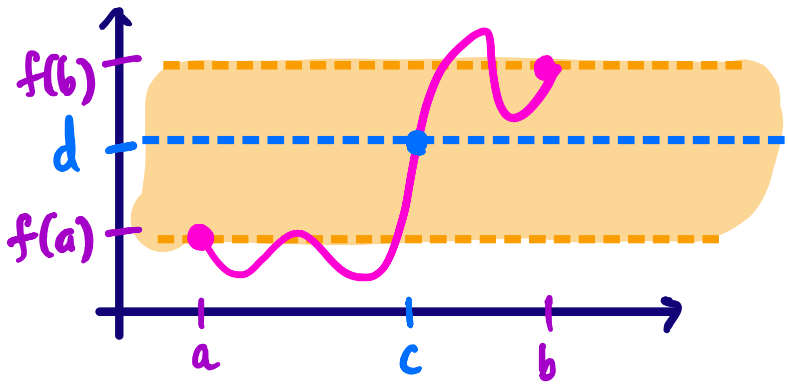
Optional: Intermediate Value Theorem & Applications

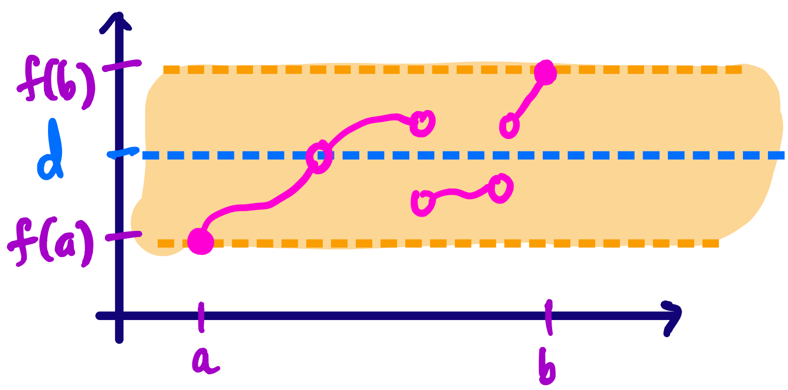
Thm: (IVT) Suppose f is continuous on $[a,b]$. Then given any number d between $f(a)$ and $f(b)$, there exists a number c in (a,b) such that $f(c)=d$.

What does this mean graphically?



Our intuitive notion of continuity implies that continuous functions cannot skip any y-values.

Challenge: Given $(a, f(a))$ and $(b, f(b))$, can we draw the graph of f so that some y-values between $f(a)$ and $f(b)$ are skipped?



Easy! As long as we are allowed to make f not continuous on $[a,b]$!

The IVT says this challenge is impossible if f has to be continuous.

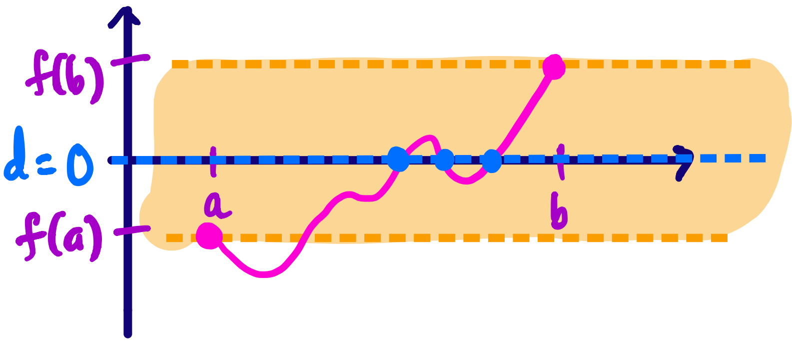
Proving an equation has a solution

Suppose we knew the following:

- f is continuous on $[a,b]$

- $f(a)$ and $f(b)$ have opposite signs (one is positive and the other is negative)

What can we say about the equation "$f(x)=0$"?



The IVT tells us that there must be at least one solution to "$f(x)=0$" in the interval (a,b)!

Ex. 1

Prove that the equation $\cos(x) = x^3 - x$ has a solution in $[\pi/4, \pi/2]$.

<u>Solution</u>:

Equivalently, we show $\cos(x) - x^3 + x = 0$ has a solution.

Let $f(x) = \cos(x) - x^3 + x$. Then observe:

- f is continuous on $[\pi/4, \pi/2]$

$$\left.\begin{aligned} &\bullet\ f(\pi/4) = \frac{1}{\sqrt{2}} - \frac{\pi^3}{64} + \frac{\pi}{4} \approx 1.008 \\ &\bullet\ f(\pi/2) = 0 - \frac{\pi^3}{8} + \frac{\pi}{2} \approx -2.305 \end{aligned}\right\} \text{opposite signs}$$

So by the IVT, the equation $f(x)=0$ has a solution in $(\pi/4, \pi/2)$.

Solving nonlinear inequalities

Suppose we want to solve "$f(x) > 0$" where f is a rational function (ratio of two polynomials, with no common factors). Recall the general method.

Ex.2

Solve $\dfrac{2x-1}{x+2} > 1$.

Solution:

$\dfrac{2x-1}{x+2} - 1 > 0$ } move all terms to one side

$\dfrac{x-3}{x+2} > 0$ } simplify left side; write as quotient

$(x=3,\ x=-2)$ } determine "cut points" by setting each of numerator and denominator to 0

} draw a number line and mark cut points

$-3 \quad 0 \quad 4$ } choose one test point from each interval

$x=-3:\ \dfrac{x-3}{x+2} = 6 > 0$

$x=0:\ \dfrac{x-3}{x+2} = -\dfrac{3}{2} < 0$

$x=4:\ \dfrac{x-3}{x+2} = \dfrac{1}{6} > 0$

} in each interval, test the truth of the inequality using one test point only.

$x \in (-\infty, -2) \cup (3, \infty)$ } final answer is union of intervals for which inequality is true

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Why does this work? Specifically, why can we test only one point per interval?

Q: At what values of x can a function f change sign?

A: The IVT tells us $f(x)$ can change sign at $x=c$ only if either $f(c)=0$ or f is not continuous at c. Otherwise, $f(x)$ has a single sign in each interval determined by these values of c.

When we find cut points for rational functions, we are finding these values of c where f can change sign.

- $f(c) = 0$
  ↪ set numerator to 0, solve for x
- f is not continuous at c
  ↪ set denominator to 0, solve for x

So we only test <u>one</u> point per interval because IVT tells us $f(x)$ has just one sign for the entire interval!