Section 7.4: Integrals with Hyperbolic Functions

* We will revisit each integration method but with hyperbolic functions

Note: Be sure to go over the "Hyperbolic Functions Review Sheet".

(A) Substitution

(B) Integration by Parts

↪ When choosing u, treat hyperbolic and inverse hyperbolic functions as you would treat trigonometric and inverse trigonometric functions

Priority for u

L: logarithms

I: inverse trigonometric
inverse hyperbolic

A: algebraic functions

T: trigonometric
hyperbolic

E: exponentials

(C) Hyperbolic Integrals

↪ Treat powers of hyperbolic functions as you would treat trigonometric functions. The strategies are identical.

(D) Hyperbolic Substitution

↪ Similar to trigonometric substitution

Expression	Trig. Substitution	Hyp. Substitution
$\sqrt{a^2-x^2}$	$x = a\sin(\theta)$	$x = a\tanh(u)$
$\sqrt{a^2+x^2}$	$x = a\tan(\theta)$	$x = a\sinh(u)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta)$	$x = a\cosh(u)$

$\boxed{\text{Ex. 1}}$

Write $\sinh^{-1}(x)$ in terms of logs.

Solution:

Suppose $y = \sinh^{-1}(x)$.

$$\sinh(y) = x$$

$$\frac{e^{y} - e^{-y}}{2} = x$$

$$e^y - e^{-y} = 2x$$

$$e^{2y} - 1 = 2xe^y$$

$$\boxed{e^{2y} - 2xe^y - 1 = 0}$$

$$(e^y)^2 - 2x(e^y)^1 - 1 = 0$$

↪ This a quadratic equation in e^y

$$e^y = \frac{2x \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2}$$

$$\underbrace{e^y}_{\text{positive}} = x \pm \sqrt{x^2+1}$$

$$\underbrace{\sqrt{x^2} - \sqrt{x^2+1}}$$

so negative root gives negative number.

$$e^y = x + \sqrt{x^2+1}$$

$$\boxed{\begin{aligned} y &= \ln\left(x + \sqrt{x^2+1}\right) \\ y &= \sinh^{-1}(x) \end{aligned}}$$

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* See reference sheet for similar formulas for other inverse functions.

Ex. 2

Calculate $\int x^2 \cosh(x)\,dx$

Solution:

(LIATE) We will use tabular integration-by-parts.

x^2	$+$	$\cosh(x)$
$2x$	$-$	$\sinh(x)$
2	$+$	$\cosh(x)$
0		$\sinh(x)$

$$\int x^2\cosh(x)\,dx = x^2\sinh(x) - 2x\cosh(x) + 2\sinh(x) + C$$

Ex. 2

Calculate $\int \cosh(x)^2\,dx$.

Solution:

Since exponents on both $\cosh(x)$ and $\sinh(x)$ are even, we use integration by parts.

$$u = \cosh(x) \qquad dv = \cosh(x)\,dx$$

$$du = \sinh(x)\,dx \qquad v = \sinh(x)$$

$$\int \cosh(x)^2\,dx = \cosh(x)\sinh(x) - \int \underbrace{\sinh(x)^2}\,dx$$

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

$$\sinh(x)^2 = \cosh(x)^2 - 1$$

$$\underbrace{\int \cosh^2 x\,dx}_{=I} = \cosh x \sinh x - \underbrace{\int \cosh^2 x\,dx}_{=I} + \int 1\,dx$$

$$I = \int \cosh^2 x\,dx = \frac{1}{2}\cosh(x)\sinh(x) + \frac{x}{2} + C$$

Alternatively, use the "double-angle" formulas:

$$\cos(\theta)^2 = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

$$\hookrightarrow \cosh(\theta)^2 = \frac{1}{2} + \frac{1}{2}\cosh(2\theta)$$

$$\sin(\theta)^2 = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

$$\hookrightarrow -\sinh(\theta)^2 = \frac{1}{2} - \frac{1}{2}\cosh(2\theta)$$

$$\sinh(\theta)^2 = -\frac{1}{2} + \frac{1}{2}\cosh(2\theta)$$

$\boxed{\text{Ex. 3}}$

Calculate $\int \sinh(x)^4 \cosh(x)^3 \, dx$.

Solution:

$$\int \sinh(x)^4 \cosh(x)^3 \, dx =$$

$$= \int \sinh(x)^4 \cosh(x)^2 \underbrace{\cosh(x)\, dx}_{= du}$$

$u = \sinh(x)$

$$= \int \sinh(x)^4 \left(1 + \sinh(x)^2\right) \cosh(x)\, dx$$

$$= \int u^4 (1 + u^2)\, du = \int (u^4 + u^6)\, du$$

$$= \frac{1}{5}\sinh(x)^5 + \frac{1}{7}\sinh(x)^7 + C$$

$\boxed{\text{Ex. 4}}$

Calculate $\displaystyle\int \sqrt{x^2+16}\,dx.$

(a) trigonometric substitution

(b) hyperbolic substitution

Solution:

(a) We substitute

$$x = 4\tan(\theta)$$
$$dx = 4\sec(\theta)^2\,d\theta$$
$$\sqrt{x^2+16} = 4\sec(\theta)$$

$$\int \sqrt{x^2+16}\,dx = \int 4\sec(\theta)\cdot 4\sec(\theta)^2\,d\theta$$
$$= 16\int \sec(\theta)^3\,d\theta$$

Now compute $\int \sec(\theta)^3\,d\theta$. We use integration-by-parts.

$u = \sec(\theta)$ $\qquad dv = \sec(\theta)^2\,d\theta$

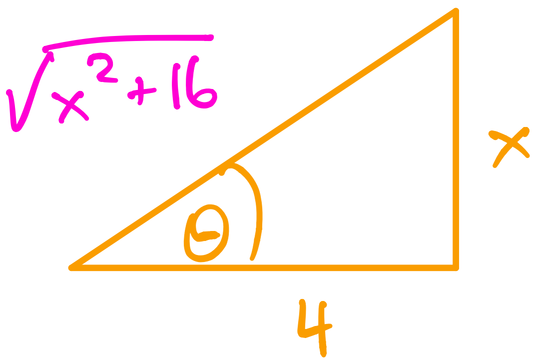
$du = \sec(\theta)\tan(\theta)\,d\theta$ $\qquad v = \tan(\theta)$

$$\int \sec(\theta)^3 \, d\theta = \sec(\theta)\tan(\theta) - \int \sec(\theta)\underbrace{\tan(\theta)^2}_{= \sec(\theta)^2 - 1} d\theta$$

$$\underbrace{\int \sec^3\theta \, d\theta}_{I} = \sec\theta \tan\theta - \underbrace{\int \sec^3\theta \, d\theta}_{I} + \underbrace{\int \sec\theta \, d\theta}_{\text{memorize}}$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

Now rewrite everything in terms of x.



$\tan(\theta) = x/4$

$\sec(\theta) = \dfrac{\sqrt{x^2+16}}{4}$

So our final answer is:

$$\int \sqrt{x^2+16}\, dx = 8 \cdot \frac{\sqrt{x^2+16}}{4} \cdot \frac{x}{4} + 8\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

(b) We will substitute

$$x = 4\sinh(u)$$

$$dx = 4\cosh(u)\,du$$

$$\sqrt{x^2+16} = \sqrt{16\sinh(u)^2+16}$$
$$= 4\sqrt{\sinh(u)^2+1}$$
$$= 4\sqrt{\cosh(u)^2}$$
$$= 4\cosh(u)$$

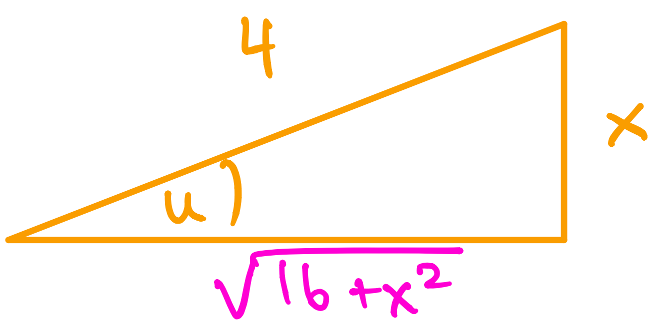
$$\int\sqrt{x^2+16}\,dx = \int 4\cosh(u)\cdot 4\cosh(u)\,du$$

$$= 16\int \cosh(u)^2\,du$$ (Now use Exampe 2)

$$I = \int \cosh^2 x\,dx = \frac{1}{2}\cosh(x)\sinh(x) + \frac{x}{2} + C$$

$$= 8\cosh(u)\sinh(u) + 8u + C$$

Now we write in terms of x:



$$\sinh(u) = \frac{x}{4}$$
$$\cosh(u) = \frac{\sqrt{16+x^2}}{4}$$

$$(ADJ)^2 - (OPP)^2 = (HYP)^2$$
$$(ADJ)^2 = 16 + x^2$$

So our final answer is:

$$\int \sqrt{16+x^2}\,dx = 8 \cdot \frac{\sqrt{16+x^2}}{4} \cdot \frac{x}{4} + 8\sinh^{-1}\left(\frac{x}{4}\right) + C$$

$\boxed{\text{Ex. 5}}$

Calculate $\displaystyle\int_1^{\sqrt{26}} \frac{\sqrt{x^2-1}}{x^2}\,dx.$

(Use hyperbolic substitution)

<u>Solution:</u>

We will substitute

$$x = \cosh(u)$$

convention for $\cosh^{-1}$

$$dx = \sinh(u)\,du$$

$$\sqrt{x^2-1} = \sqrt{\cosh(u)^2 - 1}$$
$$= \sqrt{\sinh(u)^2}$$
$$= |\sinh(u)|$$
$$= \sinh(u) \qquad (u \geq 0)$$

<u>Limits of integration:</u> $x = \cosh(u)$

$$x = 1 \implies u = \cosh^{-1}(1) = 0$$

$x = \sqrt{26} \implies u = \cosh^{-1}(\sqrt{26}) := a$

$$\int_1^{\sqrt{26}} \frac{\sqrt{x^2-1}}{x^2}\,dx = \int_0^a \frac{\sinh(u)}{\cosh(u)^2}\sinh(u)\,du$$

$$= \int_0^a \tanh(u)^2\,du = \int_0^a (1 - \operatorname{sech}(u)^2)\,du$$

$\cosh(u)^2 - \sinh(u)^2 = 1$

$1 - \tanh(u)^2 = \operatorname{sech}(u)^2$

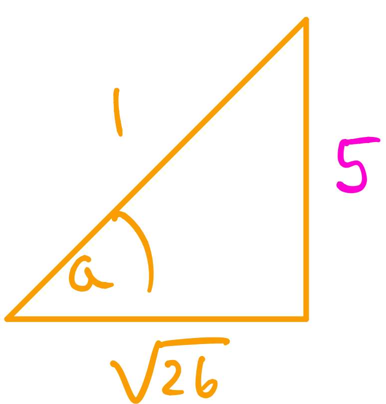
$1 - \operatorname{sech}(u)^2 = \tanh(u)^2$

$$= \left(u - \tanh(u)\right)\Big|_0^a$$

$$= (a - \tanh(a)) - (0 - \tanh(0))$$

$$= a - \tanh(a)$$

Now write as an exact answer.



$a = \cosh^{-1}(\sqrt{26})$

$\cosh(a) = \sqrt{26}$

$\tanh(a) = \dfrac{5}{\sqrt{26}}$

$$(\text{ADJ})^2 - (\text{OPP})^2 = (\text{HYP})^2$$

$$(\sqrt{26})^2 - (5)^2 = 1$$

So our final answer is:

$$\int_1^{\sqrt{26}} \frac{\sqrt{x^2-1}}{x^2}\,dx = \cosh^{-1}(\sqrt{26}) - \frac{5}{\sqrt{26}}$$

$\boxed{\text{Ex. 6}}$

Calculate $\displaystyle\int \frac{\sqrt{9+x^2}}{x}\,dx$

<u>Solution:</u>

We can substitute

<u>Trigonometric:</u> $x = 3\tan(\theta)$

<u>Hyperbolic:</u> $x = 3\sinh(u)$

$$dx = 3\cosh(u)\,du$$

$$\sqrt{9+x^2} = 3\cosh(u)$$

$$\left(1 + \sinh(u)^2 = \cosh(u)^2\right)$$

$$\int \frac{\sqrt{9+x^2}}{x}\,dx = \int \frac{3\cosh(u)}{3\sinh(u)} \cdot 3\cosh(u)\,du$$

$$= 3\int \frac{\cosh(u)^2}{\sinh(u)}\,du \qquad \left(\int \cos(x)^2 \sin(x)^{-1}\,dx\right)$$

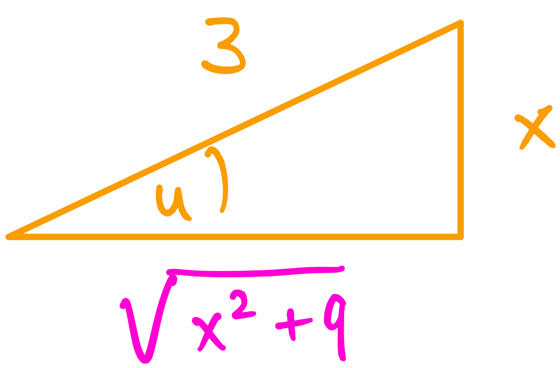
$$= 3\int \frac{1+\sinh(u)^2}{\sinh(u)}\,du$$

$$= 3\int \left(\operatorname{csch}(u) + \sinh(u)\right) du$$

$$= \underbrace{3\int \operatorname{csch}(u)\,du}_{= -\ln\left|\operatorname{csch}(u)+\coth(u)\right|} + \underbrace{3\int \sinh(u)\,du}_{= \cosh(u)}$$

$$= -3\ln\left|\operatorname{csch}(u)+\coth(u)\right| + 3\cosh(u) + C$$

Now rewrite in terms of x:



$$\sinh(u) = x/3$$

$$\operatorname{csch}(u) = \frac{3}{x}$$

$$\coth(u) = \frac{\sqrt{x^2+9}}{x}$$

$$\cosh(u) = \frac{\sqrt{x^2+9}}{3}$$

So our final answer is:

$$\int \frac{\sqrt{9+x^2}}{x}\,dx = -3\ln\left|\frac{3}{x} + \frac{\sqrt{9+x^2}}{x}\right| + \sqrt{9+x^2} + C$$