, distributive) in use for the addition and multiplication of real numbers, and the fact that $i^2 = -1$.

Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, their product is obtained as follows:

$$\begin{aligned}(x_1 + iy_1) \cdot (x_2 + iy_2) &= x_1 \cdot x_2 + x_1 \cdot (iy_2) + (iy_1) \cdot x_2 + (iy_1) \cdot (iy_2) + \\ x_1 \cdot x_2 + i(x_1 \cdot y_2) + i(x_2 \cdot y_1) + i^2(y_1 \cdot y_2) &= (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1) \quad (9)\end{aligned}$$

In conclusion

$$\operatorname{Re}(z_1 \cdot z_2) = x_1 \cdot x_2 - y_1 \cdot y_2 \quad \text{and} \quad \operatorname{Im}(z_1 \cdot z_2) = x_1 \cdot y_2 + x_2 \cdot y_1$$

It can be easily seen that $z_1 \cdot z_2 = z_2 \cdot z_1$. Moreover, we can multiply any three complex numbers z_1, z_2, z_3 by performing successive multiplications of two complex numbers at a time, in any order, that is $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3 = (z_1 \cdot z_3) \cdot z_2$. Obviously, this can be extended to any number of complex numbers.

In particular, for any natural number n , the power z^n is obtained through multiplication of z by itself n times.

In the applications below, instead of using formula (9) we multiply any two complex numbers directly, using in each problem the common properties of addition and multiplication.

Example 3 - If $z_1 = 1 - i$, $z_2 = -3 + 2i$, find the rectangular form of

$$(a) z_1 \cdot z_2, \quad (b) z_1^2, \quad (c) z_1^3$$

Solution -

(a) Using the distributive property, the fact that $i^2 = -1$, and the associative property, we have that $z_1 \cdot z_2 = (1 - i) \cdot (-3 + 2i) = -3 + 2i + 3i - 2(i)^2 = (-3 + 2) + i(2 + 3) = -1 + 5i$

(b) Similarly, $z_1^2 = (1 - i) \cdot (1 - i) = 1 - 2i + i^2 = -2i$

(c) Using the result in part (b), $z_1^3 = (1 - i) \cdot (1 - i)^2 = (1 - i) \cdot (-2i) = -2i - 2$. \square

It is useful to notice that

$$i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = i \cdot i^2 = -i,$$

$$i^4 = i \cdot i^3 = -i^2 = 1, \quad i^5 = i, \quad i^6 = -1 \quad \dots$$

and all the subsequent powers of i repeat these same numbers again and again, so we can conclude that i^n , for $n = 0, 1, 2, \dots$, is equal to one of the numbers $1, i, -1, -i$. In particular, for $n \geq 4$, if q and r are the quotient and the remainder in the division by 4, namely $n = q \cdot 4 + r$, then

$$i^n = i^{q \cdot 4 + r} = (i^4)^q \cdot i^r = 1^q \cdot i^r = i^r \quad (10)$$

because $i^4 = 1$.

For example, $i^{95} = i^{4 \cdot 23 + 3} = i^3 = -i$.

1.3 EXERCISES

Find the rectangular form of the given complex numbers

- | | |
|--|---|
| 1. $(1 + i) \cdot (\sqrt{3} - 2i)$ | 2. $(1 + i) \cdot i$ |
| 3. $(-1 + i)^2$ | 4. $(2 + 2i)^2 \cdot (1 - i)$ |
| 5. i^{26} | 6. $(-i)^{49} \cdot (-1 + i)^3$ |
| 7. $(1 + i) \cdot (1 - i)$ | 8. $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$ |
| 9. $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^2 \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$ | 10. $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2$ |
-

1.4 Complex Conjugate and Division

Definition 2 - The **complex conjugate** of $z = x + iy$ is defined as $\bar{z} = x + i\bar{y} = x - iy$. The **modulus** of $z = x + iy$ is defined as $|z| = \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2}$.

From this definition, it follows that a complex number and its conjugate have the same real part and opposite imaginary parts

$$\operatorname{Re}(\bar{z}) = \operatorname{Re}(z) \quad \text{and} \quad \operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$$

We also have that $|z| \geq 0$.

Note that if $\operatorname{Im}(z) = 0$, so that z is a real number, then $|z|$ reduces to the absolute value of z .

Example 4 - Find the complex conjugate and the modulus of

$$(a) \quad -1 + \frac{3}{4}i, \quad (b) \quad -1, \quad (c) \quad -i, \quad (d) \quad 3 + 2i$$

Solution -

$$(a) \quad \overline{-1 + \frac{3}{4}i} = -1 - \frac{3}{4}i, \quad \text{and} \quad |-1 + \frac{3}{4}i| = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}.$$

$$(b) \quad \overline{-1} = -1, \quad \text{and} \quad |-1| = 1.$$

$$(c) \quad \overline{-i} = i, \quad \text{and} \quad |-i| = 1.$$

$$(d) \quad \overline{3 + 2i} = 3 - 2i, \quad \text{and} \quad |3 + 2i| = \sqrt{13}. \quad \square$$

Properties of the Complex Conjugate and the Modulus - Given any complex numbers z , z_1 , and z_2 , we have

1. $\overline{\overline{z}} = z$
2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
3. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
4. $z \cdot \overline{z} = |z|^2$
5. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
6. $|\operatorname{Re}(z)| \leq |z|$ and $|\operatorname{Im}(z)| \leq |z|$.

Proof. Throughout this proof, $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$

1. This property is true because if $z = x + iy$, then $\overline{\overline{z}} = \overline{x - iy} = x + iy = z$.
2. By adding z_1 and z_2 , we find $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, so $\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2}$, which proves Property 2.
3. From (9) we have $\overline{z_1 \cdot z_2} = \overline{(x_1 + iy_1) \cdot (x_2 + iy_2)} = \overline{(x_1 \cdot x_2 - y_1 \cdot y_2) - i(x_1 \cdot y_2 + x_2 \cdot y_1)} = (x_1 \cdot x_2 - (-y_1) \cdot (-y_2)) + i(x_1 \cdot (-y_2) + x_2 \cdot (-y_1)) = (x_1 - iy_1) \cdot (x_2 - iy_2) = \overline{z_1} \cdot \overline{z_2}$.
4. From the definition of conjugate complex number $z \cdot \overline{z} = (x + iy) \cdot (x - iy) = x^2 - (iy)^2 = x^2 + y^2$.
5. To prove this property it is enough to show that $|z_1 \cdot z_2|^2 = |z_1|^2 \cdot |z_2|^2$. Using Property 4, we have $|z_1 \cdot z_2|^2 = (z_1 \cdot z_2) \cdot \overline{(z_1 \cdot z_2)} = z_1 \cdot z_2 \cdot \overline{z_1} \cdot \overline{z_2} = z_1 \cdot \overline{z_1} \cdot z_2 \cdot \overline{z_2} = |z_1|^2 \cdot |z_2|^2$, which implies the wanted result.
6. This property is a consequence of the following inequalities $|\operatorname{Re}(z)| = |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |z|$ and $|\operatorname{Im}(z)| = |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} = |z|$. \square

As with the real numbers, the **reciprocal** of a complex number $z \neq 0$, denoted by $\frac{1}{z}$ or z^{-1} , is the complex number such that $z \cdot \frac{1}{z} = 1$. We now give a way to find the rectangular form of $\frac{1}{z}$ in terms of the real and imaginary parts of z using Property 4.

Multiplying and dividing $\frac{1}{z}$ by \overline{z} , the complex conjugate of z , we obtain

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{\overline{x + iy}}{(x + iy) \cdot \overline{(x + iy)}} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} = \frac{\overline{z}}{|z|^2} \quad (11)$$

and the general formulae for the real and imaginary parts are

$$\operatorname{Re}\left(\frac{1}{z}\right) = \frac{\operatorname{Re}(z)}{|z|^2} = \frac{\operatorname{Re}(z)}{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}, \quad \operatorname{Im}\left(\frac{1}{z}\right) = -\frac{\operatorname{Im}(z)}{|z|^2} = -\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$$

therefore

$$\frac{1}{z} = \frac{\operatorname{Re}(z)}{|z|^2} - i \frac{\operatorname{Im}(z)}{|z|^2} = \frac{\operatorname{Re}(z)}{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2} - i \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$$

Example 5 - Find the rectangular form of

$$(a) \frac{1}{i}, \quad (b) \frac{1}{1-i}, \quad (c) \frac{1}{-2+3i}$$

Solution -

(a) From (11), because $\bar{i} = -i$, and $|i| = 1$, then multiplying and dividing $\frac{1}{i}$ by $-i$, we obtain $\frac{1}{i} = -i$

(b) Given that, $\overline{1-i} = 1+i$, $|1-i| = \sqrt{2}$, then $\frac{1}{1-i} = \frac{1+i}{2} = \frac{1}{2} + i\frac{1}{2}$

(c) $\overline{-2+3i} = -2-3i$, $|-2+3i|^2 = 13$, so $\frac{1}{-2+3i} = \frac{-2-3i}{13} = -\frac{2}{13} - i\frac{3}{13}$ \square

We now define the **quotient** of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$ as the product of z_1 and z_2^{-1} . The result is denoted by $\frac{z_1}{z_2}$

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = z_1 \cdot \frac{\overline{z_2}}{|z_2|^2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i \frac{x_1y_2 - x_2y_1}{x_2^2 + y_2^2} \quad (12)$$

Cancellations can be made like with real numbers.

Example 6 - Find the rectangular form of $\frac{2-6i}{-1+i}$

Solution -

$$\begin{aligned} \frac{2-6i}{-1+i} &= (2-6i) \cdot (-1+i)^{-1} = (2-6i) \cdot \frac{(-1-i)}{|-1+i|^2} \\ &= (2-6i) \cdot \left(-\frac{1}{2} - i\frac{1}{2}\right) = -4 + 2i \end{aligned}$$

\square

1.4 EXERCISES

In Exercises 1-6, find (a) the conjugate and (b) the modulus of the given complex numbers

1. $2 - 2i$

2. $1 + i$

3. $-3 + 2i$

4. $\frac{3}{2} + \frac{1}{2}i$

5. $(2 - 5i)(-3 + i)$

6. $(1 - 2i)(1 + 2i)$

In Exercises 7-12, for each one of the given complex numbers, (a) find its rectangular form, (b) find its conjugate, and (c) find its modulus

7. $\frac{2}{\sqrt{2} - \sqrt{2}i}$

8. $-\frac{1}{3i}$

9. $\frac{1}{(1-i)^2}$

10. $\frac{1}{2}z_1 + 3z_2$

11. $\frac{1-i}{2+2i}$

12. $\frac{1+i}{1-i}$

2 The Geometry of Complex Numbers

2.1 Rectangular Coordinates and points in the Plane

The function that maps complex numbers z to the ordered pairs $(Re(z), Im(z))$ is one-to-one. Hence, by setting $x = Re(z)$ and $y = Im(z)$ we establish a one-to-one map between complex numbers z and points (x, y) in the x, y -plane, which enables us to speak of complex numbers as of points in the plane, and represent the set of complex numbers \mathbb{C} with the entire x, y -plane, called now the **complex plane**.

Because of this geometric interpretation, we can plot each complex number z with the point $(x, y) = (Re(z), Im(z))$ in the x, y -plane (see Figure 1), and $Re(z)$ and $Im(z)$ will be called **rectangular coordinates** of z .

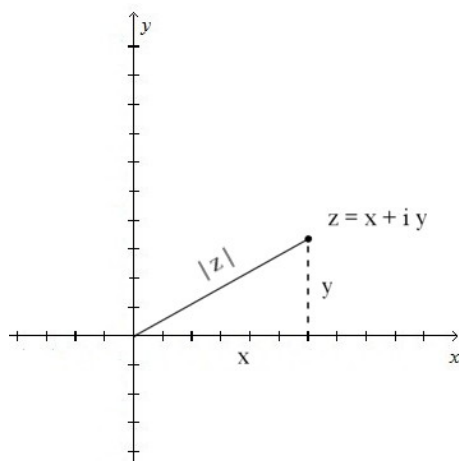


Figure 1: Complex numbers as points in the plane

The plots of complex numbers in the x,y -plane are called *Argand Diagrams* (see Figure 2 (a)). In such diagrams, the x -axis is called *real axis*, and the y -axis is called *imaginary axis*.

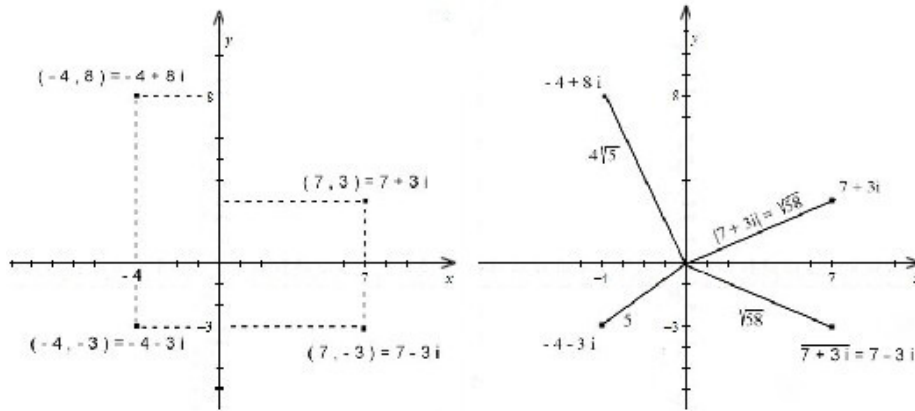


Figure 2: (a) Argand Diagram; (b) Modulus of Complex Numbers

The real numbers (complex numbers with a zero imaginary part) are plotted with points on the real axis, in particular $z = 1$ is plotted with $(1, 0)$. Imaginary numbers (complex numbers with a zero real part) are represented by points on the imaginary axis, in particular the imaginary unit $z = i$ is plotted with $(0, 1)$.

In the complex plane, the modulus of a complex number $z = x + iy$, defined as $|z| = \sqrt{x^2 + y^2}$, is therefore the distance of the point (x, y) from the origin $(0, 0)$, which we call the distance between z and 0 (see Figure 2 (b)).

2.1 EXERCISES

In Exercises 1-8, find the rectangular coordinates of the given complex numbers and plot them on the x,y -plane

- | | |
|-----------------------|------------------------|
| 1. -1 | 2. i |
| 3. $2 - 2i$ | 4. $1 + i$ |
| 5. $(2 - 5i)(-3 + i)$ | 6. $(1 - 2i)(1 + 2i)$ |
| 7. $(-1 + 2i)^2$ | 8. $\frac{20i}{3 + i}$ |
-

2.2 Complex Numbers in Polar Coordinates

In this section and in the next one we will use and often refer to the material covered by the sections 10.7 and 11.3 in the *Rogawski* textbook.

A point $P = (x, y)$ in the x,y -plane has polar coordinates (r, θ) , where r (*radial coordinate*) is the distance to the origin and θ (*angular coordinate*) is the angle between the positive x -axis and the segment \overline{OP} measured in the counterclockwise direction.

Therefore, we can assign to a complex number $z = x + iy$ the polar coordinates of $P = (x, y)$, and conversely, we can assign to a pair of polar coordinates (r, θ) the complex number $z = x + iy$, via the point $P = (x, y)$.

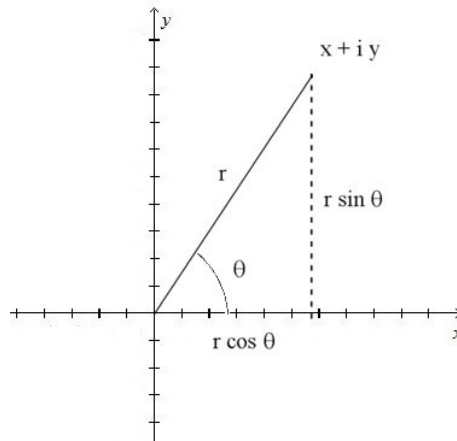


Figure 3: Polar Coordinates

Here we place the restriction $r > 0$. The angles θ are measured in radians.

The equations that allow to find the polar coordinates of a complex number z from its rectangular coordinates (x, y) are given below (see section 11.3, *Rogawski*).

If $z = x + iy \neq 0$, we set

$$r = |z| = \sqrt{x^2 + y^2} \quad (13)$$

$$\theta = \begin{cases} \tan^{-1} \frac{y}{x}, & \text{if } x > 0 \\ \tan^{-1} \frac{y}{x} + \pi, & \text{if } x < 0 \\ \pm \frac{\pi}{2}, & \text{if } x = 0 \end{cases} \quad (14)$$

If $z = 0$, then $r = 0$ and θ can be any angle.

As (13) shows, the radial coordinate of a complex number is its modulus.

Example 7 - Find the polar coordinates of $z = -2\sqrt{2} + 2\sqrt{2}i$.

Solution - From (13), we derive that $|z| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$, and because $\operatorname{Re}(z) < 0$ from (14) it follows that $\theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$, so the polar coordinates are $(4, \frac{3\pi}{4})$ (see Figure 4 below). \square

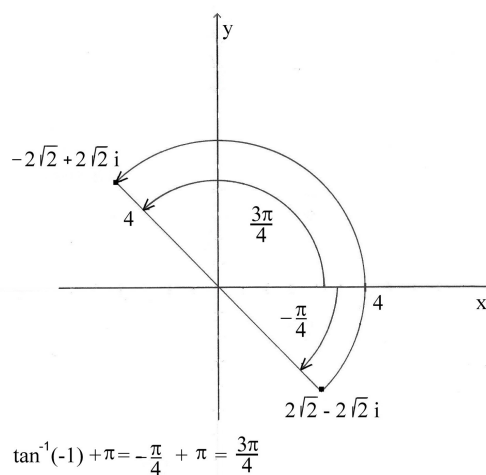


Figure 4: Polar coordinates of $-2\sqrt{2} + 2\sqrt{2}i$. To reflect its negative real part the angle θ is obtained adding π to $\tan^{-1}(-1) = -\frac{\pi}{4}$.

As with a point in the x, y -plane, the polar coordinates of a complex number are not unique. In fact, if (r, θ) are polar coordinates of z , also $(r, \theta + 2k\pi)$, for any $k = 0, \pm 1, \pm 2, \pm 3, \dots$, are polar coordinates of z . Because of that, we will

be concerned with finding one only pair of polar coordinates, and with an abuse of language we will refer to it as *the* polar coordinates of z .

Example 8 - Find the polar coordinates of

- (a) -1 , (b) $2i$, (c) $-\frac{3\sqrt{3}}{2} + \frac{3}{2}i$, (d) $-1 + i$.

Solution -

(a) -1 is plotted with the point $(-1, 0)$, which is located on the negative part of the x-axis at a distance 1 from the origin, so its polar coordinates are $(1, \pi)$.

(b) The complex number $2i$ is plotted with the point $(0, 2)$ located on the positive part of the imaginary axis at a distance 2 from the origin, so its polar coordinates are $(2, \frac{\pi}{2})$.

(c) Referring to *Example 1* from section 11.3 of the Rogawski textbook, the complex number $-\frac{3\sqrt{3}}{2} + \frac{3}{2}i$ is represented by the point $(-\frac{3\sqrt{3}}{2}, \frac{3}{2})$, so it has polar coordinates $(3, \frac{5\pi}{6})$.

(d) Because $-1 + i$ is plotted with the point $(-1, 1)$, then from *Example 3*, section 11.3 of the Rogawski textbook, it follows that its polar coordinates are $(\sqrt{2}, -\frac{\pi}{4})$ \square

As a consequence of (11) and the fact that the conjugate \bar{z} of a complex number z is its reflection in the x-axis, if z has polar coordinates (r, θ) , then \bar{z} has polar coordinates $(r, -\theta)$. For instance, using the example 8(c), one can state that polar coordinates of $-\frac{3\sqrt{3}}{2} - \frac{3}{2}i$ are $(3, -\frac{5\pi}{6})$.

2.2 EXERCISES

Find the polar coordinates of the given complex number

1. 5π

2. $-\pi i$

3. $\sqrt{3} - i$

4. $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

5. $-2 + 2\sqrt{3}i$

6. $-1 - i$

2.3 The Exponential Notation - Euler's Formula

The equations that allow to pass from the polar coordinates (r, θ) to the rectangular coordinates (x, y) of a complex number z are (see section 11.3, *Rogawski*)

$$x = r \cos \theta = |z| \cos \theta \qquad y = r \sin \theta = |z| \sin \theta \qquad (15)$$

For a complex number that has rectangular form $z = x + iy$, using (15) we derive the so called **polar form** of z

$$z = r(\cos \theta + i \sin \theta) = |z|(\cos \theta + i \sin \theta) \qquad (16)$$

Like the polar coordinates, the polar form of a complex number is not unique.

Example 9 - Find the polar form of

$$(a) \quad -2\sqrt{2} + 2\sqrt{2}i, \quad (b) \quad 3i, \quad (c) \quad 2\sqrt{3} - 2i, \quad (d) \quad -2$$

Solution -

(a) From *Example 7* it follows that the polar coordinates of $-2\sqrt{2} + 2\sqrt{2}i$ are $(4, \frac{3\pi}{4})$, and from (16) we have that $-2\sqrt{2} + 2\sqrt{2}i = 4(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$.

(b) The polar coordinates of $3i$ are $(3, \frac{\pi}{2})$, so $3i = 3(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$

(c) Because $|2\sqrt{3} - 2i| = 4$ and the angular coordinate of $2\sqrt{3} - 2i$ is $\theta = \tan^{-1}(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$, then $2\sqrt{3} - 2i = 4(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))$

(d) The polar coordinates of -2 are $(2, \pi)$, therefore $-2 = 2(\cos \pi + i \sin \pi)$. \square

The Taylor expansion of e^x , for all real numbers x , is given by (see Rogawski, section 10.7)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (17)$$

We now use (17) as starting point to define the exponential function of the complex variable z

Definition 3 - For any complex number z , the exponential function is defined as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots \quad (18)$$

EULER'S FORMULA - A complex number z with polar coordinates (r, θ) can be expressed as

$$z = re^{i\theta} \quad \text{or} \quad z = |z|e^{i\theta} \quad (19)$$

called the **exponential form** of z .

Proof. Writing (18) for $z = i\theta$, and using the recurrence of the powers of the imaginary unit (10), we have

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\ &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \end{aligned} \quad (20)$$

In (20) only the odd powers of θ are multiplied by i .

We recall now that the Taylor expansions of $\cos \theta$ and $\sin \theta$, for any real number θ , are (see Example 2, section 10.7 Rogawski)

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \quad (21)$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \quad (22)$$

Multiplying (22) by i

$$i \sin \theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \dots \quad (23)$$

and adding (23) to (21) we obtain that $\cos \theta + i \sin \theta$ is equal to the right end side of (20), therefore

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (24)$$

Euler's Formula (19) follows from (24) and the polar form of a complex number (16). \square

It is worth noting that, on one hand, because of its own definition (18), e^z is equal to the real valued function e^x for $z = x$ real, on the other hand e^z is a true extension of e^x because its outputs are all complex numbers. In particular, negative real numbers can be outputs of e^z but not of e^x . For example, from (24) written for $z = i\pi$ we have $e^{i\pi} = -1$ (see also example 10d below).

We should also point out that in (19) the coefficient of the exponential function is the modulus of z , which cannot be negative, therefore an expression like $-e^{i\theta}$ is not acceptable as exponential form of a complex number z . Using the example mentioned above, the exponential form of -1 can only be $e^{i\pi}$, where the minus sign of -1 results from $\cos \pi = -1$ in (24).

Example 10 - Find the exponential form of

$$(a) -2\sqrt{2} + 2\sqrt{2}i, \quad (b) 3i, \quad (c) 2\sqrt{3} - 2i, \quad (d) -2$$

Solution - Based on the results of Example 9 and on the Euler's formula (19), we have

$$(a) -2\sqrt{2} + 2\sqrt{2}i = 4e^{\frac{3\pi}{4}i}. \quad (b) 3i = 3e^{\frac{\pi}{2}i} \quad (c) 2\sqrt{3} - 2i = 4e^{-\frac{\pi}{6}i} \quad (d) -2 = 2e^{\pi i}. \quad \square$$

2.3 EXERCISES

In Exercises 1-6, find the rectangular form of the complex number with the given polar coordinates

- | | |
|---|---|
| 1. $\left(2, \frac{\pi}{3}\right)$ | 2. $\left(4, \frac{\pi}{4}\right)$ |
| 3. $\left(\pi, \frac{3\pi}{2}\right)$ | 4. (π, π) |
| 5. $\left(\frac{2}{3}, \frac{3\pi}{4}\right)$ | 6. $\left(\frac{1}{2}, \frac{7\pi}{6}\right)$ |

In Exercises 7-12, write the given complex number in rectangular form

- | | |
|-------------------------------------|-------------------------------|
| 7. $2e^{i\frac{\pi}{4}}$ | 8. $5e^{i\frac{\pi}{3}}$ |
| 9. $\frac{1}{3}e^{-i\frac{\pi}{3}}$ | 10. $\pi e^{-i\frac{\pi}{4}}$ |
| 11. $6e^{i\frac{3\pi}{4}}$ | 12. $3e^{i\frac{5\pi}{4}}$ |

In Exercises 13-18, find the polar form of the given complex number

- | | |
|----------------------|---|
| 13. 5 | 14. $-\pi i$ |
| 15. $\sqrt{3} - i$ | 16. $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ |
| 17. $2 + 2\sqrt{3}i$ | 18. $-1 - i$ |

In Exercises 19-24, find the exponential form of the given complex number

- | | |
|----------------------|-----------------------------|
| 19. -3 | 20. $-2i$ |
| 21. $-\sqrt{3} + i$ | 22. $-\sqrt{2} - \sqrt{2}i$ |
| 23. $2 + 2\sqrt{3}i$ | 24. $-1 + i$ |

2.4 The Product and the Quotient of Complex Numbers in Polar Coordinates - De Moivre's Formulas

If the complex numbers z_1 and z_2 have respective polar coordinates (r_1, θ_1) and (r_2, θ_2) , then

$$z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

By expanding and then using the addition and subtraction formulas for sine and cosine, we have that

$$\begin{aligned} (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) &= \\ &= \cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \sin \theta_2 \cos \theta_1 - \sin \theta_1 \sin \theta_2 \\ &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2), \end{aligned}$$

therefore

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right) \quad (25)$$

Formula (25) means that the product of two complex numbers z_1 and z_2 has for radial coordinate the product of the radial coordinates of z_1 and z_2 , and for angular coordinate the sum of the angular coordinates of z_1 and z_2 .

The formula for the product of two complex numbers in exponential form follows from (24) and (25). If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

but this same formula can also be obtained via the straightforward use of the properties of the exponential function (the same properties we are accustomed to for e^x hold for e^z): $z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{(i\theta_1 + i\theta_2)} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.

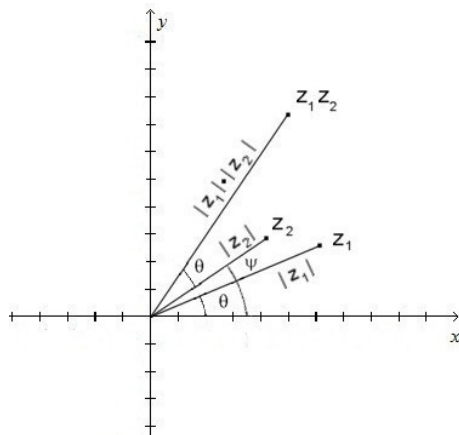


Figure 5: The Product in Polar Coordinates

It is now possible to derive from (25) the formula for the quotient of two complex numbers z_1 and z_2 ($z_2 \neq 0$) with respective polar coordinates (r_1, θ_1) and (r_2, θ_2) .

Because $\frac{1}{z_2} = \frac{\bar{z}_2}{|z_2|^2} = \frac{\bar{z}_2}{(r_2)^2}$ and because from (11) \bar{z}_2 has polar coordinates $(r_2, -\theta_2)$, therefore $\frac{1}{z_2}$ has polar coordinates $(\frac{r_2}{(r_2)^2}, -\theta_2) = (\frac{1}{r_2}, -\theta_2)$. From this fact and from (25) we can state that the quotient of two complex numbers z_1 and z_2 has for radial coordinate the quotient of the radial coordinates of z_1 and z_2 , and for angular coordinate the difference between the angular coordinates of z_1 and z_2 .

This can be expressed using the polar and the exponential forms as follows

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right) = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \quad (26)$$

Example 11 - Use (25) and (26) to find the polar coordinates

$$(a) (-3) \cdot i, \quad (b) \frac{-3}{i}, \quad (c) \left(\frac{\sqrt{3}}{4} - \frac{1}{4}i \right) \cdot (2 + 2\sqrt{3}i) \quad (d) \frac{\frac{\sqrt{3}}{4} - \frac{1}{4}i}{2 + 2\sqrt{3}i}$$

then verify your results first by finding the rectangular form of the product and quotient, and then deriving their polar coordinates using (13) and (14).

Solution -

(a) The polar coordinates of -3 and i are respectively $(3, \pi)$ and $(1, \frac{\pi}{2})$, therefore, from (25) we have that the polar coordinates of $(-3) \cdot i$ are $(3, \pi + \frac{\pi}{2}) = (3, \frac{3\pi}{2})$. In order to verify this result by proceeding directly, we find that $-3i$ has modulus 3, and from (14) we derive that its angular coordinate is $-\frac{\pi}{2}$. This implies that the polar coordinates are $(3, -\frac{\pi}{2})$, which is in accordance with the result found above, because $(3, \frac{3\pi}{2})$ and $(3, -\frac{\pi}{2})$ represent the same complex number since the angles differ by 2π .

(b) From (26) we have that the polar coordinates of $\frac{-3}{i}$ are $(3, \pi - \frac{\pi}{2}) = (3, \frac{\pi}{2})$. To verify this, we proceed like in section 1.3 to find that the rectangular form $\frac{-3}{i} = 3i$, which indeed has polar coordinates $(3, \frac{\pi}{2})$.

(c) The polar coordinates of $\frac{\sqrt{3}}{4} - \frac{1}{4}i$ are $(\frac{1}{2}, \tan^{-1}(-\frac{1}{\sqrt{3}})) = (\frac{1}{2}, -\frac{\pi}{6})$. On the other hand, the polar coordinates of $2 + 2\sqrt{3}i$ are $(4, \tan^{-1}(\sqrt{3})) = (4, \frac{\pi}{3})$. According to (25), the product has for polar coordinates $(\frac{1}{2} \cdot 4, -\frac{\pi}{6} + \frac{\pi}{3}) = (2, \frac{\pi}{6})$. To check our result, we perform the product directly

$$(\frac{\sqrt{3}}{4} - \frac{1}{4}i)(2 + 2\sqrt{3}i) = \frac{\sqrt{3}}{4} \cdot 2 + \frac{\sqrt{3}}{4} \cdot 2\sqrt{3}i - 2 \cdot \frac{1}{4}i - \frac{1}{4} \cdot 2\sqrt{3}i^2 = \sqrt{3} + i$$

and we can verify that the polar coordinates of this complex number are $(2, \tan^{-1} \frac{1}{\sqrt{3}}) = (2, \frac{\pi}{6})$, which confirms the result found above.

(d) From (26), it follows that the polar coordinates of $\frac{\frac{\sqrt{3}}{4} - \frac{1}{4}i}{2 + 2\sqrt{3}i}$ are $(\frac{1}{4}, -\frac{\pi}{6} - \frac{\pi}{3}) = (\frac{1}{8}, -\frac{\pi}{2})$.

To check our result, we perform the quotient directly

$$\frac{\frac{\sqrt{3}}{4} - \frac{1}{4}i}{2 + 2\sqrt{3}i} = \frac{(\frac{\sqrt{3}}{4} - \frac{1}{4}i)(2 - 2\sqrt{3}i)}{(2 + 2\sqrt{3}i)(2 - 2\sqrt{3}i)} = -\frac{2i}{16} = -\frac{1}{8}i$$

which indeed has polar coordinates $(\frac{1}{8}, -\frac{\pi}{2})$. □

From (25) and (26) we derive the

DE MOIVRE'S FORMULAS - If $z \neq 0$ has polar coordinates $(|z|, \theta)$, then for any integer $n \geq 0$

$$z^n = |z|^n e^{in\theta} = |z|^n (\cos n\theta + i \sin n\theta) \quad (27)$$

and

$$z^{-n} = |z|^{-n} e^{-in\theta} = |z|^{-n} (\cos n\theta - i \sin n\theta). \quad (28)$$

In particular, if $z = \cos \theta + i \sin \theta$ (i.e. $|z| = 1$), we have

$$(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)$$

$$(\cos \theta + i \sin \theta)^{-n} = (\cos n\theta - i \sin n\theta).$$

Example 12 - Find the (a) exponential, (b) polar, and (c) rectangular forms of $(1+i)^3$ and $(1+i)^{-3}$.

Solution -

(a) $1+i$ has polar coordinates $(\sqrt{2}, \frac{\pi}{4})$, therefore the exponential forms are $(1+i)^3 = (\sqrt{2})^3 e^{\frac{3\pi}{4}i}$, and $(1+i)^{-3} = (\sqrt{2})^{-3} e^{-\frac{3\pi}{4}i}$.

(b) From (a) it follows that $(1+i)^3 = 2\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$ and $(1+i)^{-3} = \frac{\sqrt{2}}{4}(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4})$.

(c) From (b) it follows that $(1+i)^3 = 2\sqrt{2}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = -2+2i$, and $(1+i)^{-3} = 2\sqrt{2}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = -\frac{1}{4} - \frac{1}{4}i$. \square

2.4 EXERCISES

In Exercises 1-6, write the given complex number in rectangular form

- | | |
|--|---|
| 1. $2e^{i\frac{\pi}{6}} e^{i\frac{\pi}{3}}$ | 2. $\frac{1}{3}e^{i\frac{\pi}{4}} e^{i\frac{\pi}{2}}$ |
| 3. $\frac{2e^{i\frac{\pi}{4}}}{3e^{i\frac{\pi}{2}}}$ | 4. $\frac{3e^{i\frac{\pi}{6}}}{2e^{i\frac{\pi}{3}}}$ |
| 5. $(2e^{i\frac{\pi}{3}})^3$ | 6. $(\frac{1}{\sqrt{2}}e^{i\frac{\pi}{6}})^{-3}$ |

In Exercises 7-12, find the exponential form of (a) $z_1 z_2$, and (b) $\frac{z_1}{z_2}$, where

- | | |
|---|--|
| 7. $z_1 = 2(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$, $z_2 = (\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$ | 8. $z_1 = 3e^{i\pi}$, $z_2 = \frac{1}{2}e^{i\frac{\pi}{3}}$ |
| 9. $z_1 = 3(\cos \pi + i \sin \pi)$, $z_2 = 2(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3})$ | 10. $z_1 = e^{i\frac{\pi}{4}}$, $z_2 = 2e^{i\frac{\pi}{3}}$ |
| 11. $z_1 = (\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}))$, $z_2 = \sqrt{5}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$ | 12. $z_1 = -2$, $z_2 = e^{i\frac{\pi}{6}}$ |

2.5 Roots of Complex Numbers

In this section, at first we need to take into account all possible polar coordinates of a complex number. If θ is the angular coordinate of z in $[0, 2\pi)$, we consider all polar coordinates $(r, \theta + 2k\pi)$, for $k = 0, \pm 1, \pm 2, \pm 3, \dots$

Let n be any natural number (aka *counting number*: 1, 2, 3, ...).

Definition 4 - The n^{th} roots of a complex number z are all complex numbers ζ such that $\zeta^n = z$. They will be denoted by $\zeta = z^{\frac{1}{n}}$ or $\zeta = \sqrt[n]{z}$

Given z with polar coordinates $(|z|, \theta + 2k\pi)$, for $k = 0, \pm 1, \pm 2, \pm 3, \dots$, our aim is now to find the polar coordinates of its n^{th} roots. Those roots by definition are complex numbers ζ such that $\zeta^n = z$. If we call (ρ, ψ) the polar coordinates of ζ , then using the De Moivre's Formula (27), we have that ζ^n has polar coordinates $(\rho^n, n\psi)$. From $\zeta^n = z$ it follows that $(\rho^n, n\psi)$ and $(|z|, \theta + 2k\pi)$ must coincide. In particular

$$\rho^n = |z|, \quad n\psi = \theta + 2k\pi, \quad \text{for } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

that yield

$$\rho = |z|^{\frac{1}{n}} = \sqrt[n]{|z|}, \quad \psi = \frac{\theta + 2k\pi}{n}, \quad \text{for } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

We conclude that the polar coordinates of the n^{th} roots of z have polar coordinates

$$(\rho, \psi) = \left(|z|^{\frac{1}{n}}, \frac{\theta + 2k\pi}{n} \right), \quad \text{for } k = 0, \pm 1, \pm 2, \pm 3, \dots \quad (29)$$

Here $|z|^{\frac{1}{n}} = \sqrt[n]{|z|}$ denotes the unique real n^{th} root of the real number $|z|$, which exists in \mathbb{R} for any n because $|z| \geq 0$.

Example 13 - Find the polar coordinates of the 5^{th} roots of $z = 1 + i$.

Solution - The polar coordinates of $z = 1 + i$ are $(\sqrt{2}, \frac{\pi}{4})$. From (29) we have that its 5^{th} roots have polar coordinates $\left(\sqrt[5]{\sqrt{2}}, \frac{\pi/4 + 2k\pi}{5} \right) = \left(\sqrt[10]{2}, \frac{\pi}{20} + \frac{2k\pi}{5} \right)$, for $k = 0, \pm 1, \pm 2, \pm 3, \dots$ \square

Remark 1 - For $z \neq 0$, from (29) we obtain $\psi = \frac{\theta}{n}$ for $k = 0$, and $\frac{\theta}{n} + 2\pi$ for $k = n$, therefore the corresponding points in the plane coincide and so do the respective complex numbers. We can repeat the same reasoning for any multiple of n , $k = hn$, and affirm that all n^{th} roots obtained in (29) with $k = hn$ coincide with the one obtained with $k = 0$.

It is possible to verify a similar occurrence for each $k = 0, 1, \dots, n-1$. Namely, all angles ψ obtained in (29) for all integers $k = 0, \pm 1, \pm 2, \pm 3, \dots$ differ from the n angles obtained for $k = 0, 1, \dots, n-1$ by multiples of 2π .

Geometrically, this implies that only n distinct points are obtained by plotting in the plane the complex numbers with polar coordinates given in (29) with all possible values $k = 0, \pm 1, \pm 2, \pm 3, \dots$. The same n distinct points in the plane can be obtained by choosing for k only the n values $k = 0, 1, \dots, n-1$. \square

We can now state the following result

Theorem - Any complex number $z \neq 0$ with polar coordinates $(|z|, \theta)$ has exactly n distinct n^{th} roots whose polar coordinates are

$$\left(|z|^{\frac{1}{n}}, \frac{\theta}{n} + \frac{2k\pi}{n} \right), \quad \text{for } k = 0, 1, \dots, n-1. \quad (30)$$

Their corresponding polar form is then

$$z^{\frac{1}{n}} = |z|^{\frac{1}{n}} \left(\cos \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) \right), \quad \text{for } k = 0, 1, \dots, n-1. \quad (31)$$

Example 14 - Find the rectangular form of the two distinct 2^{nd} (square) roots of -1 .

Solution - The polar coordinates of -1 are $(|-1|, \pi)$. It follows that the radial coordinate of $\sqrt{-1}$ is equal to 1. The angular coordinates that will yield the two distinct square roots are obtained from (30) with $n = 2$, $\theta = \pi$, and for $k = 0, 1$, namely

$$\left(1, \frac{\pi}{2} + k\pi\right), \text{ for } k = 0, 1$$

that is

$$\left(1, \frac{\pi}{2}\right) \quad \text{and} \quad \left(1, \frac{3\pi}{2}\right)$$

Using (16), it follows that the square roots of -1 in rectangular form are

$$z_1 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \quad \text{and} \quad z_2 = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i \quad \square$$

Example 15 - Find the rectangular form of the three distinct 3^{rd} (cubic) roots of $z = -2 - 2i\sqrt{3}$.

Solution - $|z| = \sqrt{4 + 12} = 4$. From (14), because $Re(z) = -2 < 0$, then the angular coordinate of z is $\theta = \tan^{-1} \sqrt{3} + \pi = \frac{4\pi}{3}$.

From (30) it follows that the *cubic* roots of z have polar coordinates

$$\left(\sqrt[3]{4}, \frac{4\pi}{3} + \frac{2k\pi}{3}\right), \quad k = 0, 1, 2$$

therefore, their rectangular forms are

$$\begin{aligned} z_1 &= \sqrt[3]{4} \left(\cos \left(\frac{4\pi}{3} \right) + i \sin \left(\frac{4\pi}{3} \right) \right) \\ &\approx 0.276 + 1.563i \\ z_2 &= \sqrt[3]{4} \left(\cos \left(\frac{4\pi}{3} + \frac{2\pi}{3} \right) + i \sin \left(\frac{4\pi}{3} + \frac{2\pi}{3} \right) \right) = \sqrt[3]{4} \left(\cos \left(\frac{10\pi}{3} \right) + i \sin \left(\frac{10\pi}{3} \right) \right) \\ &\approx -1.492 - 0.543i \\ z_3 &= \sqrt[3]{4} \left(\cos \left(\frac{4\pi}{3} + \frac{4\pi}{3} \right) + i \sin \left(\frac{4\pi}{3} + \frac{4\pi}{3} \right) \right) = \sqrt[3]{4} \left(\cos \left(\frac{16\pi}{3} \right) + i \sin \left(\frac{16\pi}{3} \right) \right) \\ &\approx 1.216 - 1.02i \quad \square \end{aligned}$$

Remark 2 - In the set \mathbb{R} , the n^{th} roots of a real number are at most two. If n is odd, there is only one root, for example, $\sqrt[3]{-8} = -2$ is the only real number that satisfies $(-2)^3 = -8$. If n is even, there are two, one, or no roots, depending on whether the number is positive, zero, or negative. For example for $n = 4$, $\pm \sqrt[4]{16} = \pm 2$ are the only real numbers that solve $x^4 = 16$, while 0 is the

only real number that solves $x^4 = 0$, and there is no real number that solves $x^4 = -16$.

In the set \mathbb{C} of the complex numbers, the situation is very different. As previously seen, any complex number $z \neq 0$, has n different n^{th} roots. This applies also to the complex numbers with a zero imaginary part, namely the real numbers: any real number has n distinct n^{th} roots in \mathbb{C} , contrary to what happens in \mathbb{R} .

Example 16 - Find the rectangular form of the 6 distinct 6^{th} roots of 1.

Solution - The polar coordinates of 1 are $(1, 0)$, hence from (31)

$$\sqrt[6]{1} = \cos\left(0 + \frac{2k\pi}{6}\right) + i \sin\left(0 + \frac{2k\pi}{6}\right), \quad k = 0, 1, 2, 3, 4, 5$$

therefore the six distinct 6^{th} roots of 1 are

$$\begin{aligned} z_1 &= \cos 0 + i \sin 0 = 1, & \text{for } k = 0 \\ z_2 &= \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i, & \text{for } k = 1 \\ z_3 &= \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, & \text{for } k = 2 \\ z_4 &= \cos \pi + i \sin \pi = -1, & \text{for } k = 3 \\ z_5 &= \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, & \text{for } k = 4 \\ z_6 &= \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i, & \text{for } k = 5 \end{aligned}$$

□

Remark 3 - Formula (31) shows that the points in the x,y-plane representing the n^{th} distinct roots of a complex number are located on a circle of radius $|z|^{\frac{1}{n}}$ centered at $(0, 0)$. In addition, these points, for $k = 0, 1, \dots, n-1$, have angular coordinates that differ by the same angle $\frac{2\pi}{n}$. This means that these points are at the vertexes of a regular n-sided polygon inscribed in the circle of radius $|z|^{\frac{1}{n}}$ centered at the origin.

Example 17 - Find the rectangular form of the three distinct 3^{rd} roots of $8i$.

Solution - The polar coordinates of $8i$ are $(8, \frac{\pi}{2})$. Using (30), we find that the polar coordinates of its cubic roots are

$$\left(8^{\frac{1}{3}}, \frac{\frac{\pi}{2}}{3} + \frac{2k\pi}{3}\right), \quad k = 0, 1, 2$$

that is

$$\left(2, \frac{\pi}{6}\right), \quad \left(2, \frac{5\pi}{6}\right), \quad \left(2, \frac{3\pi}{2}\right)$$

It follows that

$$z_1 = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$z_2 = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$z_3 = 2\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right) = -2i$$

□

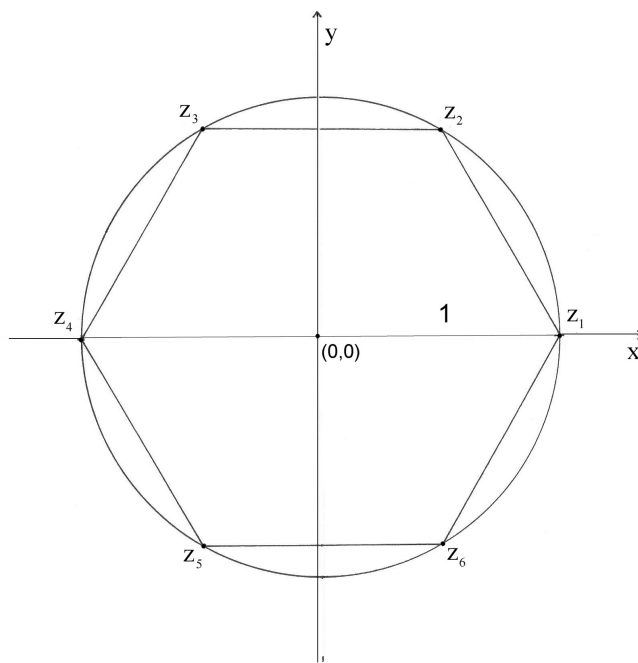


Figure 6: The 6th roots of 1

Example 18 - Solve the equations

(a) $z^2 + 1 = 0$, (b) $z^4 + 16 = 0$, (c) $z^6 - 1 = 0$, (d) $z^3 - 8i = 0$

Solution - (a) The solutions are the complex numbers z that satisfy $z^2 = -1$, namely $z = \sqrt{-1}$, the square roots of -1 . From *Example 14*, it follows that $z = \pm i$.

(b) Similarly, the solutions to $z^4 + 16 = 0$ are the complex numbers $z = \sqrt[4]{-16}$, that is the 4th roots of -16 .

Because -16 has polar coordinates $(16, \pi)$, from (30) it follows that the polar coordinates of $\sqrt[4]{-16}$ are

$$\left(2, \frac{\pi}{4} + \frac{k\pi}{2}\right) \quad \text{for } k = 0, 1, 2, 3$$

therefore the solutions to equation (b) are

$$\begin{aligned} z_1 &= 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2} \quad \text{for } k = 0 \\ z_2 &= 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\sqrt{2} + i\sqrt{2}, \quad \text{for } k = 1 \\ z_3 &= 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}, \quad \text{for } k = 2 \\ z_4 &= 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2} - i\sqrt{2}, \quad \text{for } k = 3 \end{aligned}$$

(c) The solutions to the equation $z^6 - 1 = 0$ are the complex numbers $z = \sqrt[6]{1}$, that is the 6th roots of 1, which have been found in *Example 16*:

$$z_1 = 1, z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, z_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, z_4 = -1, z_5 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, z_6 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

(d) The solutions to the equation $z^3 - 8i = 0$ are the complex numbers $z = \sqrt[3]{8i}$, that is the cubic roots of $8i$, which have been found in *Example 17*:

$$z_1 = \sqrt{3} + i, z_2 = -\sqrt{3} + i, z_3 = -2i \quad \square$$

So far, we have solved equations in one variable z of the form $z^n + w = 0$, where n is a given natural number and w a given complex number. Because z has to satisfy $z^n = -w$, the solutions are found to be $z = \sqrt[n]{-w}$, namely the n complex roots of $-w$.

Here we are interested in solving polynomial equations of a more general form, and for that we need to introduce a few general notions and results.

A *polynomial* in one variable z is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0 \quad (32)$$

where the constants $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are called the *coefficients*, n the *degree*, a_0 the *constant term*, and a_n the *leading coefficient* of $P(z)$.

A polynomial with real coefficients a_i , $i = 0, \dots, n$, is called *real polynomial*. Here we will limit our attention to real polynomials, and from now on we will simply refer to those as *polynomials*.

A number ζ is called *root* of the polynomial $P(z)$ if $P(\zeta) = 0$, that is if ζ is *solution* to the polynomial equation $P(z) = 0$.

The equation $P(z) = 0$ is said to have *degree* n if the polynomial $P(z)$ has degree n . In particular we use the words *linear*, *quadratic*, *cubic*, *etc* for both equations and polynomials with degree 1, 2, 3, etc.

Complex numbers are essential in solving polynomial equations. In fact, as noticed at the very beginning of these lecture notes, the equation (1)

$$z^2 + 1 = 0$$

has no solutions in \mathbb{R} , while its solutions in \mathbb{C} are $z = \pm i$.

The following result gives the exact number of roots of a polynomial (or solutions to a polynomial equation)

THE FUNDAMENTAL THEOREM OF ALGEBRA - Any polynomial equation $P(z) = 0$ of degree $n \geq 1$ has exactly n solutions (some of which may be repeated).

For example, the only real solutions to the equation $z^6 - 1 = 0$, are $z = \pm 1$. Then, the *Fundamental Theorem of Algebra* tells us that there must be other four non-real solutions. These complex numbers are provided in *Example 18c*.

Remark 4 - The common algebraic operations with polynomials in the real variable x , including the polynomial division and factorization, still work for polynomials in the complex variable z . As a consequence, also the quadratic formula that expresses the solutions to a quadratic equation is valid. In the few examples below, we will make use of these tools.

Example 19 - Solve the equations

$$\begin{array}{ll} (a) z^5 - z^2 = 0 & (b) z^4 + 2z^2 + 1 = 0 \\ (c) z^2 + 4z + 13 = 0 & (d) z^5 - 3z^4 + 3z^3 - 3z^2 + 2z = 0 \end{array}$$

Solution - (a) By factoring out z^2 , we derive that the solutions are given by both the solutions to $z^2 = 0$ and $z^3 - 1 = 0$. The equation $z^2 = 0$ has one only solution $z = 0$ repeated twice. The equation $z^3 - 1 = 0$ is solved by finding the cubic roots of 1, which are $z_1 = 1$, $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

(b) Because $z^4 + 2z^2 + 1 = (z^2 + 1)^2$, and because the solutions to $z^2 + 1 = 0$ are $z = \pm i$, then each of i and $-i$ is repeated twice.

(c) Using the quadratic formula we find that the solutions are $z = 2 \pm 3i$.

(d) By splitting the term $3z^3 = 2z^3 + z^3$, we have that $z^5 - 3z^4 + 3z^3 - 3z^2 + 2z = z^5 - 3z^4 + 2z^3 + z^3 - 3z^2 + 2z = z^3(z^2 - 3z + 2) + z(z^2 - 3z + 2) = (z^3 + z)(z^2 - 3z + 2) = z(z^2 + 1)(z^2 - 3z + 2) = 0$, therefore the solutions are given by the roots of each factor. The roots of the first two factors are $0, \pm i$, and the roots of $z^2 - 3z + 2$ are $z = 1, 2$, which can be found using the quadratic formula. \square

The following result on real polynomials is fundamental

Complex Conjugate Root Theorem - If $P(z)$ is a polynomial in one variable with real coefficients, and $\zeta = \alpha + \beta i$ is root of $P(z)$ with α and β real numbers, then its complex conjugate $\bar{\zeta} = \alpha - \beta i$ is also root of $P(z)$.

In short, complex roots of a real polynomial must occur in conjugate pairs. For example, if $-3 - 10i$ is solution to an equation, then also $-3 + 10i$ is solution.

Example 19 has also provided us with a few examples of this result.

Here are few of the many consequences of the *Complex Conjugate Root Theorem* for real polynomial equations

- If a cubic equation has two real solutions, the third one must be real too.
 - No equation can have an odd number of complex (with non zero imaginary part) solutions.
 - An equation of odd degree must have at least one real solution
 - An equation of even degree cannot have an odd number of real solutions. - If $\alpha + i\beta$ is root of a polynomial, then the polynomial is divisible by $(z - (\alpha + i\beta))(z - (\alpha - i\beta)) = ((z - \alpha)^2 + \beta^2) = (z^2 - 2\alpha z + \beta^2)$.
-

2.5 EXERCISES

In Exercises 1-4, find the polar coordinates of the distinct n^{th} roots of the complex number whose polar coordinates are given

- | | |
|---|---|
| 1. $n = 3, \left(5, \frac{\pi}{4}\right)$ | 2. $n = 4, \left(7, \frac{\pi}{4}\right)$ |
| 3. $n = 5, \left(1, \frac{\pi}{3}\right)$ | 4. $n = 2, \left(1, \frac{\pi}{6}\right)$ |

In Exercises 5-10, find the rectangular form of the n^{th} roots of the given complex number

- | | |
|---------------------|-------------------------|
| 5. $n = 3, z = i$ | 6. $n = 3, z = -i$ |
| 7. $n = 4, z = -16$ | 8. $n = 3, z = 8$ |
| 9. $n = 6, z = -1$ | 10. $n = 4, z = -1 + i$ |

In Exercises 11-14, find the rectangular form of all the complex numbers that solve the given equation

- | | |
|-------------------|-------------------|
| 11. $z^4 + 1 = 0$ | 12. $z^3 + 8 = 0$ |
| 13. $z^3 - 8 = 0$ | 14. $z^6 + 1 = 0$ |

In Exercises 15-18, find all solutions to the given equation, knowing that it has the given solution

- | | |
|--|--|
| 15. $z^3 - 4z^4 + z + 26; z = -2$ | 16. $z^4 + z^3 + 4z^2 + 4z = 0; z = -1$ |
| 17. $z^3 + z^2 - z + 15 = 0; z = 1 + 2i$ | 18. $z^4 - 2z^3 - 2z^2 + 8 = 0; z = 1 + i$ |

ANSWERS TO ODD-NUMBERED EXERCISES

Section 1.1

1. (a) 2, (b) -2 ; 3. (a) 1, (b) 0; 5. (a) 0, (b) -1

7. $2 + i$; 9. $\frac{3}{2}$; 11. $-\frac{5}{3} - \frac{2}{3}i$

Section 1.2

1. (a) 5, (b) $-1 - 4i$; 3. (a) $-2\sqrt{2} + i$, (b) $4\sqrt{2} - i$; 5. (a) $-1 - 4i$, (b) $5 - 6i$

7. $6 - 12i$; 9. $6 - 2i$; 11. $1 - 5i$

Section 1.3

1. $(\sqrt{3} + 2) + (\sqrt{3} - 2)i$; 3. $-2i$; 5. -1 ; 7. 2; 9. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

Section 1.4

1. (a) $2 + 2i$, (b) $2\sqrt{2}$; 3. (a) $-3 - 2i$, (b) $\sqrt{13}$; 5. (a) $-1 + 17i$, (b) $\sqrt{290}$

7. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$; 9. $\frac{1}{2}i$; 11. $-\frac{1}{2}i$.

Section 2.1

1. $(-1, 0)$; 3. $(2, -2)$; 5. $(-1, 17)$; 7. $(-3, -4)$.

Section 2.2

1. $(5\pi, 0)$; 3. $(2, -\frac{\pi}{6})$ or $(2, \frac{11\pi}{6})$; 5. $(4, \frac{2\pi}{3})$.

Section 2.3

1. $1 + \sqrt{3}i$; 3. $-\pi i$; 5. $-\frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6}i$.

7. $\sqrt{2} + i\sqrt{2}$; 9. $\frac{1}{6} - i\frac{\sqrt{3}}{6}$; 11. $-3\sqrt{2} + 3\sqrt{2}i$.

13. $5(\cos 0 + i \sin 0)$; 15. $2(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))$; 17. $4(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$.

19. $3e^{i\pi}$; 21. $2e^{\frac{5\pi}{6}i}$; 23. $4e^{\frac{\pi}{3}i}$.

Section 2.4

1. $2i$; 3. $\frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3}i$; 5. -8 ; 7. (a) $2e^{i(\frac{\pi}{4} + \frac{\pi}{6})} = 2e^{i\frac{5\pi}{12}}$, (b) $2e^{i\frac{\pi}{12}}$.

9. (a) $6e^{i\frac{2\pi}{3}}$, (b) $\frac{3}{2}e^{i\frac{4\pi}{3}}$; 11. (a) $\sqrt{5}e^{i\frac{\pi}{4}}$, (b) $\frac{\sqrt{5}}{5}e^{-i\frac{3\pi}{4}}$.

Section 2.5

1. $(\sqrt[3]{5}, \frac{\pi}{12})$, $(\sqrt[3]{5}, \frac{9\pi}{12})$, $(\sqrt[3]{5}, \frac{17\pi}{12})$; 3. $(1, \frac{\pi}{15})$, $(1, \frac{7\pi}{15})$, $(1, \frac{13\pi}{15})$, $(1, \frac{19\pi}{15})$, $(1, \frac{5\pi}{3})$.

5. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $-i$; 7. $\sqrt{2} + i\sqrt{2}$, $-\sqrt{2} + i\sqrt{2}$, $-\sqrt{2} - i\sqrt{2}$, $\sqrt{2} - i\sqrt{2}$.

9. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, i , $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$, $-i$, $\frac{\sqrt{3}}{2} - \frac{1}{2}i$.

11. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

13. 2 , $-1 + \sqrt{3}i$, $-1 - \sqrt{3}i$.

15. $-2, 3 \pm 2i$.

17. $-3, 1 \pm 2i$.