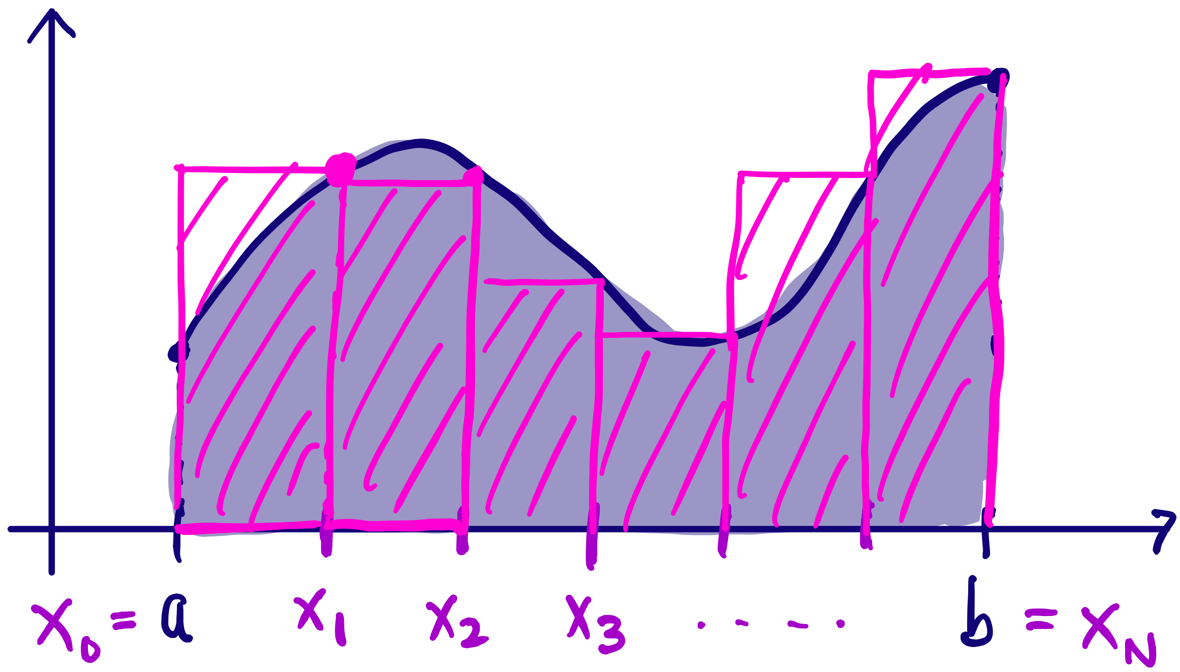
More on Riemann Sums

We want to approximate the area under the graph of $y = f(x)$ and above the interval $[a, b]$ on the x-axis

(We will assume $f(x) \geq 0$ and f is continuous.)



- We will use rectangles whose bases lie on the x-axis to estimate the area.

- First divide $[a, b]$ into N equal-length subintervals. These subintervals are the bases of the N rectangles.

- Each rectangle has width

$$\Delta x = \frac{b-a}{N}$$ ← total length of $[a,b]$ / ← # of rectangles

- Determine the endpoints of each of the N subintervals.

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$ $\quad (x_2 = x_1 + \Delta x)$

$x_3 = a + 3\Delta x$ $\quad (x_3 = x_2 + \Delta x)$

$\vdots$

$x_N = a + N\Delta x = b$

- We will choose the height of each rectangle to be the function value at the right endpoint of the corresponding subinterval.
- Total area of rectangles estimates the area under the graph.

(As # of rectangles increases, the approximation gets better.)

This is called a <u>Riemann sum</u> using right endpoints with N rectangles.

Methods for determining heights

subinterval: $[x_{k-1}, x_k]$

right endpoints: $h_k = f(x_k)$

left endpoints: $h_k = f(x_{k-1})$

midpoint: $h_k = f\left(\underbrace{\frac{x_{k-1}+x_k}{2}}\right)$

midpoint of $[x_{k-1}, x_k]$

* On HW, you have to do all three
* On final exam, you have to do only right endpoints

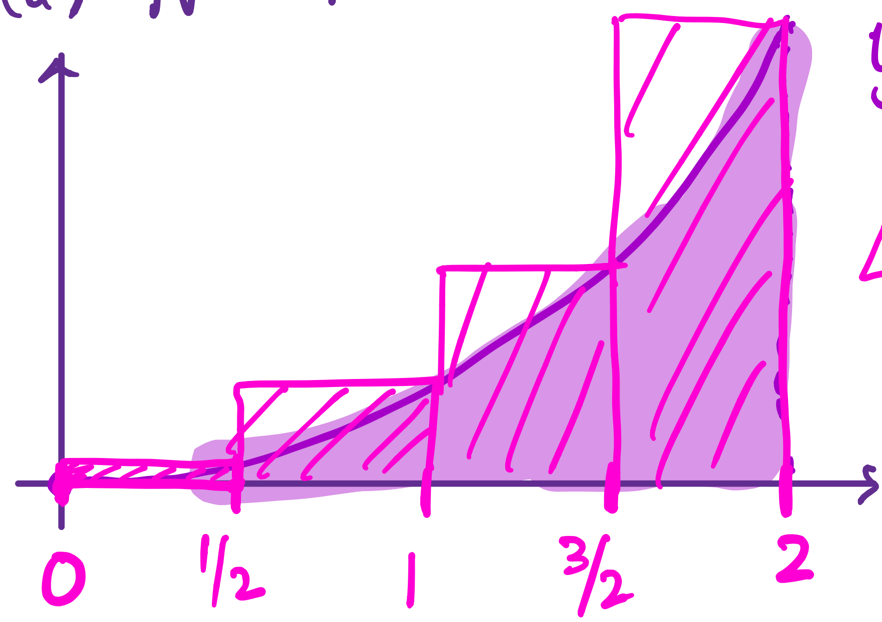
Ex. 1

Let $f(x) = x^3$. Approximate the area under the graph of $y = f(x)$ and above the interval $[0, 2]$ on the x-axis, using right endpoint values and ....

(a) 4 rectangles (b) 6 rectangles

Solution:

(a) $N = 4$



$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

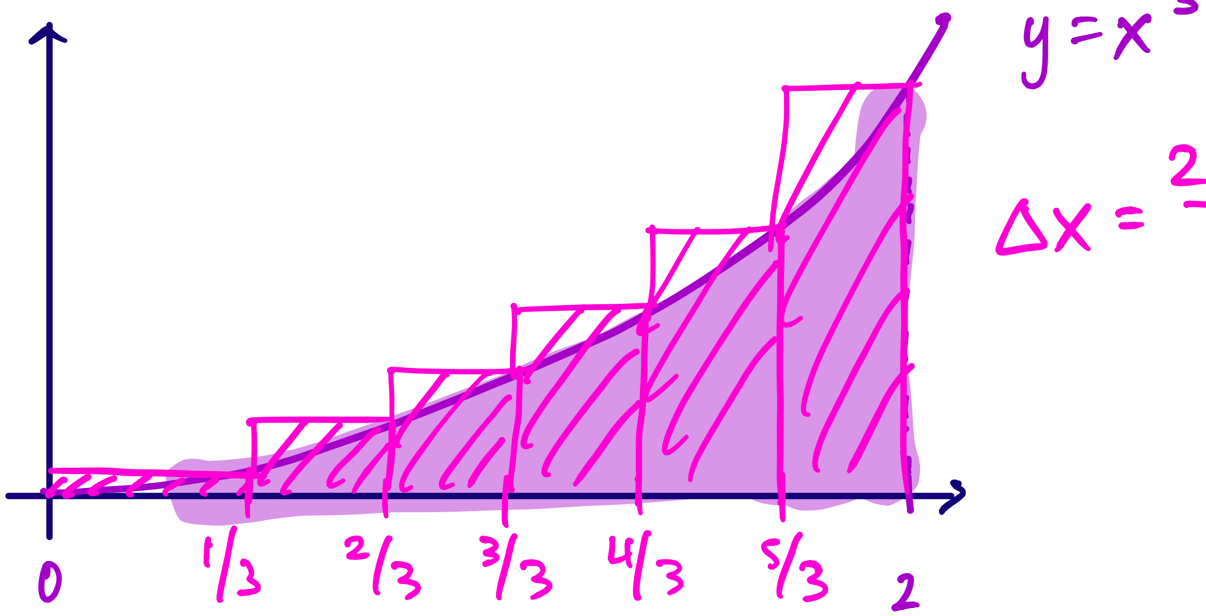
Rectangle #	Width	x-value right endpoint	$f(x) = x^3$ height	area
1	$1/2$	$1/2$	$1/8$	$1/16$
2	$1/2$	$2/2$	$8/8$	$8/16$
3	$1/2$	$3/2$	$27/8$	$27/16$
4	$1/2$	$4/2$	$64/8$	$64/16$

The approximate area under the graph is the total area of the rectangles.

$$R_4 = \frac{1+8+27+64}{16} = \frac{100}{16} = \frac{25}{4}$$

right endpoint sum, 4 rectangles

(b) $N = 6$



$$\Delta x = \frac{2-0}{6} = \frac{1}{3}$$

Rectangle #	width	right endpoint	height ($y=x^3$)	area
1	$1/3$	$1/3$	$1/27$	$1/81$
2	$1/3$	$2/3$	$8/27$	$8/81$
3	$1/3$	$3/3$	$27/27$	$27/81$
4	$1/3$	$4/3$	$64/27$	$64/81$
5	$1/3$	$5/3$	$125/27$	$125/81$
6	$1/3$	$6/3$	$216/27$	$216/81$

$$R_6 = \frac{1+8+27+64+125+216}{81} = \frac{441}{81}$$

Special Notation for Sums

$$\sum_{k=m}^{n} c_k = c_m + c_{m+1} + c_{m+2} + \dots + c_n$$

c_k : terms of sum

k : index

m: lower index

n: upper index

$\boxed{\text{Ex:}}$

$$\sum_{k=3}^{6} c_k = c_3 + c_4 + c_5 + c_6$$

$$\sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2$$