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1 Fall 2017
Midterm Exam #1

8 pts 1. Find an equation of the line tangent to the graph of \( y = 2x^2 - 3x + 1 \) at \( x = 1 \).

Solution
Differentiating \( y \) gives
\[
y' = 4x - 3
\]
If \( x = 1 \), then \( y = 2(1)^2 - 3(1) + 1 = 0 \) and \( y' = 4(1) - 3 = 1 \). Hence the equation of the tangent line is
\[
y - 0 = 1(x - 1)
\]

10 pts 2. Show that the equation
\[
x^{2/3} = 2x^2 + 2x - 2
\]
has at least one solution in the interval \([0, 1]\). Explain your answer.

Solution
Let \( f(x) = 2x^2 + 2x - 2 - x^{2/3} \). Since \( f \) is a sum of power functions, \( f \) is continuous on its domain, which includes the interval \([0, 1]\). Observe that \( f(0) = -2 \) and \( f(1) = 1 \). Since 0 is between -2 and 1, the intermediate value theorem guarantees the existence of a number \( c \) in the interval \([0, 1]\) such that \( f(c) = 0 \). The same number \( c \) satisfies the desired equation.

10 pts 3. Find all real solutions to the following equation.
\[
2 \ln(x) = \ln \left( \frac{x^5}{5-x} \right) - \ln \left( \frac{x^3}{2+x} \right)
\]

Solution
Using the logarithm law \( a \ln(b) = \ln(b^a) \) on the left side of the equation and the logarithm law \( \ln(a) - \ln(b) = \ln \left( \frac{a}{b} \right) \) on the right side of the equation gives
\[
\ln \left( x^2 \right) = \ln \left( \frac{x^5}{x^3} \frac{5-x}{2+x} \right)
\]
Simplifying the right side gives
\[
\ln \left( x^2 \right) = \ln \left( \frac{x^2(2+x)}{5-x} \right)
\]
Exponentiating each side gives
\[
x^2 = \frac{x^2(2+x)}{5-x}
\]
Cross-multiplication and some algebra gives
\[
0 = x^2(5-x) - x^2(2+x) = x^2(5-x-2-x) = x^2(3-2x)
\]
Observe that \( x = 0 \) cannot be a solution to the original equation since \( \ln(0) \) is undefined.
Hence \( x^2 \neq 0 \) and we must have \( 3 - 2x = 0 \). The only solution is \( x = \frac{3}{2} \).

4. Find the values of the constants \( a \) and \( b \) so that the following function is continuous for all \( x \).
If this is not possible, explain why.

\[
f(x) = \begin{cases} 
ax + b, & x < 1 \\
-2, & x = 1 \\
3\sqrt{x} + b, & x > 1 
\end{cases}
\]

**You must give a full, clear justification for your answer. You must use proper methods taught in this course.**

**Solution**
The first two “pieces” of \( f(x) \) are continuous for all \( x \) regardless of the values of \( a \) and \( b \) since polynomials are continuous for all \( x \). The “piece” \( 3\sqrt{x} + b \) is continuous regardless of the value of \( b \) as long as \( x \geq 0 \). Hence each piece is continuous on each of its “pieces” separately on the respective intervals. We need only force continuity at \( x = 1 \) to guarantee \( f \) is continuous for all \( x \). Hence we must choose \( a \) and \( b \) such that

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) \\
\lim_{x \to 1^-} (ax + b) = \lim_{x \to 1^+} (3\sqrt{x} + b) = -2 \\
a + b = 3 + b = -2
\]

Hence \( b = -5 \) and \( a = 3 \).

5. Suppose \( f(x) \) is a function about which the following is known.

- the points \((1, 2), (3, 3), \) and \((5, 2)\) lie on the graph of \( f \)
- \( \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 1 \)
- \( \lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) + \infty \)
- \( \lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) - \infty \)
- \( f'(1) = f'(3) = 0 \)
- \( f'(x) < 0 \) on the intervals \((-\infty, -2), (-2, 1), \) and \((3, \infty)\)
- \( f'(x) > 0 \) on the intervals \((1, 2)\) and \((2, 3)\)
- \( f''(5) = 0 \)
- \( f''(x) < 0 \) on the intervals \((-\infty, -2)\) and \((2, 5)\)
- \( f''(x) > 0 \) on the interval \((-2, 1), (1, 2), \) and \((5, \infty)\)
Sketch a possible graph of $f$ on the axes provided. Label each asymptote by its equation. For each relative extremum or inflection point, identify its coordinates and label the point “rel. min”, “rel. max”, or “infl. pt.” as appropriate.

6. For each part, calculate $f'(x)$.

*After calculating the derivative, do not simplify your answer.*

6 pts
(a) $f(x) = \frac{7x^3}{3x^{1/2}x^5}$
6 pts
(b) $f(x) = -\cos(x)\ln(x)$
6 pts
(c) $f(x) = \frac{\csc(x) + 4x^3}{e^x - e^5}$

**Solution**

(a) Observe that $f(x) = \frac{7}{3}x^{-5/2}$. Hence $f'(x) = \frac{7}{3} \left( -\frac{5}{2} \right) x^{-7/2}$

(b) Use product rule.

$$f'(x) = -\left( \cos(x) \cdot \frac{1}{x} + (-\sin(x)) \cdot \ln(x) \right)$$

(c) Use quotient rule.

$$f'(x) = \frac{(-\csc(x)\cot(x) + 12x^2)(e^x - e^5) - (\csc(x) + 4x^3)(e^x)}{(e^x - e^5)^2}$$

7. The parts of this question are independent of each other.

2 pts
(a) Given the function $g(x)$, state the definition of $g'(x)$.

12 pts
(b) Let $f(x) = \sqrt{6x + 1}$. Calculate $f'(1)$ directly from the definition. Show all work.

*If you simply quote a rule, you will receive no credit. You must use the definition of derivative.*

**Solution**

(a) $g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h}$
(b) Start with the definition of derivative, then simplify and cancel.

\[
\begin{align*}
  f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} \\
  &= \lim_{h \to 0} \left( \frac{\sqrt{6(1 + h) + 1} - \sqrt{7}}{h} \right) \\
  &= \lim_{h \to 0} \left( \frac{(6h + 7) - 7}{h(\sqrt{6h + 7} + \sqrt{7})} \right) \\
  &= \lim_{h \to 0} \frac{6}{\sqrt{6h + 7} + \sqrt{7}} = \frac{6}{\sqrt{7} + \sqrt{7}} = \frac{3}{\sqrt{7}}.
\end{align*}
\]

8. For each limit, calculate the value or show that it does not exist. Show all work.

(a) \( \lim_{x \to 7} \left( \frac{\frac{x}{7} - \frac{1}{x} - 7}{x - 7} \right) \)

(b) \( \lim_{x \to 0} \left( \frac{\sin(7x)}{\tan(2x)} \right) \)

(c) \( \lim_{x \to -1} \left( \frac{|x + 1|}{x + 1} \right) \)

Solution

(a) We have the following work.

\[
\begin{align*}
  \lim_{x \to 7} \left( \frac{\frac{x}{7} - \frac{1}{x} - 7}{x - 7} \right) &= \lim_{x \to 7} \left( \frac{x - 7}{7x(x - 7)} \right) = \lim_{x \to 7} \left( \frac{1}{7x} \right) = \frac{1}{49}.
\end{align*}
\]

(b) We have the following work.

\[
\begin{align*}
  \lim_{x \to 0} \left( \frac{\sin(7x)}{7x} \cdot \frac{2x}{\sin(2x)} \cdot \cos(2x) \cdot \frac{7x}{2x} \right) &= \left( \lim_{x \to 0} \frac{\sin(7x)}{7x} \right) \cdot \left( \lim_{x \to 0} \frac{2x}{\sin(2x)} \right) \cdot \left( \lim_{x \to 0} \cos(2x) \cdot \frac{7}{2} \right) \\
  &= 1 \cdot 1 \cdot 1 \cdot \frac{7}{2} = \frac{7}{2}.
\end{align*}
\]

(c) We have the following work.

\[
\begin{align*}
  \lim_{x \to -1^-} \left( \frac{|x + 1|}{x + 1} \right) &= \lim_{x \to -1^-} \left( \frac{-(x + 1)}{x + 1} \right) = -1 \\
  \lim_{x \to -1^+} \left( \frac{|x + 1|}{x + 1} \right) &= \lim_{x \to -1^+} \left( \frac{+(x + 1)}{x + 1} \right) = +1.
\end{align*}
\]

The one-sided limits are not equal, thus the desired limit does not exist.
Midterm Exam #2

9. A camera is located 5 feet from a straight wire along which a bead is moving at 6 feet per second. The camera automatically turns so that it is pointed at the bead at all times. How fast is the camera turning 2 seconds after the bead passes closest to the camera? You must express your final answer as an exact rational number and indicate its units.

Solution

Suppose the bead travels along the x-axis. Let O = (0, 0) be the origin and let C = (0, 5) be the coordinates of the camera. Let P = (x, 0) be the coordinates of the bead and let \( \theta = \angle OCP \). Then \( x = 5 \tan(\theta) \), whence \( \frac{dx}{dt} = 5 \sec(\theta)^2 \frac{d\theta}{dt} \). The bead moves to the right at a rate of 6 ft/s (\( \frac{dx}{dt} = 6 \)), and so the bead is at \( Q = (12, 0) \) 2 seconds after it has passed closest to the camera. Hence at this time we have

\[
12 = 5 \tan(\theta) \quad 6 = 5 \sec(\theta)^2 \frac{d\theta}{dt}
\]

Note that in \( \triangle OCQ \), the hypotenuse has length \( \sqrt{12^2 + 5^2} = 13 \) and the side adjacent to \( \theta \) has length 5. Hence \( \sec(\theta) = \frac{13}{5} \). Substitution into the second equation then gives us

\[
6 = 5 \left( \frac{13}{5} \right)^2 \frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{30}{169}
\]

(The units are radians per second.)

10. Calculate each of the following limits or show it does not exist. Show all work.

(a) \( \lim_{x \to 0} (1 - \sin(4x))^{6/x} \)

(b) \( \lim_{x \to 1} \left( \frac{xe^{4x} + 4e^4 - 5e^4x}{(x - 1)^2} \right) \)

Solution

(a) Substitution of \( x = 0 \) gives the indeterminate form \( 1^\infty \). Let \( L \) be the desired limit and consider \( \ln(L) \). Then we have

\[
\ln(L) = \ln \left( \lim_{x \to 0} (1 - \sin(4x))^{6/x} \right) = \lim_{x \to 0} \ln \left( (1 - \sin(4x))^{6/x} \right) = \lim_{x \to 0} \left( \frac{6 \ln(1 - \sin(4x))}{x} \right)
\]

We have used continuity of the logarithm and logarithm identities. Substitution of \( x = 0 \) now gives the indeterminate form \( \frac{0}{0} \), whence we may use L’Hospital’s Rule.
(and for any subsequent indeterminate forms of $0/0$).

\[
\lim_{x \to 0} \left( \frac{6 \ln(1 - \sin(4x))}{x} \right) = \lim_{x \to 0} \left( \frac{6 \cdot \frac{1}{1 - \sin(4x)} \cdot (-4 \cos(4x))}{1} \right) = \frac{6 \cdot \frac{1}{1 - 0} \cdot (-4)}{1} = -24
\]

Hence $\ln(L) = -24$, and so $L = e^{-24}$.

(b) Substitution of $x = 0$ gives the indeterminate form $0/0$, whence we may use L'Hospital's Rule (and for any subsequent indeterminate forms of $0/0$).

\[
\lim_{x \to 1} \left( \frac{xe^{4x} + 4e^4 - 5e^4x}{(x - 1)^2} \right) = \lim_{x \to 1} \left( \frac{4xe^{4x} + e^4x - 5e^4}{2(x - 1)} \right) = \lim_{x \to 0} \left( \frac{16xe^{4x} + 8e^4}{2} \right) = \frac{24e^4}{2} = 12e^4
\]

11. Define the function $f$ by

\[f(x) = \frac{4 - 3e^{-2x}}{6 - 5e^{-2x}}\]

(a) Find all horizontal asymptotes of $f(x)$.

(b) Find all vertical asymptotes of $f(x)$. Then, at each vertical asymptote, calculate both one-sided limits of $f(x)$.

**Solution**

(a) Observe the following.

\[
\lim_{x \to \infty} \frac{4 - 3e^{-2x}}{6 - 5e^{-2x}} = \frac{4 - 0}{6 - 0} = \frac{2}{3}
\]

\[
\lim_{x \to -\infty} \frac{4 - 3e^{-2x}}{6 - 5e^{-2x}} \stackrel{H}{=} \lim_{x \to -\infty} \frac{6e^{-2x}}{10e^{-2x}} = \frac{6}{10} = \frac{3}{5}
\]

Hence the horizontal asymptotes are $y = \frac{2}{3}$ and $y = \frac{3}{5}$.

(b) The function $f$ is continuous on its domain. The only $x$-value not in the domain of $f$ is that $x$-value such that $6 - 5e^{-2x} = 0$, or $x = -\frac{1}{2} \ln \left( \frac{6}{5} \right)$. Hence the only candidate vertical asymptote is the line $x = -\frac{1}{2} \ln \left( \frac{6}{5} \right)$. (From now on, let $a = -\frac{1}{2} \ln \left( \frac{6}{5} \right)$.)

If $x = a$, then $4 - 3e^{-2x} = 4 - 3 \left( \frac{6}{5} \right) = \frac{2}{5} > 0$. Now note that $g(x) = 6 - 5e^{-2x}$ is an increasing function because $g'(x) = 10e^{-2x} > 0$. Hence $6 - 5e^{-2x} < 0$ if $x < a$ and $6 - 5e^{-2x} > 0$ if $x > a$. Now we have

\[
\lim_{x \to a^-} \frac{4 - 3e^{-2x}}{6 - 5e^{-2x}} = \frac{2/5}{0^-} = -\infty
\]

\[
\lim_{x \to a^+} \frac{4 - 3e^{-2x}}{6 - 5e^{-2x}} = \frac{2/5}{0^+} = +\infty
\]
12. Find all points on the graph of the equation

\[ 2x^2 - 4xy + 7y^2 = 45 \]

at which the tangent line is horizontal.

**Hint:** Find a second equation that such points must satisfy. Then solve a system of two equations in the two unknowns \( x \) and \( y \).

**Solution**

The tangent line is horizontal at points where \( y' = 0 \). Using implicit differentiation we have

\[ 4x - 4xy' - 4y + 14yy' = 0 \]

Setting \( y' = 0 \) gives the equation \( 4x - 4y = 0 \), or \( x = y \). Hence the desired points must satisfy both \( x = y \) and the original equation. Substituting \( x = y \) into the original equation gives

\[ 2x^2 - 4x^2 + 7x^2 = 45 \]

Hence \( 5x^2 = 45 \), or \( x = \pm 3 \). The points on the graph where the tangent line is horizontal are \((-3, -3)\) and \((3, 3)\).

13. The parts of this question are independent.

**3 pts**

(a) Complete the statement of the mean value theorem (MVT).

Suppose \( f(x) \) is \__________ for all \( x \) in \([a, b]\) and \__________ for all \( x \) in \__________. Then there exists a number \( c \) in the interval \__________ such that the following equation is satisfied (write the equation in the space below):

(b) For each of the following parts, determine whether the MVT applies. If MVT does not apply, explain why. If MVT does apply, find all values of \( c \) guaranteed to exist by the MVT.

**3 pts**

(i) \( f(x) = (x^2 - 2x)^{1/3} \) on \([-2, 4]\)

(ii) \( f(x) = \frac{x + 4}{x - 4} \) on \([-3, 3]\)

**Solution**

(a) Suppose \( f(x) \) is continuous for all \( x \) in \([a, b]\) and differentiable for all \( x \) in \((a, b)\). Then there exists a number \( c \) in the interval \((a, b)\) such that the following equation is satisfied:

\[ \frac{f(b) - f(a)}{b - a} = f'(c) \]

(b) (i) The function \( g(x) = x^{1/3} \) is not differentiable at \( x = 0 \), hence \( f(x) \) is not differentiable whenever \( x^2 - 2x = 0 \), or at \( x = 0 \) and \( x = 2 \). These values of \( x \) are in the interval \([-2, 4]\), hence the MVT does not apply.

(ii) The equation guaranteed to have a solution by the MVT is given by

\[ -\frac{8}{(c - 4)^2} = f'(c) = \frac{f(3) - f(-3)}{3 - (-3)} = \frac{-7 - \frac{-1}{7}}{6} = \frac{-8}{7} \]
Hence \((c-4)^2 = 7\), or \(c = 4 \pm \sqrt{7}\). However, the value \(4 + \sqrt{7}\) is not in the given interval. So the only value of \(c\) guaranteed to exist by the MVT is \(c = 4 - \sqrt{7}\).

14. Use a linear approximation to estimate the value of \(\sqrt{33}\). Express your answer as an exact rational number.

Solution
If \(\Delta x\) is small, then we may use the formula

\[
f(a + \Delta x) \approx f(a) + f'(a)\Delta x
\]

Let \(f(x) = \sqrt{x}\), \(a = 36\), and \(\Delta x = -3\). Then in this language, our formula is

\[
f(33) \approx f(36) + f'(36)(-3)
\]

Observe that \(f(36) = 6\) and \(f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12}\). Hence we have

\[
\sqrt{33} \approx 6 + \frac{1}{12}(-3) = 6 - \frac{1}{4} = \frac{23}{4}
\]

15. A wire of length 51 cm is cut into two pieces. One piece is bent into a square. The other piece is bent into a rectangle whose length is two times its width. How should the wire be cut and the pieces assembled so that the total area enclosed by both pieces is a minimum? You must include correct units as part of your answer.

You must give a full justification for your answer using methods taught in this course. You must also demonstrate that your answer really does give the minimum area.

Solution
Let \(x\) be the side length of the square and let \(y\) be the width of the rectangle (so that the length of the rectangle is \(2y\)). The total area of the square and rectangle is

\[
A = x^2 + 2y^2
\]

Now note that the perimeter of the square is \(P_S = 4x\) and the perimeter of the rectangle is \(P_R = 6y\). The total perimeter must equal the length of the wire, hence

\[
4x + 6y = 51
\]

Solving for \(y\) gives

\[
y = \frac{51 - 4x}{6}
\]

Hence our area function only in terms of \(x\) is

\[
A(x) = x^2 + 2\left(\frac{51 - 4x}{6}\right)^2
\]

Our goal is to find the minimum value of \(A(x)\) on the interval \(x \in [0, \frac{51}{4}]\). Since \(A(x)\) is differentiable on this interval, the minimum value must occur at an \(x\)-value where \(A'(x) = \)
0. \[ 0 = A'(x) = 2x + 4 \left( \frac{51 - 4x}{6} \right) \left( -\frac{4}{6} \right) \]

Solving for \( x \) gives the following.

\[ 0 = 2x + 4 \left( \frac{51 - 4x}{6} \right) \left( -\frac{4}{6} \right) \]
\[ 0 = 72x - 16(51 - 4x) \]
\[ 0 = 72x - 16(51) + 64x \]
\[ 0 = 136x - 16(51) \]
\[ 0 = 17x - 2(51) \]
\[ 0 = x - 2(3) \]

Hence \( x = 6 \). This means that the wire should be cut into a piece 24 cm long (which is bent into a square) and a piece 27 cm long (which is bent into the described rectangle). Now observe that the second derivative of our area function is

\[ A''(x) = 2 + 4 \left( -\frac{4}{6} \right) \left( -\frac{4}{6} \right) = 2 - \frac{64}{36} \]

which is strictly positive for all \( x \). Hence the graph of \( A(x) \) is concave up on the entire interval \([0, \frac{51}{3}]\). This means that the only critical point we found must give a global minimum of \( A(x) \).

16. Consider the function \( f(x) = (x - 5)(x + 10)^2 = x^3 + 15x^2 - 500 \).

(a) Calculate all \( x \)- and \( y \)-intercepts of \( f \).

(b) Find where \( f \) is increasing and find where \( f \) is decreasing. Then calculate the \( x \)- and \( y \)-coordinates of all local extrema, classifying each as either a local minimum or a local maximum.

(c) Find where \( f \) is concave up and find where \( f \) is concave down. Then calculate the \( x \)- and \( y \)-coordinates of all inflection points.

(d) Sketch the graph of \( y = f(x) \) on the provided grid using your previous answers. Label any asymptote by its equation. Label each transition point by its \( x \)- and \( y \)-coordinates and as “rel. min”, “rel. max”, or “infl. pt.”, as appropriate.

Make sure to draw and label axes. Your graph need not to be to scale, but it must have the correct shape.

Solution

(a) The equation \( f(x) = 0 \) has solutions \( x = 5 \) and \( x = -10 \), whence the \( x \)-intercepts are \((5, 0)\) and \((-10, 0)\). The \( y \)-intercept is \((0, -500)\).

(b) The first derivative is

\[ f'(x) = 3x^2 + 30x = 3x(x + 10) \]

The first derivative exists everywhere in the domain and vanishes when \( x = 0, -10 \).
Now we calculate a sign chart for $f'$.

$x < -10$ : $f'(-11) = (-)(-) > 0$

$-10 < x < 0$ : $f'(-1) = (-)(+) < 0$

$0 < x$ : $f'(1) = (+)(+) > 0$

Hence we find the following about the function $f$.

- $f$ decreasing on: $(-10, 0)$
- $f$ increasing on: $(-\infty, -10), (0, \infty)$
- Local minimum value at: $(0, -500)$
- Local maximum value at: $(-10, 0)$

(c) The second derivative is

$$f''(x) = 6x + 30 = 6(x + 5)$$

The first derivative exists everywhere in the domain and vanishes when $x = -5$. Now we calculate a sign chart for $f''$.

$x < -5$ : $f'(-6) = (+)(-) < 0$

$-5 < x$ : $f'(0) = (+)(+) > 0$

Hence we find the following about the function $f$.

- $f$ concave down on: $(-\infty, -5)$
- $f$ concave up on: $(-5, \infty)$
- Point of inflection at: $(-5, -250)$

(d) Using the previous solutions, we have the following sketch.
Midterm Exam #1

17. The parts of this question are independent of each other.

(a) Given the function \( g(x) \), state the definition of \( g'(4) \).

(b) Let \( F(x) = \frac{1}{3x - 5} \). Calculate \( F'(2) \) directly from the definition. Show all work.

\textbf{If you simply quote a rule, you will receive no credit. You must use the definition of derivative.}

\textbf{Solution}

(a) \( g'(4) = \lim_{h \to 0} \left( \frac{g(4 + h) - g(4)}{h} \right) \)

(b) Start with the definition of derivative, then simplify and cancel.

\[ F'(2) = \lim_{h \to 0} \left( \frac{F(2 + h) - F(2)}{h} \right) = \lim_{h \to 0} \left( \frac{\frac{1}{3(2 + h) - 5} - 1}{h} \right) \]

\[ = \lim_{h \to 0} \left( \frac{-3}{3h + 1} \right) = \frac{-3}{0 + 1} = -3 \]

18. For each part, calculate \( f'(x) \).

\textbf{After calculating the derivative, do not simplify your answer.}

(a) \( f(x) = \frac{x^{-1}x^{8/3}}{4\sqrt{x^2}} \)

(b) \( f(x) = (x + \sqrt{5x - 6})^{1/4} \)

(c) \( f(x) = \frac{x^2e^x}{\ln(x) - \cos(x)} \)

\textbf{Solution}

(a) Simplifying the exponents, we observe that \( f(x) = \frac{1}{4}x \). Hence \( f'(x) = \frac{1}{4} \).

(b) Use power rule and chain rule (twice!).

\[ f'(x) = \frac{1}{4} \left( x + \sqrt{5x - 6} \right)^{-3/4} \cdot \left( 1 + \frac{1}{2} (5x - 6)^{-1/2} \cdot 5 \right) \]

(c) Use quotient rule.

\[ f'(x) = \frac{(\ln(x) - \cos(x)) \cdot (x^2e^x + 2xe^x) - (x^2e^x) \cdot (\frac{1}{x} + \sin(x))}{(\ln(x) - \cos(x))^2} \]
19. For each limit, calculate the value or show that it does not exist. Show all work.

(a) \( \lim_{x \to 0} \frac{(2x + 9)^2 - 81}{x} \) 

(b) \( \lim_{x \to 3^-} \frac{|x - 3|}{x - 3} \) 

(c) \( \lim_{x \to 1} \frac{5 - \sqrt{32 - 7x}}{x - 1} \) 

\[ \text{Solution} \]

(a) We have the following work.

\[ = \lim_{x \to 0} \left( \frac{4x^2 + 36x + 81 - 81}{x} \right) = \lim_{x \to 0} \left( \frac{4x^2 + 36x}{x} \right) = \lim_{x \to 0} (4x + 46) = 36 \]

(b) If \( x \to 3^- \), then we may assume that \( x < 3 \), or \( x - 3 < 0 \). For such values of \( x \), we have that \( |x - 3| = -(x - 3) \). So now we have

\[ \lim_{x \to 3^-} \left( \frac{|x - 3|}{x - 3} \right) = \lim_{x \to 3^-} \left( \frac{-(x - 3)}{x - 3} \right) = -1 \]

(c) We have the following work.

\[ = \lim_{x \to 1} \left( \frac{5 - \sqrt{32 - 7x}}{x - 1} \cdot \frac{5 + \sqrt{32 - 7x}}{5 + \sqrt{32 - 7x}} \right) \]

\[ = \lim_{x \to 1} \left( \frac{25 - (32 - 7x)}{(x - 1)(5 + \sqrt{32 - 7x})} \right) \]

\[ = \lim_{x \to 1} \left( \frac{7}{(x - 1)(5 + \sqrt{32 - 7x})} \right) \]

\[ = \frac{7}{5 + \sqrt{32 - 7}} = \frac{7}{10} \]

20. Consider the function \( f \) and its derivatives below.

\[ f(x) = \frac{x^2}{x^2 - 1}, \quad f'(x) = \frac{-2x}{(x^2 - 1)^2}, \quad f''(x) = \frac{6x^2 + 2}{(x^2 - 1)^3} \]

(a) Find all horizontal asymptotes of \( f \).

(b) Find all vertical asymptotes of \( f \). Then at each vertical asymptote you find, calculate the corresponding one-sided limits of \( f \).

(c) Find where \( f \) is decreasing and find where \( f \) is increasing. Then calculate all points of local extrema, classifying each as either a local minimum, a local maximum, or neither.

(d) Find where \( f \) is concave down and find where \( f \) is concave up. Then calculate all points of inflection.
Solution

(a) Horizontal asymptotes are found by computing the limit of \( f \) as \( x \to \pm \infty \).

\[
\lim_{x \to \pm \infty} \left( \frac{x^2}{x^2 - 1} \right) = \lim_{x \to \pm \infty} \left( \frac{1}{1 - \frac{1}{x^2}} \right) = \frac{1}{1 - 0} = 1
\]

Hence the only horizontal asymptote is the line \( y = 1 \).

(b) Since \( f \) is continuous on its domain, the only candidate vertical asymptotes are the lines \( x = -1 \) and \( x = 1 \) (since these are the only \( x \)-values not in the domain of \( f \)). Direct substitution of either \( x = -1 \) or \( x = 1 \) into \( f(x) \) gives the expression \( \frac{1}{0} \), which is undefined by indicates that all of the corresponding one-sided limits at both \( x = -1 \) and \( x = 1 \) are infinite. Hence \( x = -1 \) and \( x = 1 \) are vertical asymptotes.

Now we may compute the limits using sign analysis.

\[
\lim_{x \to -1^-} \left( \frac{x^2}{x^2 - 1} \right) = \oplus \quad \ominus \quad \infty = \infty
\]

\[
\lim_{x \to -1^+} \left( \frac{x^2}{x^2 - 1} \right) = \oplus \quad \ominus \quad \infty = -\infty
\]

\[
\lim_{x \to 1^-} \left( \frac{x^2}{x^2 - 1} \right) = \oplus \quad \ominus \quad \infty = -\infty
\]

\[
\lim_{x \to 1^+} \left( \frac{x^2}{x^2 - 1} \right) = \oplus \quad \ominus \quad \infty = \infty
\]

(c) Since \( f \) is differentiable on its domain, the only first-order critical numbers are solutions to \( f'(x) = 0 \).

\[
\frac{-2x}{(x^2 - 1)^2} = 0 \implies -2x = 0 \implies x = 0
\]

Recall that since \( x = -1 \) and \( x = 1 \) are not in the domain of \( f \), we must include such \( x \)-values on our sign chart also.

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<td>\oplus</td>
<td>increasing</td>
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<td>(-1, 0)</td>
<td>( f'(-\frac{1}{2}) = \oplus )</td>
<td>\oplus</td>
<td>increasing</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>( f'(\frac{1}{2}) = \ominus \oplus )</td>
<td>\ominus</td>
<td>decreasing</td>
</tr>
<tr>
<td>(1, \infty)</td>
<td>( f'(2) = \ominus )</td>
<td>\ominus</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

So \( f \) is decreasing on \((0, 1)\) and \((1, \infty)\); \( f \) is increasing on \((-\infty, -1)\) and \((-1, 0)\). There is no local minimum; there is a local maximum at \((0, 0)\).
(d) Since \( f \) is twice-differentiable on its domain, the only second-order critical numbers are solutions to \( f''(x) = 0 \).

\[
\frac{6x^2 + 2}{(x^2 - 1)^3} = 0 \implies 6x^2 + 2 = 0 \implies \text{no solution}
\]

Recall that since \( x = -1 \) and \( x = 1 \) are not in the domain of \( f \), we must include such \( x \)-values on our sign chart also.

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<td>(2)</td>
<td>(\oplus)</td>
<td>concave up</td>
</tr>
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Hence \( f \) is concave down on \((-1, 1)\); \( f \) is concave up on \((-\infty, -1)\) and \((1, \infty)\). There are no inflection points.

21. Each part of this question refers to the function \( f \) below, where \( a \) and \( b \) are constants.

\[
f(x) = \begin{cases} 
\frac{\sin(ax)}{x}, & x < 0 \\
2x + 3, & 0 \leq x < 1 \\
b, & x = 1 \\
\frac{x^2 - 1}{x - 1}, & 1 < x
\end{cases}
\]

For each of the following parts, you must give a full, clear justification for your answer. You must use proper methods taught in this course.

7 pts  
(a) Find the value of the constant \( a \) so that \( f \) is continuous at \( x = 0 \). If this is not possible, explain why.

7 pts  
(b) Find the value of the constant \( b \) so that \( f \) is continuous at \( x = 1 \). If this is not possible, explain why.

Solution
(a) We require that the left-limit, right-limit, and function value all be equal at \(x = 0\).

We have the following.

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left( \frac{\sin(ax)}{x} \right) = \lim_{x \to 0^-} \left( a \cdot \frac{\sin(ax)}{ax} \right) = a \cdot 1 = a \\
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x + 3) = 3 \\
f(0) = (2x + 3)|_{x=0} = 3
\]

So we must have that \(a = 3\).

(b) We require that the left-limit, right-limit, and function value all be equal at \(x = 1\).

We have the following.

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 3) = 5 \\
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left( \frac{x^2 - 1}{x - 1} \right) = \lim_{x \to 1^+} \left( \frac{(x - 1)(x + 1)}{x - 1} \right) = \lim_{x \to 1^+} (x + 1) = 2 \\
f(0) = b
\]

So we must have that \(5 = 2 = b\), which is impossible.

(It is impossible to find such a value of \(b\) because \(\lim_{x \to 1} f(x)\) does not exist.)

22. Find an equation of each line that is both tangent to the graph of \(y = 4x^2 - 3x - 1\) and parallel to the line \(y = 13x - 5\).

**Solution**

The slope of the desired tangent line is 13 (parallel lines have equal slope). Hence we must solve the equation \(f'(x) = 13\) where \(f(x) = 4x^2 - 3x - 1\).

\[
f'(x) = 8x - 3 = 13 \quad \Rightarrow \quad x = 2
\]

Observe that \(f(2) = 9\). Hence the desired tangent line has slope 13 and passes through the point \((2, 9)\). An equation of this line is

\[
y - 9 = 13(x - 2)
\]

23. Suppose \(f(x)\) is continuous on the interval \([0, 6]\). Selected values of \(f\) are given in the table below.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>-3</td>
<td>2</td>
<td>12</td>
<td>-5</td>
<td>-1</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

On what intervals is the graph of \(y = f(x)\) guaranteed to intersect the \(x\)-axis? Explain your answer.

*List as many intervals as possible, but no two intervals can overlap.*
Solution
Since \( f \) is continuous on \([0, 6]\), the intermediate value theorem (IVT) applies to this interval and any subintervals. Hence the IVT guarantees that \( f(x) = 0 \) on any interval on which the values of \( f \) change sign (either from positive to negative or negative to positive). There are three such intervals: \((0, 1), (2, 3), \) and \((4, 5)\).

24. A coffee vendor has collected data on the price of coffee in her store over the last year. The price of coffee \( t \) weeks since when the data collection began was

\[
p(t) = 0.02t^2 - 0.1t + 6
\]
dollars per pound.

**For each part, you must give correct units as part of your answer.**

(a) How much did the price of one pound of coffee increase in the first ten weeks after the data collection began?

3 pts

(b) What was the average rate at which the price of one pound of coffee changed over the same ten-week period mentioned in part (a)?

3 pts

The vendor also found that, in a given week, the local consumers bought approximately

\[
D(p) = \frac{2500}{p^2 + 1}
\]
pounds of coffee when the price was \( p \) dollars per pound. That is, \( D \) is the *weekly demand* of the consumers.

(c) Calculate \( D'(7) \) and explain its precise meaning in the given context.

5 pts

(d) At what rate was the weekly demand for coffee changing with respect to time exactly ten weeks after data collection began?

5 pts

**Solution**

(a) The change in price is \( \Delta p = p(10) - p(1) = 7 - 6 = 1 \). The units are *dollars*.

(b) The average rate is \( \frac{\Delta p}{\Delta t} = \frac{p(10) - p(0)}{10 - 0} = \frac{1}{10} \). The units are *dollars per week*.

(c) First find the derivative of \( D \) using chain rule or quotient rule.

\[
D'(p) = -\frac{5000p}{(p^2 + 1)^2}
\]

Now substitute \( p = 7 \).

\[
D'(7) = -\frac{5000 \cdot 7}{50^2} = -\frac{5000 \cdot 7}{2500} = -14
\]

The units are *pounds per dollar*. When the price of one pound of coffee was 7 dollars, the weekly demand was decreasing at a rate of 14 pounds per dollar. In other words, if this rate were constant, this would mean that each additional 1-dollar increase in the price would result in a loss of 14 pounds in the weekly demand.
(d) Observe that $D = D(p(t))$. So we must use chain rule.

$$\frac{dD}{dt} = \frac{d}{dt} D(p(t)) = D'(p(t)) \cdot p'(t)$$

Now substitute $t = 10$.

$$\frac{dD}{dt} \Bigg|_{t=10} = D'(p(10)) \cdot p'(10)$$

Now we compute each of these numbers. The following have already been computed in previous parts.

$$p(10) = 7$$
$$D'(p(10)) = D'(7) = -14$$

The following have not already been computed.

$$p'(t) = 0.04t - 0.1$$
$$p'(10) = 0.04 \cdot 10 - 0.1 = 0.3$$

Hence we find that

$$\frac{dD}{dt} \Bigg|_{t=10} = D'(p(10)) \cdot p'(10) = (-14) \cdot (0.3) = -4.2$$

The units are pounds per week.
Math 135

Midterm Exam #2

25. At a certain factory, the daily output is

\[ Q(L) = 1500L^{2/3} \]

units, where \( L \) denotes the size of the labor force measured in worker-hours. Currently 1,000 worker-hours of labor are used each day. Use a linear approximation to estimate the effect on the daily output if the labor force is cut to 975 worker-hours.

**Solution**

The linear approximation formula is

\[ \Delta Q = Q(L_0 + \Delta L) - Q(L_0) \approx Q'(L_0) \Delta L \]

For this problem, \( L_0 = 1000 \) and \( \Delta L = 975 - 1000 = -25 \). We also have

\[ Q'(L) = 1000L^{-1/3} \implies Q'(L_0) = 100 \]

Hence we have

\[ \Delta Q \approx (100)(-25) = -2500 \]

So the output decreases by approximately 2500 units.

26. Find an equation of the line tangent to the curve

\[ x^3 + e^{xy} = 3y + 9 \]

at the point \((2, 0)\).

**Solution**

Implicitly differentiating the equation with respect to \( x \) gives

\[ 3x^2 + e^{xy}(xy' + y) = 3y' \]

Substituting \( x = 2 \) and \( y = 0 \) gives

\[ 12 + 1 \cdot (2y' + 0) = 3y' \implies y' = 12 \]

Hence the equation of the tangent line is

\[ y - 0 = 12(x - 2) \]

27. On the axes provided, draw the graph of a function \( f(x) \) with domain \([-8, 8]\) that satisfies all of the following properties.

- \( f(-2) = -3 \) and \( f'(-6) = 0 \)
- asymptotes: \( x = -2 \) and \( y = -3 \)
- \( f \) is decreasing on: \((-8, -2), (-2, 2)\)
• \( f \) is increasing on: \((2, 8)\)
• \( f \) is concave down on: \((-6, -2), (6, 8)\)
• \( f \) is concave up on: \((-8, -6), (-2, 6)\)

Solution
There are many such solutions. Here is one.

28. The total surface area of a cube is changing at a rate of 12 in\(^2\)/s when the length of one of the sides is 10 in. At what rate is the volume of the cube changing at that time?

You must include correct units as part of your answer.

Solution
Let \( x \) be the side length of the cube. Then the total surface area and volume of the cube are

\[
S = 6x^2, \quad V = x^3
\]

Differentiating with respect to time \( t \) gives

\[
\frac{dS}{dt} = 12x \frac{dx}{dt}, \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}
\]

These four equations hold for all time. Now we substitute the information relevant to the specific time, i.e., \( x = 10 \) and \( \frac{dS}{dt} = 12 \).

\[
S = 600, \quad V = 1000, \quad 12 = 120 \frac{dx}{dt}, \quad \frac{dV}{dt} = 300 \frac{dx}{dt}
\]
Solving for \( \frac{dx}{dt} \) in the third equation gives \( \frac{dx}{dt} = \frac{12}{120} = \frac{1}{10} \). Substituting into the fourth equation gives

\[
\frac{dV}{dt} = 300 \cdot \frac{1}{10} = 30
\]

Hence the volume of the cube is increasing at a rate of 30 in\(^3\)/sec.

**29.** Find the minimum and maximum values of

\[ f(x) = 2x^3 - 3x^2 - 12x + 18 \]

on the interval \([-3, 3]\).

*Note that \( f \) may also be factored as \( f(x) = (x^2 - 6)(2x - 3) \).*

**Solution**

Since \( f \) is differentiable everywhere (it is a polynomial), the only critical numbers are solutions to \( f'(x) = 0 \).

\[
0 = f'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1) \implies x = -1, 2
\]

Now we find the values of \( f \) at the critical numbers and endpoints \( x = -3, 3 \). (We may use the factored form of \( f \) to make the arithmetic easier.)

\[ f(-3) = (3)(-9) = -27 \]
\[ f(-1) = (-5)(-5) = 25 \]
\[ f(2) = (-2)(1) = -2 \]
\[ f(3) = (3)(3) = 9 \]

Hence the absolute minimum is \(-27\) and the absolute maximum is 25.

**30.** Calculate each of the following limits or show it does not exist. Show all work.

\[
\begin{align*}
\text{(a) } & \lim_{x \to 1} \left( \frac{x^{1/4} - 1}{e^{2x} - e^2} \right) \\
\text{(b) } & \lim_{x \to 1} \left( (x - 1) \tan \left( \frac{\pi x}{2} \right) \right)
\end{align*}
\]

**Solution**

(a) Standard application of L’Hospital’s Rule.

\[
\lim_{x \to 1} \left( \frac{x^{1/4} - 1}{e^{2x} - e^2} \right) = \lim_{x \to 1} \left( \frac{\frac{1}{4}x^{-3/4}}{2e^{2x}} \right) = \frac{1}{4} \cdot \frac{1}{2e^2} = \frac{1}{8e^2}
\]
(b) We write the product as a quotient and then use L'Hospital’s Rule.

\[
\lim_{{x \to 1}} \left( (x - 1) \tan \left( \frac{\pi x}{2} \right) \right) = \lim_{{x \to 1}} \left( \frac{(x - 1) \sin \left( \frac{\pi x}{2} \right)}{\cos \left( \frac{\pi x}{2} \right)} \right)
\]

\[
H = \lim_{{x \to 1}} \left( \frac{(x - 1) \cos \left( \frac{\pi x}{2} \right) \cdot \frac{\pi}{2} + \sin \left( \frac{\pi x}{2} \right)}{-\sin \left( \frac{\pi x}{2} \right)} \right)
\]

\[
= \frac{0 \cdot 1 \cdot \frac{\pi}{2} + 1}{-1 \cdot \frac{\pi}{2}} = \frac{2}{-\pi}
\]

16 pts

31. You are constructing a rectangular box with a total surface area (six sides) of 450 in². The length of the box is three times its width. Find the dimensions of the box, measured in inches, with the largest possible volume.

You must give a full justification for your answer using methods taught in this course. You must also demonstrate that your answer really does give the maximum volume.

Solution

Let \( \ell, w, \) and \( h \) denote the length, width, and height of the box, respectively. We want to maximize the volume

\[ V(\ell, w, h) = \ell wh \]

Since \( V \) is a function of 3 variables, we must eliminate 2 of the variables. We will solve for all variables in terms of the width \( w \). We immediately have that \( \ell = 3w \). The total surface area is given by

\[ S = 2\ell w + 2\ell h + 2wh \]

Substituting \( S = 450 \) and \( \ell = 3w \) gives

\[ 450 = 6w^2 + 8wh \]

Now we solve for \( h \) in terms of \( w \).

\[ h = \frac{225 - 3w^2}{4w} \]

Rewriting \( \ell \) and \( h \) in terms of \( w \) in our volume function shows that \( V \) may be written as the single variable function

\[ V(w) = 3w \cdot w \cdot \frac{225 - 3w^2}{4w} = \frac{3}{4} \left( 225w - 3w^3 \right) \]

We now maximize \( V(w) \) on the interval \( w \in (0, \sqrt{75}) \). (The interval is found by considering the extreme cases \( \ell = 0, w = 0, h = 0 \) as degenerate boxes. Neither endpoint may be included since then the surface area would be 0, not 450.)
Since $V(w)$ is differentiable everywhere (it is a polynomial), the only critical numbers are solutions to $V'(w) = 0$.

$$0 = V'(w) = \frac{3}{4} (225 - 9w^2) \implies w^2 = \frac{225}{9} = 25 \implies w = 5$$

(The solution $w = -5$ is not physical since width cannot be negative.) Now observe that $V''(w) = -\frac{3}{4} (18w) < 0$ for all $w > 0$. Hence the graph of $V(w)$ is concave down for $w > 0$, and so $w = 5$ gives a maximum value of $V(w)$.

The optimal dimensions are $\ell = 15$, $w = 5$, and $h = 7.5$ (all measured in inches).

32. Consider the function $f$ and its derivatives below.

$$f(x) = \frac{x^2}{x^2 - 1}, \quad f'(x) = \frac{-2x}{(x^2 - 1)^2}, \quad f''(x) = \frac{6x^2 + 2}{(x^2 - 1)^3}$$

(a) Find all horizontal asymptotes of $f$.

(b) Find all vertical asymptotes of $f$. Then at each vertical asymptote you find, calculate the corresponding one-sided limits of $f$.

(c) Find where $f$ is decreasing and find where $f$ is increasing. Then calculate all points of local extrema, classifying each as either a local minimum, a local maximum, or neither.

(d) Find where $f$ is concave down and find where $f$ is concave up. Then calculate all points of inflection.

Solution

(a) Horizontal asymptotes are found by computing the limit of $f$ as $x \to \pm \infty$.

$$\lim_{x \to \pm \infty} \left( \frac{x^2}{x^2 - 1} \right) = \lim_{x \to \pm \infty} \left( \frac{1}{1 - \frac{1}{x^2}} \right) = \frac{1}{1 - 0} = 1$$

Hence the only horizontal asymptote is the line $y = 1$.

(b) Since $f$ is continuous on its domain, the only candidate vertical asymptotes are the lines $x = -1$ and $x = 1$ (since these are the only $x$-values not in the domain of $f$). Direct substitution of either $x = -1$ or $x = 1$ into $f(x)$ gives the expression $\frac{1}{0}$, which is undefined by indicates that all of the corresponding one-sided limits at both $x = -1$ and $x = 1$ are infinite. Hence $x = -1$ and $x = 1$ are vertical asymptotes.

Now we may compute the limits using sign analysis.

$$\lim_{x \to -1^{-}} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = \infty$$

$$\lim_{x \to -1^{+}} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = -\infty$$

$$\lim_{x \to 1^{-}} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = -\infty$$

$$\lim_{x \to 1^{+}} \left( \frac{x^2}{x^2 - 1} \right) = \bigoplus \infty = \infty$$
(c) Since $f$ is differentiable on its domain, the only first-order critical numbers are solutions to $f'(x) = 0$.

\[
\frac{-2x}{(x^2 - 1)^2} = 0 \implies -2x = 0 \implies x = 0
\]

Recall that since $x = -1$ and $x = 1$ are not in the domain of $f$, we must include such $x$-values on our sign chart also.

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<tr>
<td>$(1, \infty)$</td>
<td>$f'(2)$ = $\oplus$ $\ominus$</td>
<td>decreasing</td>
<td></td>
</tr>
</tbody>
</table>

So $f$ is decreasing on $(0, 1)$ and $(1, \infty)$; $f$ is increasing on $(-\infty, -1)$ and $(-1, 0)$. There is no local minimum; there is a local maximum at $(0, 0)$.

(d) Since $f$ is twice-differentiable on its domain, the only second-order critical numbers are solutions to $f''(x) = 0$.

\[
\frac{6x^2 + 2}{(x^2 - 1)^3} = 0 \implies 6x^2 + 2 = 0 \implies \text{no solution}
\]

Recall that since $x = -1$ and $x = 1$ are not in the domain of $f$, we must include such $x$-values on our sign chart also.

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>shape of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -1)$</td>
<td>$f''(-2)$ = $\oplus$ $\ominus$</td>
<td>concave up</td>
<td></td>
</tr>
<tr>
<td>$(-1, 1)$</td>
<td>$f''(0)$ = $\oplus$ $\ominus$</td>
<td>concave down</td>
<td></td>
</tr>
<tr>
<td>$(1, \infty)$</td>
<td>$f''(2)$ = $\oplus$ $\ominus$</td>
<td>concave up</td>
<td></td>
</tr>
</tbody>
</table>

Hence $f$ is concave down on $(-1, 1)$; $f$ is concave up on $(-\infty, -1)$ and $(1, \infty)$. There are no inflection points.
3 Fall 2018
Midterm Exam #1

33. A scientist measures the temperature $T$ (measured in kelvins) of a certain metal bar $t$ seconds after the measurements have begun. The following equation models the data observed by the scientist.

$$T(t) = 30e^{-t} + 10e^t$$

(a) Calculate $T'(\ln(3))$ and explain its meaning in the context of this problem.

(b) Describe in plain English, as precisely and specifically as you can, what the quantity $Q = T(273) - T(152)$ represents in the context of this problem.

The scientist also observes that the length $L$ of the metal bar (measured in centimeters) depends on its temperature through the following equation.

$$L(T) = 0.01T^3 + \frac{64,000}{T}$$

All final answers must be integers or simplified fractions with proper units.

(c) Calculate $L'(40)$ and explain its meaning in the context of this problem.

(d) At what rate is the length of the bar changing with respect to time exactly $\ln(3)$ seconds after the measurements begin?

Solution

(a) Write $T(t) = 30e^{-t} + 10e^t$. Then we have, by chain rule,

$$T'(t) = -30e^{-t} + 10e^t$$

Hence

$$T'(\ln(3)) = -30 \cdot \frac{1}{3} + 10 \cdot 3 = 20$$

The units are kelvins per second. After $\ln(3)$ seconds since measurements begin, the rate of change of the temperature of the bar was 20 kelvins per second. In other words, if this rate were constant, this would mean that after each additional second after $\ln(3)$ seconds, the temperature of the bar would increase by 20 kelvins.

(b) The quantity $Q$ is the increase in the temperature (measured in kelvins) from 152 seconds after measurements begin to 273 seconds after measurements begin.

(c) Write $L(T) = 0.01T^3 + 64,000T^{-1}$. Then we have

$$L'(T) = 0.03T^2 - 64,000T^{-2}$$

Hence $L'(40) = 0.03 \cdot 1600 - 64,000/1600 = 8$. The units are centimeters per kelvin. If the temperature of the bar is 40 kelvins, then the rate of change of the length of the bar was 8 centimeters per kelvin. In other words, if this rate were constant, this would mean that for each additional 1-kelvin increase in the temperature beyond 40 kelvins, the length of the bar would increase by 8 centimeters.

(d) Observe that $L = L(T(t))$. So we must use chain rule.

$$\frac{dL}{dt} = \frac{d}{dt}L(T(t)) = L'(T(t)) \cdot T'(t)$$
Now substitute \( t = 40 \).

\[
\left. \frac{dL}{dt} \right|_{t=\ln(3)} = L'(T(\ln 3)) \cdot T'(\ln 3)
\]

Note that \( T(\ln 3) = 40 \), and so we have from our previous work

\[
\left. \frac{dL}{dt} \right|_{t=\ln(3)} = L'(40) \cdot T'(\ln 3) = 8 \cdot 20 = 160
\]

The units are **centimeters per second**.

34. For each limit, calculate the value or show that it does not exist. Show all work.

(a) \[ \lim_{u \to 4} \left( \frac{(u + 6)^2 - 25u}{u - 4} \right) \]

(b) \[ \lim_{s \to 1} g(s) \text{ where } g(s) = \begin{cases} \sqrt{1-s} & , s \leq 1 \\ \frac{s^2 - s}{s - 1} & , s > 1 \end{cases} \]

(c) \[ \lim_{h \to 0} \left( \frac{\sin(7 + h) - \sin(7)}{h} \right) \]

*Hint:* Recall the limit definition of the derivative!

(d) \[ \lim_{x \to 6} \left( \frac{\frac{1}{36} - x^{-2}}{x^2 - 36} \right) \]

**Solution**

(a) We have the following work.

\[
\lim_{u \to 4} \left( \frac{(u + 6)^2 - 25u}{u - 4} \right) = \lim_{u \to 4} \left( \frac{u^2 + 12u + 36 - 25u}{u - 4} \right) = \lim_{u \to 4} \left( \frac{u^2 - 13u + 36}{u - 4} \right)
\]

\[
\lim_{u \to 4} \left( \frac{(u - 9)(u - 4)}{u - 4} \right) = \lim_{u \to 4} (u - 9) = -5
\]

(b) We examine the one-sided limits.

\[
\lim_{s \to 1^-} g(s) = \lim_{s \to 1^-} (\sqrt{1-s}) = \sqrt{1-1} = 0
\]

\[
\lim_{s \to 1^+} g(s) = \lim_{s \to 1^+} \left( \frac{s^2 - s}{s - 1} \right) = \lim_{s \to 1^+} \left( \frac{s(s - 1)}{s - 1} \right) = \lim_{s \to 1^+} (s) = 1
\]

Since the left-limit and right-limit are not equal, \( \lim_{s \to 1} g(s) \) does not exist.

(c) Let \( f(x) = \sin(x) \). Then by definition of the derivative,

\[
\lim_{h \to 0} \left( \frac{\sin(7 + h) - \sin(7)}{h} \right) = f'(7) = \cos(7)
\]
(d) We have the following.

\[
\lim_{x \to 6} \left( \frac{1}{36} - x^{-2} \right) = \lim_{x \to 6} \left( \frac{1}{36} - x^{-2} \cdot \frac{36x^2}{36x^2} \right) = \lim_{x \to 6} \left( \frac{x^2 - 36}{36x^2(x^2 - 36)} \right) = \lim_{x \to 6} \left( \frac{1}{36x^2} \right) = \frac{1}{1296}
\]

35. The position of a particle along the x-axis at time t is given by

\[x(t) = t^3 - 6t^2 + 9t + 10\]

(a) When is the particle retreating? When is the particle advancing?

Write your answer using interval notation.

(b) What is the total distance traveled by the particle during the period 0 ≤ t ≤ 7? (The table below shows selected values of the position.)

Write your answer as an integer or simplified fraction.

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>x(t)</td>
<td>10</td>
<td>14</td>
<td>12</td>
<td>10</td>
<td>14</td>
<td>30</td>
<td>64</td>
<td>122</td>
</tr>
</tbody>
</table>

Solution

(a) The velocity of the particle is

\[v(t) = x'(t) = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)\]

We now find where \(v(t) < 0\) and where \(v(t) > 0\). To that end, we find that \(v(t) = 0\) if \(t = 1\) or \(t = 3\). An elementary sign analysis shows the following.

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(−∞, 1)</td>
<td>v(0) = 9</td>
<td>⊕</td>
<td>particle advancing</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>v(2) = −3</td>
<td>⊖</td>
<td>particle retreating</td>
</tr>
<tr>
<td>(3, ∞)</td>
<td>v(4) = 9</td>
<td>⊕</td>
<td>particle advancing</td>
</tr>
</tbody>
</table>

Hence the particle is retreating on (1, 3) and advancing on (−∞, 1) ∪ (3, ∞).

(b) On the interval 0 ≤ t ≤ 7, the particle moves forward for 0 ≤ t < 1, then moves backward for 1 < t < 3, and then moves forward for 3 < t ≤ 7. Hence the total distance traveled is the sum of the distances traveled on each of these three intervals.

\[d = |x(1) - x(0)| + |x(3) - x(1)| + |x(7) - x(3)| = 4 + 4 + 112 = 120\]
36. Find the values of $a$ and $b$ which would make the following function continuous at $x = 0$.

$$f(x) = \begin{cases} 
\frac{4 - \sqrt{16 + 49x^2}}{ax^2}, & x < 0 \\
-23, & x = 0 \\
\tan(2bx)/x, & x > 0 
\end{cases}$$

Write “does not exist” for your answer if appropriate. Otherwise, your answers should be integers or simplified fractions. You must use calculus to give a full, clear justification for your answer.

**Solution**

We require that the left-limit, right-limit, and function value all be equal to $x = 0$. We have the following.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left( \frac{4 - \sqrt{16 + 49x^2}}{ax^2} \right) = \lim_{x \to 0^-} \left( \frac{16 - (16 + 49x^2)}{ax^2(4 + \sqrt{16 + 49x^2})} \right)$$

$$= \lim_{x \to 0^-} \left( \frac{-49}{a(4 + \sqrt{16 + 49})} \right) = \frac{-49}{a(4 + 16 + 0)} = -\frac{49}{8a}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left( \frac{\tan(2bx)}{x} \right) = \lim_{x \to 0^+} \left( \frac{\sin(2bx)}{2bx} \cdot \frac{2b}{\cos(2bx)} \right)$$

$$= \left( \lim_{x \to 0^+} \frac{\sin(2bx)}{2bx} \right) \left( \lim_{x \to 0^+} \frac{2b}{\cos(2bx)} \right) = 1 \cdot \frac{2b}{1} = 2b$$

$f(0) = -23$

Hence we must have that

$$-\frac{49}{8a} = -23 = 2b$$

It follows that $f$ is continuous at $x = 0$ if the constants $a$ and $b$ have the following values.

$$a = \frac{49}{184}, \quad b = -\frac{23}{2}$$

37. Let $f(x) = \frac{1}{3}x^3$ and let $g(x) = x^2 + 15x - 3$. Find all values of $a$ for which the tangent lines to $y = f(x)$ and $y = g(x)$ at $x = a$ are parallel.

**Solution**

If the tangent lines at $x = a$ are parallel, then their slopes are equal, whence it follows that we must solve the equation $f'(a) = g'(a)$.

$$f'(a) = g'(a) \implies a^2 = 2a + 15 \implies 0 = a^2 - 2a - 15 = (a + 3)(a - 5)$$

Hence $a = -3$ or $a = 5$. 


38. Calculate $f'(x)$ where $f$ is the function below.

$$f(x) = \left( \frac{x^8 \sin(3x)}{\ln(x) - \ln(11)} \right)^{2/3}$$

*After calculating the derivative, do not simplify your answer.*

**Solution**

Use power rule, followed by chain rule. The derivative of the expression inside the power “$\frac{2}{3}$” is given by quotient rule.

$$f'(x) = \frac{2}{3} \left( \frac{x^8 \sin(3x)}{\ln(x) - \ln(11)} \right)^{-1/3} \cdot \frac{(\ln(x) - \ln(11))(8x^7 \sin(3x) + 3x^8 \cos(3x)) - x^8 \sin(3x) \cdot \frac{1}{x}}{(\ln(x) - \ln(11))^2}$$

39. Suppose $f$ and $g$ are differentiable for all $x$. For each part, use the table below or explain why there is not enough information.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>-4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-3</td>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
<td>3</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

(a) Let $F(x) = \frac{f(x)}{g(x)}$. Calculate $F'(0)$.

(b) Let $G(x) = f(xg(x))$. Calculate $G'(1)$.

**Solution**

(a) First use quotient rule.

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Now substitute $x = 0$ and use the table of values.

$$F'(0) = \frac{(-4)(4) - (-1)(2)}{4^2} = \frac{-16 + 2}{16} = -\frac{7}{8}$$

(b) First use chain rule, then product rule.

$$G'(x) = \frac{d}{dx}f(xg(x)) = f'(xg(x)) \cdot \frac{d}{dx}(xg(x))$$

$$= f'(xg(x)) \cdot (1 \cdot g(x) + xg'(x))$$
Now substitute $x = 1$ and use the table of values.

\[
G''(1) = f'(1 \cdot g(1)) \cdot (g(1) + 1 \cdot g'(1)) \\
= f'(2) \cdot (g(1) + g'(1)) \\
= 3 \cdot (2 + (-4)) = -6
\]
Midterm Exam #2

40. For each limit, calculate the value or show that it does not exist. If the limit is $+\infty$ or $-\infty$, that should be your answer instead of “does not exist”. Show all work.

(a) \( \lim_{x \to 0} \left( \frac{1 - \cos(9x)}{x^2} \right) \)

(b) \( \lim_{x \to 0} (1 - 3x)^{5/x} \)

Solution

(a) Direct substitution of \( x = 0 \) gives the indeterminate form \( \frac{0}{0} \), whence we may use L'Hospital’s Rule (twice).

\[
\lim_{x \to 0} \left( \frac{1 - \cos(9x)}{x^2} \right) \overset{H}{=} \lim_{x \to 0} \left( \frac{9 \sin(9x)}{2x} \right) \overset{H}{=} \lim_{x \to 0} \left( \frac{81 \cos(9x)}{2} \right) = \frac{81}{2}
\]

(b) Direct substitution of \( x = 0 \) gives the indeterminate form \( 1^{\pm \infty} \), whence we let \( L \) be the desired limit and consider \( \ln(L) \).

\[
\ln(L) = \ln \left( \lim_{x \to 0} (1 - 3x)^{5/x} \right) = \lim_{x \to 0} \ln \left( (1 - 3x)^{5/x} \right) = \lim_{x \to 0} \left( \frac{5 \ln(1 - 3x)}{x} \right)
\]

Direct substitution of \( x = 0 \) now gives the indeterminate form \( \frac{0}{0} \), whence we may use L'Hospital’s Rule.

\[
\lim_{x \to 0} \left( \frac{5 \ln(1 - 3x)}{x} \right) \overset{H}{=} \lim_{x \to 0} \left( \frac{5 \cdot \frac{1}{1-3x} \cdot (-3)}{1} \right) = -15
\]

Hence \( \ln(L) = -15 \), and so \( L = e^{-15} \).

41. Find an equation of the line tangent to the curve

\[
\sin \left( \frac{\pi x}{y} \right) = x - 8y
\]

at the point \((8, 1)\).

Solution

First we use implicitly differentiate each side of the equation with respect to \( x \).

\[
\cos \left( \frac{\pi x}{y} \right) \cdot \left( \frac{y \cdot \frac{\pi}{y} - \pi x \cdot y'}{y^2} \right) = 1 - 8y'
\]

Now we substitute the point \((x, y) = (8, 1)\).

\[
1 \cdot \left( \frac{\pi - 8\pi y'}{1} \right) = 1 - 8y'
\]
Solving for \( y' \) gives \( y' = \frac{1}{8} \), which is the slope of the desired tangent line. Hence an equation of the tangent line is

\[
y - 1 = \frac{1}{8}(x - 8)
\]

42. A person 5 feet tall stands stationary 8 feet from the point \( P \), which is directly beneath a lantern that falls toward the ground. At the moment when the lantern is 15 feet above the ground, the lantern is falling at a speed of 4 feet per second. At what rate is the length of the person’s shadow changing at this moment?

**You must give correct units as part of your answer.**

![Diagram of the problem](image)

**Solution**

Let \( x \) be the distance from the person to the tip of the person’s shadow and let \( h \) be the height of the lantern above the ground. Then by similar triangles, we have the relation

\[
\frac{h}{5} = \frac{x + 8}{x} = 1 + \frac{8}{x}
\]

This relation holds for all time, and so we may differentiate this equation with respect to time \( t \) to obtain an equation involving rates of change which also holds for all time.

\[
\frac{1}{5} \frac{dh}{dt} = -\frac{8}{x^2} \frac{dx}{dt}
\]

We now substitute the values of the variables at the given, specific time. That is, we substitute \( h = 15 \) and \( \frac{dh}{dt} = -4 \) into our two equations.

\[
3 = 1 + \frac{8}{x}
\]

\[
-\frac{4}{5} = -\frac{8}{x^2} \frac{dx}{dt}
\]

Solving for \( x \) in the first of these equations gives \( x = 4 \). Substituting \( x = 4 \) into the second equation and solving for \( \frac{dx}{dt} \) gives \( \frac{dx}{dt} = \frac{8}{5} = 1.6 \). Hence the length of the person’s shadow is changing at a rate of 1.6 feet per second.

43. The concentration of a certain drug in the bloodstream \( t \) hours after the drug is injected is modeled by the following formula.

\[
C(t) = \frac{100t}{t^2 + 1}
\]
(The concentration is measured in micrograms per milliliter.) Use a linear approximation to estimate the change in the concentration over the time period from 2 to 2.1 hours after injection. Also indicate whether the concentration increases or decreases.

**Solution**

Recall that the tangent line approximation tells us that if \( t \) is close to 2, then

\[
C(t) \approx C(2) + C'(2)(t - 2)
\]

Using this approximation, we find that the change in concentration over the interval \([2, 2.1]\) is approximately

\[
\Delta C = C(2.1) - C(2) \approx C(2) + C'(2)(2.1 - 2) - C(2) = 0.1 C'(2)
\]

So all that is left is to calculate \( C'(2) \). Using quotient rule, we have

\[
C'(t) = \frac{(t^2 + 1) \cdot 100 - 100t \cdot 2t}{(t^2 + 1)^2} = \frac{-100(t^2 - 1)}{(t^2 + 1)^2}
\]

Hence we find

\[
C'(2) = \frac{-100(4 - 1)}{(4 + 1)^2} = \frac{-300}{25} = -12
\]

So the change in concentration is approximately

\[
\Delta C \approx 0.1 \cdot (-12) = -1.2
\]

(The negative sign denotes a decrease in concentration.)

44. Consider the function \( f \) and its derivatives below.

\[
f(x) = \frac{2x^3 + 3x^2 - 1}{x^3}, \quad f'(x) = \frac{3 - 3x^2}{x^4}, \quad f''(x) = \frac{6x^2 - 12}{x^5}
\]

*Intervals should be given in a comma-separated list and should be as inclusive as possible. For each part, write “does not exist” as your answer if appropriate. You must show all work.*

(a) Find all horizontal asymptotes of \( f \).

(b) Find all vertical asymptotes of \( f \). Then at each vertical asymptote you find, calculate the corresponding one-sided limits of \( f \).

(c) Find where \( f \) is decreasing and find where \( f \) is increasing. Then calculate the \( x \)-coordinates of all points of local extrema.

(d) Find where \( f \) is concave down and find where \( f \) is concave up. Then calculate the \( x \)-coordinates of all points of inflection.
Solution

(a) Horizontal asymptotes are found by computing the limit of \( f \) as \( x \to \pm\infty \).

\[
\lim_{x \to \pm\infty} \left( \frac{2x^3 + 3x^2 - 1}{x^3} \right) = \lim_{x \to \pm\infty} \left( 2 + \frac{3}{x} - \frac{1}{x^3} \right) = 2 + 0 - 0 = 2
\]

Hence the only horizontal asymptote is the line \( y = 2 \).

(b) Since \( f \) is continuous on its domain, the only candidate vertical asymptote is the line \( x = 0 \) (found by setting the denominator of \( f \) equal to 0). Direct substitution of \( x = 0 \) into \( f(x) \) gives the expression \( \frac{-1}{0} \), which is undefined but indicates that the corresponding one-sided limits at \( x = 0 \) are infinite. Hence the line \( x = 0 \) is a true vertical asymptote. Now we may compute the limits using sign analysis.

\[
\begin{align*}
\lim_{x \to 0^-} \left( \frac{2x^3 + 3x^2 - 1}{x^3} \right) &= \begin{cases} 
-\infty & \text{if } x < 0 \\
\infty & \text{if } x > 0
\end{cases} = \infty \\
\lim_{x \to 0^+} \left( \frac{2x^3 + 3x^2 - 1}{x^3} \right) &= \begin{cases} 
\infty & \text{if } x < 0 \\
-\infty & \text{if } x > 0
\end{cases} = -\infty
\end{align*}
\]

(c) Since \( f \) is differentiable on its domain, the only first-order critical numbers are solutions to \( f'(x) = 0 \).

\[
\frac{3 - 3x^2}{x^4} = 0 \implies 3 - 3x^2 = 0 \implies x = -1, 1
\]

Recall that since \( x = 0 \) is not in the domain of \( f \), we must include \( x = 0 \) on our sign chart for \( f'(x) \).

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>shape of ( f )</th>
</tr>
</thead>
</table>
| \(( -\infty, -1)\) | \( f'(-2) = \begin{cases} 
\odot & \text{if } x < 0 \\
\ominus & \text{if } x > 0
\end{cases} = \ominus \\
\ominus \quad \text{decreasing}
| \(( -1, 0)\)    | \( f'(-\frac{1}{2}) = \begin{cases} 
\odot & \text{if } x < 0 \\
\oplus & \text{if } x > 0
\end{cases} = \oplus \\
\oplus \quad \text{increasing}
| \(( 0, 1)\)     | \( f'(\frac{1}{2}) = \begin{cases} 
\ominus & \text{if } x < 0 \\
\oplus & \text{if } x > 0
\end{cases} = \ominus \\
\ominus \quad \text{increasing}
| \(( 1, \infty)\)| \( f'(2) = \begin{cases} 
\ominus & \text{if } x < 0 \\
\odot & \text{if } x > 0
\end{cases} = \odot \\
\ominus \quad \text{decreasing}

So \( f \) is decreasing on \((-\infty, -1]\) and \([1, \infty)\); \( f \) is increasing on \([-1, 0)\) and \((0, 1]\). There is a local minimum at \( x = -1 \); there is a local maximum at \( x = 1 \).

(d) Since \( f \) is twice-differentiable on its domain, the only second-order critical numbers are solutions to \( f''(x) = 0 \).

\[
\frac{6x^2 - 12}{x^5} = 0 \implies 6x^2 - 12 = 0 \implies x = -\sqrt{2}, \sqrt{2}
\]
Recall that since $x = 0$ is not in the domain of $f$, we must include $x = 0$ on our sign chart for $f''(x)$.

<table>
<thead>
<tr>
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<th>sign</th>
<th>shape of $f$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$\ominus$</td>
<td>concave down</td>
</tr>
<tr>
<td>$(-\sqrt{2}, 0)$</td>
<td>$f'(-1) = \ominus$</td>
<td>$\oplus$</td>
<td>concave up</td>
</tr>
<tr>
<td>$(0, \sqrt{2})$</td>
<td>$f'(1) = \ominus$</td>
<td>$\oplus$</td>
<td>concave down</td>
</tr>
<tr>
<td>$(\sqrt{2}, \infty)$</td>
<td>$f'(2) = \ominus$</td>
<td>$\oplus$</td>
<td>concave up</td>
</tr>
</tbody>
</table>

So $f$ is concave down on $(-\infty, -\sqrt{2}]$ and $(0, \sqrt{2})$; $f$ is concave up on $[-\sqrt{2}, 0)$ and $[\sqrt{2}, \infty)$. There are inflection points at $x = -\sqrt{2}$ and $x = \sqrt{2}$.

45. Let $f(x) = 4(x - 3)^{1/3} - \frac{1}{3}x + 1$. Note: The domain of $f$ is $(-\infty, \infty)$.

(a) Calculate all critical numbers of $f$. For each number you find, you must clearly indicate in your work why it is a critical number.

(b) What are the global extreme values of $f$ on the interval $[2, 30]$?

**Solution**

(a) Note that $f$ is continuous for all $x$. So the critical numbers of $f$ are those values of $x$ for which either $f'(x)$ does not exist or $f'(x) = 0$. We first note that $(x - 3)^{1/3}$ is not differentiable at $x = 3$, hence $x = 3$ is also a critical number of $f$. The derivative is

$$f'(x) = \frac{4}{3}(x - 3)^{-2/3} - \frac{1}{3}$$

Solving the equation $f'(x) = 0$ gives us the solutions $x = -5$ and $x = 11$. So, in summary, $f$ has three critical numbers: $x = -5$, $x = 3$, and $x = 11$.

(b) Since $f$ is continuous on the closed and bounded interval $[2, 30]$, the extreme values of $f$ exist and must be critical values. Checking the critical numbers and endpoints, we find that $f(2) = -\frac{11}{3}$, $f(3) = 0$, $f(11) = \frac{16}{3}$, and $f(30) = 3$. Hence the global minimum value of $f$ on $[2, 30]$ is $-\frac{11}{3}$ and the global maximum value is $\frac{16}{3}$. 
Find the maximum possible area of a rectangle inscribed in the region between the graph of \( f(x) = e^{-x^2/12} \) and the x-axis. Note: The graph of \( y = f(x) \) has no x-intercepts.

You must clearly demonstrate that your answer really is the maximum area!

Solution

Let the upper right vertex of the rectangle be the point \((a, b)\), so that the width of the rectangle is \(2a\) and the height is \(b\). Hence the area is

\[
A(a, b) = 2ab
\]

Since the point \((a, b)\) lies on the graph of \( y = f(x) \), we have that \( b = e^{-a^2/12} \). Hence our goal is to find the maximum value of the function

\[
g(a) = 2ae^{-a^2/12}
\]

on the interval \(0 \leq a < \infty\). Since \( g \) is differentiable on its domain, the only critical numbers are solutions to \( g'(a) = 0 \). First we calculate and simplify \( g'(a) \).

\[
g'(a) = 2 \cdot e^{-a^2/12} + 2a \cdot e^{-a^2/12} \cdot \frac{-a}{6} = 2e^{-a^2/12} \left( 1 - \frac{a^2}{6} \right)
\]

Now we solve \( g'(a) = 0 \). (Observe that \( e^{-a^2/12} > 0 \) for all \( a > 0 \).)

\[
g'(a) = 0 \implies 1 - \frac{a^2}{6} = 0 \implies a = -\sqrt{6}, \sqrt{6}
\]

The only critical number in the interval \([0, \infty)\) is \( a = \sqrt{6} \).

Now we examine the nature of this critical number using the first derivative test. On the interval \([0, \sqrt{6})\) we choose the test point \( a = 1 \) and on the interval \((\sqrt{6}, \infty)\) we choose the
test point \( a = 3 \). Observe the following.

\[
g'(a) = 2e^{-a^2/12} \left( 1 - \frac{a^2}{6} \right)
\]

\[
g'(1) = \bigoplus \left( 1 - \frac{1}{6} \right) = \bigoplus \bigoplus = \bigoplus
\]

\[
g'(3) = \bigoplus \left( 1 - \frac{9}{6} \right) = \bigoplus \bigotimes = \bigotimes
\]

Hence \( g \) is increasing on \([0, \sqrt{6}]\) and decreasing on \([\sqrt{6}, \infty)\). It follows that the absolute maximum of \( g \) on the interval \([0, \infty)\) occurs at \( a = \sqrt{6} \). This maximum value (and hence the maximum area) is

\[
g(\sqrt{6}) = 2\sqrt{6}e^{-6/12} = 2\sqrt{\frac{6}{e}}
\]
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Midterm Exam #1

47. For each part, write “TRUE” or “FALSE”. You are not required to show work, but you may use the provided space for scratch work. For each part, there is no partial credit.

(a) \( \ln(3) - \ln(11) = \frac{\ln(3)}{\ln(11)} \)

(b) The domain of \( f(x) = \sqrt[3]{x-4} \) is all real numbers.

(c) The lines \( 9x + y = 1 \) and \( x - 9y = 4 \) are perpendicular to each other.

(d) The equations \( 2\ln(x) = 0 \) and \( \ln(x^2) = 0 \) have the same solutions.

(e) \( \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} \)

Solution

(a) **False.** The correct identity is \( \ln(3) - \ln(11) = \ln\left(\frac{3}{11}\right) \).

(b) **True.** Every real number has an odd root.

(c) **True.** The slope of the line \( 9x + y = 1 \) is \( m_1 = -9 \) and the slope of the line is \( x - 9y = 4 \) is \( m_2 = \frac{1}{9} \). Since \( m_1m_2 = -1 \), the lines are perpendicular.

(d) **False.** The equation \( 2\ln(x) = 0 \) has solution \( x = 1 \). The equation \( \ln(x^2) = 0 \) has solutions \( x = 1 \) and \( x = -1 \). (The identity \( \ln(x^b) = b\ln(x) \) is true only if \( x > 0 \).)

(e) **True.** The reference angle for \( \frac{5\pi}{6} \) is \( \frac{\pi}{6} \), and \( \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \). Since the angle \( \frac{5\pi}{6} \) lies in the second quadrant, its cosine is negative.

48. Calculate \( \lim_{x \to 5} \left( \frac{x - 5}{x^2 - 2x - 15} \right) \) or determine that the limit does not exist.

Solution

Cancel common factors.

\[
\lim_{x \to 5} \left( \frac{x - 5}{x^2 - 2x - 15} \right) = \lim_{x \to 5} \left( \frac{x - 5}{(x - 5)(x + 3)} \right) = \lim_{x \to 5} \left( \frac{1}{x + 3} \right) = \frac{1}{8}
\]

49. Calculate the derivative of \( f(x) = e^x \sin(x) \). Do not simplify your answer.

Solution

Use product rule.

\[ f'(x) = e^x \sin(x) + e^x \cos(x) \]

50. Find the slope of the line tangent to the graph of \( y = 3\ln(x) - 6\sqrt{x} \) at \( x = 3 \).

Solution

Observe that

\[
\frac{dy}{dx} = 3 \cdot \frac{1}{x} - 6 \cdot \frac{1}{2} \cdot \frac{1}{x} = \frac{3}{x} - \frac{3}{\sqrt{x}}
\]
Hence the slope of the tangent line is
\[
\frac{dy}{dx}\bigg|_{x=3} = \frac{3}{3} - \frac{3}{\sqrt{3}} = 1 - \sqrt{3}
\]

5 pts 51. The number \( N \) of bacteria at time \( t \) grows exponentially, so that \( N(t) = N_0 e^{kt} \). Suppose an initial population of 100 bacteria grows to 500 after 2 hours. How many hours does it take for an initial population of 150 bacteria to grow to 300?

*Your answer should be written exactly in terms of logarithms.*

**Solution**

We are given that if \( N_0 = 100 \) then \( N(2) = 500 \). Hence \( 500 = 100 e^{2k} \), and solving for \( k \) gives \( k = \frac{1}{2} \ln(5) \). Now we want to solve the equation \( 300 = 150 e^{kt} \) for \( t \) with the known value of \( k \). This gives \( t = \frac{2 \ln(2)}{\ln(5)} \).

5 pts 52. Find the value of \( k \) that makes \( f(x) \) continuous at \( x = 1 \). If no such value of \( k \) exists, write “does not exist”.

\[
f(x) = \begin{cases} 
  k \cos(\pi x) - 3x^2, & x \leq 1 \\
  8e^x - k \ln(x), & x > 1
\end{cases}
\]

**Solution**

We require that the left-limit, right-limit, and function value at \( x = 1 \) be equal to ensure continuity at \( x = 1 \).

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (k \cos(\pi x) - 3x^2) = k \cos(\pi) - 3 = -k - 3 \\
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (8e^x - k \ln(x)) = 8e^1 - k \ln(1) = 8e \\
f(1) = (k \cos(\pi x) - 3x^2)|_{x=1} = k \cos(\pi) - 3 = -k - 3
\]

Hence we must have \(-k - 3 = 8e\), or \( k = -8e - 3 \).

5 pts 53. Calculate \( \lim_{x \to 0} \left( \frac{\sin(9x)}{\sin(16x)} \right) \) or determine that the limit does not exist.

**Solution**

Use the special limit \( \lim_{\theta \to 0} \left( \frac{\sin(\theta)}{\theta} \right) = 1 \) and some algebra.

\[
\lim_{x \to 0} \left( \frac{\sin(9x)}{\sin(16x)} \right) = \lim_{x \to 0} \left( \frac{\sin(9x)}{9x} \cdot \frac{16x}{\sin(16x)} \cdot \frac{9}{16} \right) = 1 \cdot 1 \cdot \frac{9}{16} = \frac{9}{16}
\]

5 pts 54. Suppose \( f(4) = -8 \) and \( f'(4) = 3 \). Let \( g(x) = f \left( \frac{1}{4} x^2 \right) \). Find \( g'(4) \). If it is impossible to find \( g'(4) \) with the given information, write “not enough information”.

\[
g'(x) = f' \left( \frac{1}{4} x^2 \right) \cdot \frac{1}{4} \cdot 2x = f' \left( \frac{1}{4} x^2 \right) \cdot x
\]

\[
g'(4) = f' \left( \frac{1}{4} \cdot 4^2 \right) \cdot 4 = f'(4) \cdot 4 = 3 \cdot 4 = 12
\]
Solution
Use chain rule.

\[ g'(x) = f'\left(\frac{1}{4}x^2\right) \cdot \frac{1}{2}x \]

Hence \( g'(4) = f'(4) \cdot 2 = 6 \).

5 pts 55. The graph of \( y = f(x) \) is given below. Find all values of \( a \) in the interval \((-4, 4)\) for which \( \lim_{x \to a} f(x) \) does not exist. If there are no such values of \( a \), write “does not exist”.

![Graph of f(x)](image)

Solution
The values of \( a \) for which \( \lim_{x \to a} f(x) \) does not exist are \( a = -1 \) and \( a = 1 \) only. (At both of these values of \( a \), the left-limit and right-limit are not equal.)

10 pts 56. Find an equation of the line tangent to the curve

\[ \frac{5x}{y} = 4x + y^3 \]

at the point \((1, 1)\). *Any form of the equation of a line is acceptable.*

Solution
Differentiate each side of the equation with respect to \( x \) using implicit differentiation.

\[ \frac{5 \cdot y - 5x \cdot y'}{y^2} = 4 + 3y^2 \cdot y' \]

Substituting the point \((x, y) = (1, 1)\) gives \( 5 - 5y' = 4 + 3y' \), whence \( y' = \frac{1}{8} \). Hence the tangent line has equation

\[ y - 1 = \frac{1}{8}(x - 1) \]
57. Let \( g(x) = 6 - \frac{9}{x} \). Calculate \( g'(3) \) directly from the limit definition of the derivative. If you simply quote a rule, you will receive no credit. You must use the limit definition and show all work.

**Solution**

Start with the definition of derivative and compute the limit using algebra.

\[
g'(3) = \lim_{h \to 0} \left( g(3 + h) - g(3) \right) = \lim_{h \to 0} \left( \frac{6 - \frac{9}{3+h} - (6 - \frac{9}{3})}{h} \right) = \lim_{h \to 0} \left( \frac{3 - \frac{9}{3+h}}{h} \right) \\
= \lim_{h \to 0} \left( \frac{3(3+h) - 9}{h(3+h)} \right) = \lim_{h \to 0} \left( \frac{3h}{(3+h)} \right) = \lim_{h \to 0} \left( \frac{3}{3+h} \right) = \frac{3}{3+0} = 1
\]

58. Show that the following equation is satisfied for at least one value of \( x \) in the interval \((0, 1)\).

\[
\sqrt{12x^4 + 9x^2 + 4} = 4x^3 + 2x
\]

*You must identify any theorem you use by name and clearly justify its use.*

**Solution**

Equivalently, we show that the equation \( \sqrt{12x^4 + 9x^2 + 4} - 4x^3 - 2x = 0 \) has a solution in the interval \((0, 1)\). To that end, let \( f(x) = \sqrt{12x^4 + 9x^2 + 4} - 4x^3 - 2x \). Observe that \( f \) is continuous on \([0, 1]\), \( f(0) = 2 \), and \( f(1) = -1 \). Since 0 is a value between \( f(0) \) and \( f(1) \), the Intermediate Value Theorem (IVT) guarantees that \( f(x) = 0 \) has a solution in \((0, 1)\).

59. Consider the function \( f(x) \) below.

\[
f(x) = \begin{cases} 
4 - \sqrt{2x + 10} & , \ x \neq 3 \\
\frac{4}{x-3} & , \ x = 3
\end{cases}
\]

Is \( f(x) \) continuous at \( x = 3 \)? Explain your answer. *You must use proper calculus to give a complete and clear justification for your answer.*

**Solution**

First we calculate the limit of \( f(x) \) as \( x \to 3 \).

\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} \left( \frac{4 - \sqrt{2x + 10}}{x-3} \right) = \lim_{x \to 3} \left( \frac{4 - \sqrt{2x + 10}}{x-3} \cdot \frac{4 + \sqrt{2x + 10}}{4 + \sqrt{2x + 10}} \right) \\
= \lim_{x \to 3} \left( \frac{16 - (2x + 10)}{(x-3)(4 + \sqrt{2x + 10})} \right) = \lim_{x \to 3} \left( \frac{-2(x-3)}{(x-3)(4 + \sqrt{2x + 10})} \right) \\
= \lim_{x \to 3} \left( \frac{-2}{4 + \sqrt{2x + 10}} \right) = \frac{-2}{4 + \sqrt{2 \cdot 3 + 10}} = -\frac{1}{4}
\]

Observe that \( \lim_{x \to 3} f(x) \neq f(3) = 1 \), and so \( f \) is not continuous at \( x = 3 \).
60. Calculate the derivative of \( f(x) = \frac{\ln (e^{4x} + 6)}{9\tan(x) - \pi^9} \). \textit{Do not simplify your answer.}

**Solution**

Start with quotient rule. To differentiate the numerator, use chain rule twice.

\[
f'(x) = \left( \frac{1}{e^{4x} + 6} \cdot e^{4x} \cdot 4 \right) \frac{(9\tan(x) - \pi^9) - \ln(e^{4x} + 6) \cdot 9\sec(x)^2}{(9\tan(x) - \pi^9)^2}
\]
61. Calculate \( \lim_{x \to 0} \left( \frac{e^{2x} - 1}{\sin(5x)} \right) \) or determine that the limit does not exist.

**Solution**
Direct substitution of \( x = 0 \) gives the indeterminate form \( \frac{0}{0} \), so we use L’Hospital’s Rule.

\[
\lim_{x \to 0} \left( \frac{e^{2x} - 1}{\sin(5x)} \right) = \lim_{x \to 0} \left( \frac{2e^{2x}}{5 \cos(5x)} \right) = \frac{2}{5}
\]

62. Find all (first-order) critical numbers of
\[ f(x) = x - \frac{3}{2} (x - 8)^{2/3} \]
If there are no critical numbers, write “NONE” as your answer.

**Solution**
Critical numbers are values of \( x \) at which \( f \) is not differentiable (\( x = 8 \) only) or where \( f'(x) = 0 \) (\( x = 9 \) only), see below.

\[
0 = f'(x) = 1 - (x - 8)^{-1/3} \implies 1 = (x - 8)^{3} \implies 1 = x - 8 \implies x = 9
\]

63. Use a linear approximation to estimate the value of \( \sqrt{35.9} \). Do not simplify your answer.

**Solution**
Use the tangent line to \( f(x) = \sqrt{x} \) at \( x = 36 \).

\[
y = 6 + \frac{1}{12}(x - 36)
\]
If \( x \) is near 36, then the \( y \)-values of the tangent line are approximately equal to \( \sqrt{x} \). So we have

\[
\sqrt{35.9} \approx 6 + \frac{1}{12}(35.9 - 36)
\]

64. The cost of producing \( x \) units is \( C(x) = 3x^2 + 4x + 1000 \). Use marginal analysis to estimate the cost of producing the 41st unit.

**Solution**
The approximate cost of the 41st unit is given by \( C'(40) \).

\[
C'(40) = (6x + 4) \big|_{x=40} = 6 \cdot 40 + 4 = 244
\]

65. The cost of producing \( x \) units is \( C(x) = 2x^2 + 5x + 8 \). Find the level of production (value of \( x \)) that minimizes the average cost. Recall that average cost is \( AC(x) = \frac{C(x)}{x} \).
Solution
The average cost is \( AC(x) = 2x + 5 + \frac{8}{x} \). The critical numbers are solutions to \( AC'(x) = 0 \).

\[
0 = AC'(x) = 2 - \frac{8}{x^2} \implies x = 2
\]

Since \( AC''(x) = \frac{16}{x^3} > 0 \) for all \( x > 0 \), we see that \( x = 2 \) gives the minimum value of \( AC \).

11 pts 66. Find the equation(s) of all horizontal asymptotes of \( f(x) = \frac{4x^3 - 3x^2}{2x^3 + 9x + 1} \). If there are no horizontal asymptotes, write “NONE” as your answer.

Solution
We must compute the limit of \( f(x) \) as \( x \to -\infty \) and as \( x \to \infty \). If \( x \neq 0 \), we have

\[
f(x) = \frac{4 - \frac{3}{x}}{2 + \frac{9}{x^2} + \frac{1}{x^3}}
\]

So as \( x \to \pm\infty \), each reciprocal power of \( x \) has limit 0. So both limits at infinity are \( \frac{4}{2} = 2 \).

Hence the equation of the (only) horizontal asymptote is \( y = 2 \).

18 pts 67. Consider the function \( f \) and its derivatives below.

\[
f(x) = 2x + \frac{8}{x^2} \quad , \quad f'(x) = \frac{2(x^3 - 8)}{x^3} \quad , \quad f''(x) = \frac{48}{x^4}
\]

Fill in the table below with information about the graph of \( y = f(x) \). For each part, write “NONE” as your answer if appropriate. (You may use the bottom or back of this page for scratch work.) You do not have to show work, and each part of the table will be graded with no partial credit.

Solution
<table>
<thead>
<tr>
<th><strong>vertical asymptote(s)</strong></th>
<th>( x = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>horizontal asymptote(s)</strong></td>
<td>NONE</td>
</tr>
<tr>
<td><strong>where ( f ) is decreasing</strong></td>
<td>((0, 2])</td>
</tr>
<tr>
<td><strong>where ( f ) is increasing</strong></td>
<td>((-\infty, 0), [2, \infty))</td>
</tr>
<tr>
<td><strong>( x )-coordinate(s) of local minima</strong></td>
<td>( x = 2 )</td>
</tr>
<tr>
<td><strong>( x )-coordinate(s) of local maxima</strong></td>
<td>NONE</td>
</tr>
<tr>
<td><strong>where ( f ) is concave down</strong></td>
<td>NONE</td>
</tr>
<tr>
<td><strong>where ( f ) is concave up</strong></td>
<td>((-\infty, 0), (0, \infty))</td>
</tr>
<tr>
<td><strong>( x )-coordinate(s) of inflection point(s)</strong></td>
<td>NONE</td>
</tr>
</tbody>
</table>

The derivatives of \( f \) are

\[
f(x) = 2x + \frac{8}{x^2}, \quad f'(x) = \frac{2(x^3 - 8)}{x^3}, \quad f''(x) = \frac{48}{x^4}
\]

- **asymptotes**

Observe that \( f \) is continuous on its domain, but is undefined for \( x = 0 \). Hence our candidate vertical asymptote is the line \( x = 0 \). Indeed, direct substitution of \( x = 0 \) into the term \( \frac{8}{x^2} \) gives the expression \( \frac{8}{0} \), which indicates that both one-sided limits are infinite. Hence the line \( x = 0 \) is a true vertical asymptote.

As for the horizontal asymptotes we have the following.

\[
\lim_{x \to \pm \infty} \left( 2x + \frac{8}{x^2} \right) = \pm \infty + 0 = \pm \infty
\]

Since neither limit (as either \( x \to -\infty \) or \( x \to \infty \)) is finite, there are no horizontal asymptotes.

- **first-order critical numbers**

Since \( f \) is differentiable on its domain, the only first-order critical numbers are solutions to \( f'(x) = 0 \).

\[
\frac{2(x^3 - 8)}{x^3} = 0 \implies x = 2
\]

- **intervals of increase/decrease**

We make a sign chart for \( f'(x) \). Recall that since \( x = 0 \) is not in the domain of \( f \), we must include \( x = 0 \) on our sign chart.
<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$f'(-1) = \frac{2}{-1}$</td>
<td>$\oplus$</td>
<td>increasing</td>
</tr>
<tr>
<td>$(0, 2)$</td>
<td>$f'(1) = \frac{2}{1}$</td>
<td>$\ominus$</td>
<td>decreasing</td>
</tr>
<tr>
<td>$(2, \infty)$</td>
<td>$f'(3) = \frac{2}{3}$</td>
<td>$\oplus$</td>
<td>increasing</td>
</tr>
</tbody>
</table>

Hence $f$ is decreasing on $(0, 2]$ and increasing on $(-\infty, 0)$ and $[2, \infty)$.

- **local extrema**
  There is a local minimum at $x = 2$ but no local maximum.

- **second-order critical numbers**
  Since $f$ is twice-differentiable on its domain, the only second-order critical numbers are solutions to $f''(x) = 0$.

  \[
  \frac{48}{x^3} = 0 \implies \text{no solution}
  \]

- **intervals of concavity**
  We make a sign chart for $f''(x)$. Recall that since $x = 0$ is not in the domain of $f$, we must include $x = 0$ on our sign chart.

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>shape</th>
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<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$f''(-1) = \frac{48}{-1}$</td>
<td>$\ominus$</td>
<td>concave up</td>
</tr>
<tr>
<td>$(0, \infty)$</td>
<td>$f''(1) = \frac{48}{1}$</td>
<td>$\oplus$</td>
<td>concave up</td>
</tr>
</tbody>
</table>

Hence $f$ is concave up on $(-\infty, 0)$ and $(0, \infty)$; $f$ is concave down on no intervals.

- **inflection points**
  There are no points of inflection.

- **graph sketch**
  Not required.

**10 pts** 68. A child flies a kite at a constant height of 30 feet and the wind is carrying the kite horizontally away from the child at a rate of 5 ft./sec. At what rate must the child let out the string when the kite is 50 feet away from the child?

*You must give correct units as part of your answer.*
Solution
Let $L$ be the distance from the child to the kite and let $x$ be the horizontal distance from the child to the kite. Then, for all time, we have that

$$x^2 + 30^2 = L^2$$

Differentiating with respect to time $t$ we have

$$2x \frac{dx}{dt} = 2L \frac{dL}{dt} \implies x \frac{dx}{dt} = L \frac{dL}{dt}$$

At the time of interest, we have that $\frac{dx}{dt} = 5$ and $L = 50$. Hence, at that time, we have the following.

$$x^2 + 900 = 2500$$
$$5x = 50 \frac{dL}{dt}$$

The first equation gives $x = 40$, and substitution of $x = 40$ into the second equation gives $\frac{dL}{dt} = 4$. Hence the child must let the string out at a rate of 4 ft./sec.

11 pts 69. Find the absolute minimum and absolute maximum values of the function

$$f(x) = \frac{20x}{x^2 + 4}$$

on the interval $[-4, 0]$. You must show all work.

Solution
Since $f$ is continuous and differentiable on $[-4, 0]$, the only critical numbers of $f$ are solutions to $f'(x) = 0$.

$$0 = f'(x) = \frac{(x^2 + 4)(20) - (20x)(2x)}{(x^2 + 4)^2} = \frac{-20x^2 + 80}{(x^2 + 4)^2} \implies x = -2 \text{ or } x = 2$$

Hence the only critical number in $[-4, 0]$ is $x = 2$. Now we compare the endpoint values and critical value: $f(-4) = -4$, $f(-2) = -5$, and $f(0) = 0$. Hence the absolute minimum of $f$ on $[-4, 0]$ is $-5$ and the absolute maximum is $0$.

70. The parts of this problem are related!

(a) Show that $\lim_{x \to \infty} \left( \frac{x}{x - 3} \right)^x = 1$. You must show all work.

(b) Calculate the following limit or show it does not exist. You must show all work.

$$\lim_{x \to \infty} \left( \frac{x}{x - 3} \right)^x$$

*Hint:* Use part (a) first to identify the appropriate indeterminate form.
Solution
(a) We have the following.
\[
\lim_{x \to \infty} \left( \frac{x}{x - 3} \right) = \lim_{x \to \infty} \left( \frac{1 - \frac{3}{x}}{1 - \frac{3}{x}} \right) = \frac{1}{1} = 1
\]

(b) The result of part (a) implies that as \( x \to \infty \), our limit has the indeterminate form \( 1^\infty \). Let \( L \) be the desired limit. Then we have the following.

\[
\ln(L) = \lim_{x \to \infty} \ln \left( \left( \frac{x}{x - 3} \right)^x \right) = \lim_{x \to \infty} \left[ x \ln \left( \frac{x}{x - 3} \right) \right] = \lim_{x \to \infty} \left[ \frac{\ln \left( \frac{x}{x - 3} \right)}{1/x} \right]
\]

As \( x \to \infty \), we now have the indeterminate form \( \frac{0}{0} \), so we may use L’Hospital’s Rule.

\[
\ln(L) = \lim_{x \to \infty} \left( \frac{x - 3 \cdot \frac{1}{x} - 1 \cdot 1}{\frac{1}{x^2}} \right) = \lim_{x \to \infty} \left( \frac{3x}{x - 3} \right) = 3 \lim_{x \to \infty} \left( \frac{x}{x - 3} \right) = 3
\]

We have found that \( \ln(L) = 3 \), whence \( L = e^3 \).

11 pts 71. According to postal regulations, the sum of the girth and length of a parcel may not exceed 90 inches. What are the dimensions (in inches) of the parcel with the largest possible volume that can be sent, if the parcel is a rectangular box with two square sides?

You must demonstrate that your answer really does give the largest volume.

Solution
We want to find the absolute maximum value of the objective function \( V(x, y) = x^2y \) (total volume) subject to the constraint \( 4x + y = 90 \) (sum of girth and length must be 90). Solving for \( y \) in the constraint gives \( y = 90 - 4x \), and so the total volume of the parcel is

\[
f(x) = V(x, 90 - 4x) = x^2(90 - 4x) = 90x^2 - 4x^3
\]
Note that the problem requires that \( x \geq 0 \) and \( y \geq 0 \). The condition \( y \geq 0 \) is equivalent to \( 90 - 4x \geq 0 \), or \( x \leq 22.5 \). Hence our goal is to find the absolute maximum value of
\[
f(x) = 90x^2 - 4x^3
\]
on the interval \([0, 22.5]\). Since \( f \) is differentiable on this interval, the critical numbers are the solutions to \( f'(x) = 0 \).

\[
0 = f'(x) = 180x - 12x^2 = 12x(15 - x) \implies x = 0 \text{ or } x = 15
\]

Checking the endpoint values and critical value, we get: \( f(0) = 0 \), \( f(22.5) = 0 \), and \( f(15) = 225 \cdot 30 > 0 \). Hence the volume of the parcel has an absolute maximum when \( x = 15 \) and \( y = 90 - 4 \cdot 15 = 30 \).

*Alternatively...*

Instead of finding the precise interval of allowed \( x \)-values, we may observe that the allowed interval is some subinterval of \([0, \infty)\) since lengths must be positive. Observe that
\[
f''(x) = 180 - 24x = 12(15 - 2x)
\]
and \( f''(15) = 12 \cdot (-15) < 0 \). Since \( x = 15 \) is the only critical number and \( f''(15) < 0 \), the second derivative test implies that \( f \) must have local (and hence absolute) maximum on \([0, \infty)\) at \( x = 15 \).

(We may also instead use the first derivative test to determine that \( x = 10 \) gives a local (and hence absolute) maximum value.)

### 72. This problem asks about the Mean Value Theorem (MVT).

3 pts  (a) Explain precisely why the function \( f(x) = |3x - 6| \) does not satisfy the hypotheses of the MVT on the interval \([-1, 4]\).

6 pts  (b) The function \( f(x) = \sqrt{x} \) satisfies the hypotheses of the MVT on the interval \([0, 36]\). Find all values of \( c \) guaranteed to exist by the MVT.

**Solution**

(a) The function \( f \) is not differentiable at \( x = 2 \) and \( 2 \) is in the interval \((-1, 4)\).

(b) We have to find all values of \( c \) in \((0, 36)\) that satisfy the equation
\[
f'(c) = \frac{f(36) - f(0)}{36 - 0}
\]

Hence we have
\[
\frac{1}{2\sqrt{c}} = \frac{1}{6} \implies c = 9
\]
Midterm Exam #1

5 pts 73. Calculate \( \lim_{x \to 5} \left( \frac{x^2 - 3x - 10}{x^2 - x - 20} \right) \). If the limit does not exist, write “DNE”.

Solution
Cancel common factors.

\[
\lim_{x \to 5} \left( \frac{x^2 - 3x - 10}{x^2 - x - 20} \right) = \lim_{x \to 5} \left( \frac{(x - 5)(x + 2)}{(x - 5)(x + 4)} \right) = \lim_{x \to 5} \left( \frac{x + 2}{x + 4} \right) = \frac{5 + 2}{5 + 4} = \frac{7}{9}
\]

5 pts 74. Calculate \( \lim_{x \to 0} \left( \frac{\sin^2(4x)}{x^2} \right) \). If the limit does not exist, write “DNE”.

Solution
Use the special limit \( \lim_{\theta \to 0} \left( \frac{\sin(a\theta)}{a\theta} \right) = 1 \).

\[
\lim_{x \to 0} \left( \frac{\sin^2(4x)}{x^2} \right) = \left( \lim_{x \to 0} \frac{\sin(4x)}{x} \right)^2 = \left( \lim_{x \to 0} \left( \frac{\sin(4x)}{4x} \cdot 4 \right) \right)^2 = (1 \cdot 4)^2 = 16
\]

5 pts 75. Calculate \( \lim_{x \to 4} \left( \frac{3 - \sqrt{2x + 1}}{x - 4} \right) \). If the limit does not exist, write “DNE”.

Solution
First rationalize the numerator.

\[
\lim_{x \to 4} \left( \frac{3 - \sqrt{2x + 1}}{x - 4} \right) = \lim_{x \to 4} \left( \frac{9 - (2x + 1)}{(x - 4)(3 + \sqrt{2x + 1})} \right) = \lim_{x \to 4} \left( \frac{-2(x - 4)}{(x - 4)(3 + \sqrt{2x + 1})} \right) = \\
= \lim_{x \to 4} \left( \frac{-2}{3 + \sqrt{9}} \right) = \frac{-2}{3 + 3} = -\frac{1}{3}
\]

5 pts 76. Let \( f(x) = \frac{\ln(x)}{10 - x^3} \). Calculate \( f'(x) \). Do not simplify your final answer.

Solution
Use quotient rule.

\[
f'(x) = \frac{\frac{1}{x} \cdot (10 - x^3) - \ln(x) \cdot (-3x^2)}{(10 - x^3)^2}
\]

5 pts 77. Let \( f(x) = \sqrt{\cos(3 + x^5)} \). Calculate \( f'(x) \). Do not simplify your final answer.
Solution
Use chain rule twice.
\[ f'(x) = \frac{1}{2} \left( \cos(3 + x^5) \right)^{-1/2} \cdot (-\sin(3 + x^5)) \cdot 5x^4 \]

5 pts 78. Solve the inequality \( \frac{3x - 6}{x + 4} > 0 \). Write your answer using interval notation.

Solution
We solve the inequality using the method of sign charts. The cut points for our number line are \( x = 2 \) (obtained by solving \( 3x - 6 = 0 \)) and \( x = -4 \) (obtained by solving \( x + 4 = 0 \)). Hence we examine each of the three subintervals: \( (-\infty, -4) \), \( (-4, 2) \), and \( (2, \infty) \). We test the truth of the inequality on each of these subintervals by testing one \( x \)-value in each subinterval. Testing \( x = -5 \), \( x = 0 \), and \( x = 3 \), we find that the inequality is false only for \( x = 0 \). Hence the inequality is true for all \( x \) in the set \( (-\infty, -4) \cup (2, \infty) \).

5 pts 79. Find an equation of the line tangent to the graph of \( y = \tan(2x) \) at \( x = \frac{\pi}{8} \). Any form of the equation of a line is acceptable.

Solution
The tangent line must pass through the point \( \left( \frac{\pi}{8}, f\left( \frac{\pi}{8} \right) \right) = \left( \frac{\pi}{8}, \tan\left( \frac{\pi}{4} \right) \right) = \left( \frac{\pi}{8}, 1 \right) \). Now we find the derivative using chain rule.
\[ f'(x) = \sec(2x)^2 \cdot 2 \]
Hence the slope of the tangent line is \( f'(\frac{\pi}{8}) = 2 \sec(\frac{\pi}{4})^2 = 4 \). The equation of the tangent line is:
\[ y - 1 = 4 \left( x - \frac{\pi}{8} \right) \]

5 pts 80. Find all critical numbers of \( f(x) = 2 - (x^2 - 2x)^{1/3} \) in the interval \( (-\infty, \infty) \). If there are no critical numbers, write “NONE”.

Solution
Critical numbers come in two types: where \( f'(x) \) does not exist or where \( f'(x) = 0 \). First note that since \( g(x) = x^{2/3} \) is not differentiable at \( x = 0 \), \( f(x) \) is not differentiable when \( x^2 - 2x = 0 \) (i.e., at both \( x = 0 \) and \( x = 2 \)). Now we solve \( f'(x) = 0 \).
\[ 0 = f'(x) = -\frac{2x - 2}{3(x^2 - 2x)^{2/3}} \rightarrow x = 1 \]
Hence \( f(x) \) has three critical numbers: \( x = 0 \), \( x = 1 \), and \( x = 2 \).

12 pts 81. Let \( f(x) = 2x^2 - 5x + 7 \). Use the limit definition of the derivative to calculate \( f'(x) \). If you simply quote a derivative rule without using the limit definition, you will receive no credit.
Solution

Start with the definition of derivative and compute the limit using algebra.

\[
f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} \right) = \lim_{h \to 0} \left( \frac{2(x + h)^2 - 5(x + h) + 7 - (2x^2 - 5x + 7)}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{2x^2 + 4xh + 2h^2 - 5x - 5h + 7 - 2x^2 + 5x - 7}{h} \right) = \lim_{h \to 0} \left( \frac{4xh + 2h^2 - 5h}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{h(4x + 2h - 5)}{h} \right) = \lim_{h \to 0} (4x + 2h - 5) = 4x - 5
\]

82. Find the absolute extreme values of \( f(x) = x + \frac{9}{x} \) on \([1, 18]\).

Solution

We first find the critical numbers of \( f \). Since \( f \) is differentiable on its domain, all critical numbers satisfy \( f'(x) = 0 \).

\[
0 = f'(x) = 1 - \frac{9}{x^2} \implies x = -3 \text{ or } x = 3
\]

The only critical number in the interval \((1, 18)\) is \( x = 3 \). Now we compare the critical values and the endpoint values: \( f(1) = 10, f(3) = 6, \) and \( f(18) = 18.5 \). Hence on the interval \([1, 18]\), the absolute minimum value of \( f \) is 6 and the absolute maximum value is 18.5.

83. Find all points on the graph of \( y = x \ln(x) \) where the tangent line is horizontal.

Solution

A horizontal line has slope 0 and the slope of the tangent line is given by the derivative. Hence we must solve the equation \( f'(x) = 0 \).

\[
0 = f'(x) = 1 + \ln(x) \implies x = e^{-1}
\]

Hence the point on the graph with a horizontal tangent is \((e^{-1}, e^{-1} \ln(e^{-1})) = (e^{-1}, -e^{-1})\).

84. Find the absolute extreme values of \( f(x) = (6 - x)e^x \) on \([0, 6]\). Hint: \( 2 < e < 3 \).

Solution

We first find the critical numbers of \( f \). Since \( f \) is differentiable on its domain, all critical numbers satisfy \( f'(x) = 0 \).

\[
0 = f'(x) = (5 - x)e^x \implies x = 5
\]

The only critical number in the interval \((0, 6)\) is \( x = 5 \). Now we compare the critical values and the endpoint values: \( f(0) = 6, f(5) = e^5, \) and \( f(6) = 0 \). Hence on the interval \([0, 6]\),
the absolute minimum value of $f$ is 0 and the absolute maximum value is $e^5$.

**Solution**

First we calculate the left-limit, right-limit, and function value at $x = 1$.

$$
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-3x + ax^2) = -3 + a
$$

$$
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4ax - 1) = 4a - 1
$$

$$
f(1) = b
$$

To make $f$ continuous at $x = 1$, the left-limit, right-limit, and function value at $x = 1$ must all be equal. Hence we must have

$$
-3 + a = 4a - 1 = b
$$

Solving for $a$ in $-3 + a = 4a - 1$ gives $a = -2/3$, and then solving for $b$ in $4a - 1 = b$ gives $b = -11/3$. 

Midterm Exam #2

5 pts 86. Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be \(+\infty\) or \(-\infty\) (as appropriate), instead of “does not exist”.

\[
\lim_{x\to\pi} \left( \frac{1 + \cos(x)}{(x - \pi)^2} \right)
\]

Solution
Direct substitution of \(x = \pi\) gives \(0/0\), and so we may use L’Hospital’s Rule.

\[
\lim_{x\to\pi} \left( \frac{1 + \cos(x)}{(x - \pi)^2} \right) \overset{H}{=} \lim_{x\to\pi} \left( \frac{-\sin(x)}{2(x - \pi)} \right)
\]

Direct substitution of \(x = \pi\) gives \(0/0\) again, and so we may use L’Hospital’s Rule again.

\[
\lim_{x\to\pi} \left( \frac{-\sin(x)}{2(x - \pi)} \right) \overset{H}{=} \lim_{x\to\pi} \left( \frac{-\cos(x)}{2} \right) = \frac{-\cos(\pi)}{2} = \frac{1}{2}
\]

5 pts 87. Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be \(+\infty\) or \(-\infty\) (as appropriate), instead of “does not exist”.

\[
\lim_{x\to\infty} \left( \frac{1 - \frac{12}{x}}{x} \right)^{5x}
\]

Solution
As \(x \to \infty\), we find that we have the indeterminate form \(1^\infty\). So we use logarithms and L’Hospital’s Rule. Let \(L\) be the desired limit and consider \(\ln(L)\).

\[
\ln(L) = \lim_{x\to\infty} \ln \left( \left( \frac{1 - \frac{12}{x}}{x} \right)^{5x} \right) = \lim_{x\to\infty} \left( 5x \ln \left( \frac{1 - \frac{12}{x}}{x} \right) \right) = \lim_{x\to\infty} \left( \frac{5 \ln \left( 1 - \frac{12}{x} \right)}{1/x} \right)
\]

We now have the indeterminate form \(0/0\), and so we may use L’Hospital’s Rule.

\[
\lim_{x\to\infty} \left( \frac{5 \ln \left( 1 - \frac{12}{x} \right)}{1/x} \right) \overset{H}{=} \lim_{x\to\infty} \left( \frac{5 \cdot \left( 1 - \frac{12}{x} \right) \cdot \frac{12}{x^2}}{-1/x^2} \right) = \lim_{x\to\infty} \left( \frac{-60}{1 - \frac{12}{x}} \right) = \frac{-60}{1 - 0} = -60
\]

So we have \(\ln(L) = -60\), whence \(L = e^{-60}\).

5 pts 88. Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be \(+\infty\) or \(-\infty\) (as appropriate), instead of “does not exist”.

\[
\lim_{x\to1} \left( \frac{x^3 - 2x^2 - 5x + 6}{x^3 + x^2 + x - 3} \right)
\]
Solution

Direct substitution of \( x = 1 \) gives \( 0/0 \), and so we may use L’Hospital’s Rule.

\[
\lim_{{x \to 1}} \left( \frac{x^3 - 2x^2 - 5x + 6}{x^3 + x^2 + x - 3} \right) = \lim_{{x \to 1}} \left( \frac{3x^2 - 4x - 5}{3x^2 + 2x + 1} \right) = \frac{3 - 4 - 5}{3 + 2 + 1} = -1
\]

5 pts 89. Evaluate the limit or determine that it does not exist. If the limit is infinite, then your answer should be \(+\infty\) or \(-\infty\) (as appropriate), instead of “does not exist”.

\[
\lim_{{x \to 4^+}} \left( \frac{2x - x^2}{x - 4} \right)
\]

Solution

Direct substitution of \( x = 4 \) gives \(-8/0\), which is not an indeterminate form but rather tells us that the one-sided limit is infinite. Note that if \( x \to 4^+ \), then we may assume \( x - 4 > 0 \), whence the denominator has limit 0 but remains positive. The numerator has limit \(-8\), which is negative. So the sign of our limit is \( \ominus/\ominus = \ominus \). So the limit is \(-\infty\).

5 pts 90. Find all inflection points, if any, of \( f(x) = x^3 - 12x^2 + 5x - 10 \). (For your final answer, give only the \( x \)-coordinate of each inflection point or write “NONE”.)

Solution

Observe that \( f''(x) = 6x - 24 = 6(x - 4) \), which changes sign (from negative to positive) at \( x = 4 \). Since \( f \) is also continuous at \( x = 4 \), \( f \) has an inflection point at \( x = 4 \).

5 pts 91. Find an equation of the tangent line to the graph of the following equation at the point \((0, 0)\).

\[
sin(x + 2y) + 9x + 1 = e^y
\]

Solution

Implicitly differentiating each side of the equation with respect to \( x \) gives the following.

\[
\cos(x + 2y) \cdot (1 + 2y') + 9 = e^y \cdot y'
\]

Substituting \( x = y = 0 \) gives us the equation:

\[
1 + 2y' + 9 = y'
\]

Hence \( y' = -10 \) at the point \((0, 0)\). The tangent line is thus described by the equation \( y = -10x \).

5 pts 92. If \( x \) units of a certain product are produced, the total cost is \( C(x) = 5x^2 + 104x + 80 \). Find the level of production which minimizes the average cost per unit.
Solution
The average cost per unit is

\[ AC(x) = \frac{C(x)}{x} = 5x + 104 + \frac{80}{x} \]

The minimum value of \( AC(x) \) occurs at the value of \( x \) such that \( AC''(x) = 0 \). Observe that

\[ AC''(x) = 5 - \frac{80}{x^2} \]

and \( AC''(x) = 0 \) has solutions \( x = 4 \) and \( x = -4 \). Since production must be non-negative, average cost is minimized when \( x = 4 \).

93. A spherical snowball melts in such a way that it always remains a sphere, and its volume decreases at 8 cm\(^3\)/sec. At what rate is the surface area of the snowball changing when its surface area is \( 40\pi \) cm\(^2\)?

**You must give correct units as part of your answer.**

*Hint: \( V = \frac{4}{3}\pi r^3 \), \( A = 4\pi r^2 \)*

Solution
We have the two basic equations:

\[ V = \frac{4}{3}\pi r^3 \quad , \quad A = 4\pi r^2 \]

Differentiating each equation with respect to \( t \) gives us two more equations.

\[ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad , \quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt} \]

Now we substitute the given information. Specifically, \( \frac{dV}{dt} = -8 \) and \( A = 40\pi \). The four equations above give us the following.

\[ V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad 40\pi = 4\pi r^2 \]

\[ -8 = 4\pi r^2 \frac{dr}{dt} \quad \Rightarrow \quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt} \]

Our goal is to solve for \( \frac{dA}{dt} \). The upper right equation gives us \( r = \sqrt{10} \) and substituting this into the lower left equation gives us \( \frac{dr}{dt} = -\frac{8}{40\pi} \). So now substituting everything into the lower right equation gives the final answer.

\[ \frac{dA}{dt} = 8\pi \cdot \sqrt{10} \cdot -\frac{8}{40\pi} = -\frac{64\sqrt{10}}{40} = -\frac{8\sqrt{10}}{5} \]

So the surface area is changing at a rate of \(-\frac{8\sqrt{10}}{5}\) cm\(^2\)/sec.
94. Note: The parts of this problem are not related!

(a) Use linear approximation to estimate the value of $\sqrt{79}$. Your answer should be an integer or a simplified fraction of integers.

(b) A manufacturer’s total cost to produce $x$ units is $C(x) = 25 \ln(x^2 + 16)$. Use marginal analysis to estimate the cost of the 4th unit.

**Solution**

(a) Consider the tangent line to $f(x) = \sqrt{x}$ at $x = 81$. We have $f(81) = 9$ and $f'(81) = \frac{1}{2} \cdot x^{-1/2} \bigg|_{x=81} = \frac{1}{2} \cdot 81^{-1/2} = \frac{1}{18}$. Hence the linearization is

$$L(x) = 9 + \frac{1}{18} (x - 81)$$

Recall the fundamental principle of linear approximation. If $x$ is near 81, then $f(x) \approx L(x)$. So we have

$$\sqrt{79} \approx 9 + \frac{1}{18} (79 - 81) = 9 - \frac{1}{9} = \frac{80}{9}$$

(b) Marginal analysis tells us that the approximate cost of the 4th unit is $C'(3)$. So we have:

$$C'(3) = 25 \cdot \frac{2x}{x^2 + 16} \bigg|_{x=3} = \frac{50 \cdot 3}{9 + 16} = \frac{150}{25} = 6$$
Solution
Let $x$ be the length of the square base and let $y$ be the height of the container. The constraint is that the total cost of the box must be $60$, and so our constraint equation in terms of $x$ and $y$ is:

$$2(x^2 + 4xy) + 3x^2 = 60$$

Solving for $y$ gives us

$$y = \frac{60 - 5x^2}{8x}$$

Our goal is to maximize the volume, whence our objective function is $V(x, y) = x^2y$. In terms of $x$ only, our objective is:

$$f(x) = V(x, \frac{60 - 5x^2}{8x}) = x^2 \cdot \frac{60 - 5x^2}{8x} = \frac{1}{8} (60x - 5x^3)$$

To minimize $f$, we find the critical numbers of $f$, which are solutions to $f'(x) = 0$.

$$f'(x) = \frac{1}{8} (60 - 15x^2) = \frac{15}{8} (4 - x^2)$$

Hence the only valid solution to $f'(x) = 0$ is $x = 2$. (We reject $x = -2$ since lengths must be non-negative.)

Now observe that $f''(x) = -\frac{30}{8}x$, which is negative for all positive values of $x$. Hence our objective function has one critical number and is always concave down. So our critical number must give a global maximum value.

The length of the box is $x = 2$, the height is $y = 2.5$, and the volume is $f(2) = 10$.

97. Consider the function $f$ and its derivatives below.

$$f(x) = \frac{3x^3 - 2x + 48}{x} \quad , \quad f'(x) = \frac{6(x^3 - 8)}{x^2} \quad , \quad f''(x) = \frac{6(x^3 + 16)}{x^3}$$

Find where $f$ is increasing, decreasing, concave up, and concave down. Use interval notation in your answers. Also, find all horizontal and vertical asymptotes, relative extrema, and inflection points of $f$. Write “NONE” as your answer, if appropriate.

Solution
The derivatives of $f$ are

$$f(x) = \frac{3x^3 - 2x + 48}{x}, \quad f'(x) = \frac{6(x^3 - 8)}{x^2}, \quad f''(x) = \frac{6(x^3 + 16)}{x^3}$$

- **asymptotes**

  Observe that $f$ is continuous on its domain, but is undefined for $x = 0$. Hence our candidate vertical asymptotes is the line $x = 0$. Indeed, direct substitution of $x = 0$ into $f(x)$ gives the expression $\frac{48}{0}$, which indicates that both one-sided limits are infinite. Hence the line $x = 0$ is a true vertical asymptote.

  As for the horizontal asymptotes we have the following.

  $$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \left(3x^2 - 2 + \frac{48}{x}\right) = \infty - 2 + 0 = \infty$$

  Hence there are no horizontal asymptotes.

- **first-order critical numbers**

  Since $f$ is differentiable on its domain, the only first-order critical numbers are solutions to $f'(x) = 0$.

  $$\frac{6(x^3 - 8)}{x^2} = 0 \implies x = 2$$

- **intervals of increase/decrease**

  We make a sign chart for $f'(x)$. Recall that we must include vertical asymptotes on our sign chart.
\[
\begin{array}{cccc}
\text{interval} & \text{test point} & \text{sign} & \text{shape} \\
(\infty, 0) & f'(1) = \frac{6}{6} & \ominus & \text{decreasing} \\
(0, 2) & f'(3) = \frac{6}{6} & \ominus & \text{decreasing} \\
(2, \infty) & f'(3) = \frac{6}{6} & \ominus & \text{increasing} \\
\end{array}
\]

Hence \( f \) is decreasing on \((\infty, 0)\) and \((0, 2]\); and increasing on \([2, \infty)\).

- **local extrema**
  - There is no local maximum but there is a local minimum at \( x = 2 \).

- **second-order critical numbers**
  - Since \( f \) is twice-differentiable on its domain, the only second-order critical numbers are solutions to \( f''(x) = 0 \).

  \[
  f''(x) = \frac{6(x^3 + 16)}{x^3} = 0 \implies x = -\sqrt[3]{16}
  \]

- **intervals of concavity**
  - We make a sign chart for \( f''(x) \). Recall that we must include vertical asymptotes on our sign chart.

  \[
  \begin{array}{cccc}
  \text{interval} & \text{test point} & \text{sign} & \text{shape} \\
  (\infty, -\sqrt[3]{16}) & f''(-3) = \frac{6}{6} & \oplus & \text{concave up} \\
  (-\sqrt[3]{16}, 0) & f''(-1) = \frac{6}{6} & \ominus & \text{concave down} \\
  (0, \infty) & f''(1) = \frac{6}{6} & \oplus & \text{concave up} \\
  \end{array}
  \]

Hence \( f \) is concave down on \([-\sqrt[3]{16}, 0)\); and concave up on \((\infty, -\sqrt[3]{16}]\) and \((0, \infty)\).

- **inflection points**
  - There is a point of inflection at \( x = -\sqrt[3]{16} \) since the concavity changes at those values of \( x \) and \( f \) is continuous there.

- **graph sketch**
  - Not required.
6  Spring 2020
Midterm Exam #1

5 pts 98. Solve the inequality \( \frac{3x + 6}{x - 4} < 0 \). Write your answer using interval notation.

Solution

Use the cut-point (or sign chart) method. For our sign chart, the cut points are found by setting the numerator and denominator to 0 separately. Hence the cut points are \( x = -2 \) and \( x = 4 \). Now we test the truth of the inequality using one point from each corresponding subinterval.

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign of ( \frac{3x + 6}{x - 4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (-\infty, -2) )</td>
<td>( x = -3 )</td>
<td>( \varnothing = \oplus )</td>
</tr>
<tr>
<td>( (-2, 4) )</td>
<td>( x = 0 )</td>
<td>( \oplus = \varnothing )</td>
</tr>
<tr>
<td>( (4, \infty) )</td>
<td>( x = 5 )</td>
<td>( \oplus = \oplus )</td>
</tr>
</tbody>
</table>

Checking the cut points themselves, we see the inequality is not satisfied at either cut point. So the final answer is: \( (-2, 4) \).

5 pts 99. Find the domain of the function \( f(x) = \frac{\ln(80 - x)}{\sqrt{x} - 5} \). Write your answer using interval notation.

Solution

The expression \( \ln(80 - x) \) is defined only for \( 80 - x > 0 \), or on the interval \( (-\infty, 80) \). The expression \( \sqrt{x} \) is defined only for \( x \geq 0 \), or on the interval \( [0, \infty) \). Both expressions are thus defined on the intersection of these two intervals: \( [0, 80) \). Finally, we must exclude any values of \( x \) for which \( \sqrt{x} - 5 = 0 \), so just \( x = 25 \). Hence the domain of \( f \) is \( [0, 25) \cup (25, 80) \).

5 pts 100. Evaluate \( \lim_{x \to 0} \left( \frac{(4x + 1)^2 - 1}{x} \right) \) or determine that the limit does not exist.

Solution

Expand and cancel common factors.

\[
\lim_{x \to 0} \left( \frac{(4x + 1)^2 - 1}{x} \right) = \lim_{x \to 0} \left( \frac{16x^2 + 8x + 1 - 1}{x} \right) = \lim_{x \to 0} (16x + 8) = 8
\]

5 pts 101. Evaluate \( \lim_{x \to 0} \left( \frac{9x \cos(2x)}{\sin(4x)} \right) \) or determine that the limit does not exist.
Rearrange the terms and use the special trigonometric limit and direct substitution.

\[
\lim_{x \to 0} \left( \frac{9x \cos(2x)}{\sin(4x)} \right) = \lim_{x \to 0} \left( \frac{4x}{\sin(4x)} \cdot \frac{9}{4} \cdot \cos(2x) \right) = 1 \cdot \frac{9}{4} \cdot 1 = \frac{9}{4}
\]

**5 pts** 102. Determine where \( f \) is continuous. Write your answer using interval notation.

\[
f(x) = \begin{cases} 
9 - 16x, & x < 0 \\
3x^2 - x^3, & 0 \leq x \leq 3 \\
1 - e^{x-3}, & x > 3
\end{cases}
\]

**Solution**

Observe that \( f \) is clearly continuous for all \( x \) except possibly \( x = 0 \) or \( x = 3 \). For these transition points, we check whether the corresponding left-limit, right-limit, and function value are equal. For \( x = 0 \) we have:

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (9 - 16x) = 9 - 0 = 9 \\
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3x^2 - x^3) = 0 - 0 = 0 \\
f(0) = (3x^2 - x^3)|_{x=0} = 0
\]

Since these three values are not all equal, \( f \) is discontinuous at \( x = 0 \). For \( x = 3 \) we have:

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3x^2 - x^3) = 27 - 27 = 0 \\
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (1 - e^{x-3}) = 1 - 1 = 0 \\
f(3) = (3x^2 - x^3)|_{x=3} = 27 - 27 = 0
\]

Since these three values are all equal, \( f \) is continuous at \( x = 3 \). Hence the final answer is that \( f \) is continuous on \((-\infty, 0) \cup (0, \infty)\).

**5 pts** 103. Find an equation of the line tangent to the graph of \( y = 5e^{2\cos(x)} \) at \( x = 3\pi/2 \).

**Solution**

Let \( f(x) = 5e^{2\cos(x)} \). The point of tangency is \( (\frac{3\pi}{2}, f \left( \frac{3\pi}{2} \right)) = (\frac{3\pi}{2}, 5) \). Observe that \( f'(x) = 5e^{2\cos(x)} \cdot (-2\sin(x)) \). Hence the slope of the tangent line is \( f' \left( \frac{3\pi}{2} \right) = 10 \). Thus an equation of the tangent line is

\[
y - 5 = 10 \left( x - \frac{3\pi}{2} \right)
\]

**5 pts** 104. Let \( f(x) = 2x^2 - \frac{1}{5x} - 8\sqrt{x} + 14\pi^{3/2} \). Calculate \( f'(x) \). Do not simplify your answer.
Write the function using exponents.

\[ f(x) = 2x^2 - \frac{1}{5}x^{-1} - 8x^{1/2} + 14\pi^{3/2} \]

Differentiate using power rule, noting that \(14\pi^{3/2}\) is a constant.

\[ f'(x) = 4x - \frac{1}{5}x^{-2} - 4x^{-1/2} \]

**5 pts 105.** Let \( f(x) = \left( \frac{x^4 - 20x}{x^3 + 20} \right)^{2/3} \). Calculate \( f'(x) \). Do not simplify your answer.

**Solution**

Use power rule first, then use chain rule (using quotient rule to find the derivative of the “inside” function).

\[ f'(x) = \frac{2}{3} \left( \frac{x^4 - 20x}{x^3 + 20} \right)^{-1/3} \cdot \frac{(4x^3 - 20)(x^3 + 20) - (x^4 + 20x)(3x^2)}{(x^3 + 20)^2} \]

**10 pts 106.** Let \( f(x) = \frac{x + 8}{x - 3} \). Use the limit definition of derivative to calculate \( f'(2) \). If you simply quote a derivative rule without using the limit definition, you will receive no credit.

**Solution**

Start with the definition of derivative and compute the limit using algebra.

\[
\begin{align*}
    f'(2) &= \lim_{h \to 0} \left( \frac{f(2 + h) - f(2)}{h} \right) \\
    &= \lim_{h \to 0} \left( \frac{h + 10}{h - 1} - (-10) \right) \\
    &= \lim_{h \to 0} \left( \frac{h + 10 + 10(h - 1)}{h(h - 1)} \right) \\
    &= \lim_{h \to 0} \left( \frac{11h}{h(h - 1)} \right) = \lim_{h \to 0} \left( \frac{11}{h - 1} \right) = \frac{11}{0-1} = -11
\end{align*}
\]

107. For each part, calculate \( f'(x) \). Do not simplify your answer.

(a) \( f(x) = \sin\left(12x - x^9\right) \ln(x) \)

(b) \( f(x) = \frac{e^{5\sec(6x)+1}}{7} \)

**Solution**

(a) Use product rule. When differentiating the first term, use chain rule.

\[
f'(x) = \cos(12x - x^9) \cdot (12 - 9x^8) \cdot \ln(x) + \sin(12x - x^9) \cdot \frac{1}{x}
\]
(b) Use chain rule twice. (Do not use quotient rule. The factor of $\frac{1}{7}$ is a constant coefficient.)

\[
f'(x) = \frac{1}{7}e^{5\sec(6x)+1} \cdot 5\sec(6x)\tan(6x) \cdot 6
\]

108. For each part, calculate the limit or show that it does not exist.

5 pts  
(a) \( \lim_{x \to -1} \left( \frac{4 - \sqrt{16x + 32}}{x + 1} \right) \)

5 pts  
(b) \( \lim_{x \to 3} g(x) \), where \( g(x) = \begin{cases} 
  x - 3 & , \ x < 3 \\
  x^3 - 9x & , \ x = 3 \\
  x - 2 & , \ x > 3 
\end{cases} \)

Solution
(a) Rationalize the numerator and cancel common factors.

\[
\lim_{x \to -1} \left( \frac{4 - \sqrt{16x + 32}}{x + 1} \right) = \lim_{x \to -1} \left( \frac{16 - (16x + 32)}{(x + 1)(4 + \sqrt{16x + 32})} \right) \\
= \lim_{x \to -1} \left( \frac{-16}{4 + \sqrt{16x + 32}} \right) = -\frac{16}{4} = -2
\]

(b) Compute the left- and right-limits and verify whether they are equal. For the left-limit cancel common factors. For the right-limit, use direct substitution. The function value \( g(3) \) is irrelevant to this problem.

\[
\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} \left( \frac{x - 3}{x^3 - 9x} \right) = \lim_{x \to 3^-} \left( \frac{1}{x(x + 3)} \right) = \frac{1}{18} \\
\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} \left( \frac{x - 2}{x^2 + 9} \right) = \frac{1}{18}
\]

The left- and right-limits are both equal to \( \frac{1}{18} \), hence \( \lim_{x \to 3} g(x) = \frac{1}{18} \) also.

109. Calculate the following limit or show that it does not exist. You must show and explain your work clearly.

\[
\lim_{x \to 4^-} \left( \frac{|x^2 - 16|}{4 - x} \right)
\]

Solution
Note that the limit symbol “\( x \to 4^- \)” means that we may assume that both \( x \) is arbitrarily close to 4 and \( x < 4 \). For values of \( x \) just slightly less than 4, the values of \( x^2 - 16 \) are negative. Hence under the assumptions of this limit, we have \( |x^2 - 16| = -(x^2 - 16) = \)
16 - x^2 = (4 - x)(4 + x). So now we have
\[
\lim_{{x \to 4^-}} \left( \frac{|x^2 - 16|}{4 - x} \right) = \lim_{{x \to 4^-}} \left( \frac{(4 - x)(4 + x)}{4 - x} \right) = \lim_{{x \to 4^-}} (4 + x) = 8
\]

10 pts 110. Find the x-coordinate of each point on the graph of \( y = 3x + \frac{10}{x} \) where the tangent line is parallel to the line \( y = 20 - 2x \).

Solution

Let \( f(x) = 3x + \frac{10}{x} \). The slope of the line \( y = 20 - 2x \) is \(-2\) and parallel lines have equal slopes. Hence we seek all values of \( x \) that solve the equation \( f'(x) = -2 \).

\[
f'(x) = -2 \implies 3 - \frac{10}{x^2} = -2
\]
Solving for \( x \) gives \( x = -\sqrt{2} \) or \( x = \sqrt{2} \).

10 pts 111. Find the value of \( k \) that makes \( f \) continuous at \( x = -2 \) or determine that no such value of \( k \) exists. You must use calculus to solve the problem and write your work using proper notation.

\[
f(x) = \begin{cases} 
3x^2 + k & , \ x < -2 \\
-10 & , \ x = -2 \\
kx^3 - 6 & , \ x > -2 
\end{cases}
\]

Solution

For \( f \) to be continuous at \( x = -2 \), the corresponding left-limit, right-limit, and function value must all be equal. Those three values in terms of \( k \) are given by the following:

\[
\lim_{{x \to -2^-}} f(x) = \lim_{{x \to -2^-}} (3x^2 + k) = 12 + k \\
\lim_{{x \to -2^+}} f(x) = \lim_{{x \to -2^+}} (kx^3 - 6) = -8k - 6 \\
f(-2) = -10
\]
If \( f \) is to be continuous at \( x = -2 \), we must have \( 12 + k = -8k - 6 = -10 \). This is equivalent to the following set of two equations in the single unknown \( k \).

\[
12 + k = -10 \\
-8k - 6 = -10
\]
However, this set of equations has no solution. (Indeed, the first equation gives \( k = -22 \), which does not satisfy the second equation.) Hence there is no value of \( k \) that makes \( f \) continuous at \( x = -2 \).
112. Calculate the following limit or show that it does not exist. If the limit is infinite, your answer should be $+\infty$ or $-\infty$, not “does not exist”.

$$\lim_{x \to 0^+} \left( \sqrt{12x + 9} - \sqrt{2x + 4} \right)^{1/x}$$

**Solution**
Direct substitution of $x = 0$ gives the indeterminate form $1^\infty$. So we use logarithms to write the limit as a quotient, and we use L’Hospital’s Rule.

$$L = \lim_{x \to 0^+} \left( \sqrt{12x + 9} - \sqrt{2x + 4} \right)^{1/x}$$
$$\ln(L) = \lim_{x \to 0^+} \ln \left( \left( \sqrt{12x + 9} - \sqrt{2x + 4} \right)^{1/x} \right)$$
$$\ln(L) = \lim_{x \to 0^+} \frac{\ln(\sqrt{12x + 9} - \sqrt{2x + 4})}{x}$$
$$\ln(L) = \lim_{x \to 0^+} \frac{1}{\sqrt{12x + 9} - \sqrt{2x + 4}} \cdot \left( \frac{6}{\sqrt{12x + 9}} - \frac{1}{\sqrt{2x + 4}} \right)$$

Therefore, $\ln(L) = \frac{3}{2}$, whence $L = e^{3/2}$.

113. A particle in the fourth quadrant is moving along a path described by the equation

$$x^2 + xy + 2y^2 = 16$$

such that at the moment its $x$-coordinate is 2, its $y$-coordinate is decreasing at a rate of 5 cm/sec. At what rate is its $x$-coordinate changing at that time?

**Solution**
Differentiating the given equation gives the following.

$$2x \frac{dx}{dt} + \frac{dx}{dt} y + x \frac{dy}{dt} + 4y \frac{dy}{dt} = 0$$

At the given time, we have $x = 2$ and $\frac{dy}{dt} = -5$, and we want to find $\frac{dx}{dt}$. Substituting this information into our two equations gives

$$4 + 2y + 2y^2 = 16$$
$$4 \frac{dx}{dt} + \frac{dx}{dt} y - 10 - 20y = 0$$
Solving for $y$ in the first equation gives $y = -3$ or $y = 2$. Since the particle is in the fourth quadrant, we have $y = -3$. Substituting into the second equation above gives

$$4 \frac{dx}{dt} - 3 \frac{dx}{dt} - 10 + 60 = 0$$

Hence $\frac{dx}{dt} = -50$ cm/sec.

**10 pts** 114. Use linear approximation or differentials to estimate the value of $\frac{1}{\sqrt{8.48}}$.

**Solution**

Put $f(x) = x^{-1/3}$. We use the tangent line to $f$ at $x = 8$. Observe that $f(8) = \frac{1}{2}$ and $f'(8) = -\frac{1}{3} \cdot 8^{-4/3} = -\frac{1}{48}$. Hence the desired tangent line is

$$y = \frac{1}{2} - \frac{1}{48} (x - 8)$$

The desired approximation is then

$$(8.48)^{-1/3} \approx \frac{1}{2} - \frac{1}{48} (8.48 - 8) = \frac{1}{2} - \frac{1}{100} = 0.5 - 0.01 = 0.49$$

**10 pts** 115. Find an equation of line tangent to the following curve at the origin.

$$\sin(x + 3y) + 9x + 1 = e^y$$

**Solution**

Using implicit differentiation with respect to $x$ gives the following.

$$\cos(x + 3y) \cdot \left(1 + 3 \frac{dy}{dx}\right) + 9 = e^y \cdot \frac{dy}{dx}$$

Substituting $x = 0$ and $y = 0$ gives

$$1 + 3 \frac{dy}{dx} + 9 = \frac{dy}{dx}$$

Hence the slope of the tangent line is $\frac{dy}{dx} = -5$, whence an equation of the tangent line is

$$y = -5x$$

**10 pts** 116. Find all horizontal asymptotes of

$$f(x) = \frac{12x + 5}{\sqrt{16x^2 + x + 1}}$$

or determine that there are no horizontal asymptotes.
Solution
First we do some algebra before computing the relevant limits.

\[
\frac{12x + 5}{\sqrt{16x^2 + x + 1}} = \frac{x}{\sqrt{x^2}} \cdot \frac{12 + \frac{5}{x}}{\sqrt{16 + \frac{1}{x} + \frac{1}{x^2}}} = \frac{x}{|x|} \cdot \frac{12 + \frac{5}{x}}{\sqrt{16 + \frac{1}{x} + \frac{1}{x^2}}}
\]

For the limit \( x \to \infty \), we have \(|x| = x\), whence \( \frac{|x|}{x} = 1 \). For the limit \( x \to -\infty \), we have \(|x| = -x\), whence \( \frac{|x|}{x} = -1 \). So now we have the following.

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left( \frac{12 + \frac{5}{x}}{\sqrt{16 + \frac{1}{x} + \frac{1}{x^2}}} \right) = \frac{12 + 0}{\sqrt{16 + 0 + 0}} = 3
\]

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left( -\frac{12 + \frac{5}{x}}{\sqrt{16 + \frac{1}{x} + \frac{1}{x^2}}} \right) = -\frac{12 + 0}{\sqrt{16 + 0 + 0}} = -3
\]

Hence the horizontal asymptotes of \( f \) are \( y = 3 \) and \( y = -3 \).

4 pts 117. Suppose you want to compute a limit that is in the form of a quotient, i.e., a limit of the form:

\[
\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right)
\]

Suppose you have already determined that L’Hospital’s Rule is applicable. Explain the next step in your calculation, i.e., how do you apply L’Hospital’s Rule? Your answer may contain either English, mathematical symbols, or both.

Solution
Compute the derivatives of \( f \) and \( g \) separately, and then compute the limit of the new quotient \( \frac{f'(x)}{g'(x)} \).

4 pts 118. Consider the following function, where \( a \) and \( b \) are unspecified constants.

\[ f(x) = \frac{x^2 + ax + b}{x - 2} \]

Is the line \( x = 2 \) necessarily a vertical asymptote of \( f(x) \)? Explain your answer. Your answer may contain either English, mathematical symbols, or both.

Solution
No. If \( x - 2 \) is also a factor of the numerator \( x^2 + ax + b \) (i.e., if substitution of \( x = 2 \) into the numerator gives 0), then the limit \( \lim_{x \to 2} f(x) \) would not be infinite, and so \( x = 2 \) would not be a vertical asymptote.
8 pts 119. A 6-ft tall person is initially standing 12 ft from point $P$ directly beneath a lantern hanging 42 ft above the ground, as shown in the diagram below. The person then begins to walk towards point $P$ at 5 ft/sec. Let $D$ denote the distance between the person’s feet and the point $P$. Let $S$ denote the length of the person’s shadow.

(a) Write an equation that relates $D$ and $S$.

(b) Write an equation that expresses the English sentence “The person then begins to walk towards point $P$ at 5 ft/sec.”

(c) Is the length of the person’s shadow increasing, decreasing or remaining constant?

(d) At what rate is the length of the person’s shadow changing when the person is 8 ft from point $P$? Include correct units as part of your answer.

Solution
(a) Use similar triangles to obtain $\frac{D+S}{S} = \frac{42}{6}$. (We may simplify this to $D = 6S$.)

(b) $\frac{dD}{dt} = -5$. (The equation $D = 12 - 5t$ is also acceptable.)

(c) The length of the shadow is decreasing.

(d) The equation $D = 6S$ gives $\frac{dD}{dt} = 6\frac{dS}{dt}$ and we have $\frac{dD}{dt} = -5$, whence $\frac{dS}{dt} = -\frac{5}{6}$ ft/sec.

10 pts 120. Consider the curve described by the equation

$$3x^2 + 2xy + 4y^2 = 132$$

At any point on this curve, we have

$$\frac{dy}{dx} = \frac{-3x-y}{x+4y}$$

Your task is to find the $x$-coordinate of each point on the curve where the tangent line is horizontal.

(a) Describe in two or three sentences the steps you should take to find the $x$-coordinate of each point on the curve where the tangent line is horizontal. Your answer may contain either English, mathematical symbols, or both.

(b) What is the $x$-coordinate of the rightmost (i.e., greatest $x$-coordinate) point on the curve where the tangent line is horizontal?
(c) What is the \( y \)-coordinate of the rightmost (i.e., greatest x-coordinate) point on the curve where the tangent line is horizontal?

(d) Describe in one or two sentences how this problem would change if instead you wanted to find the points where the tangent line is vertical. You do not have to solve the problem again, but only describe generally what you would do differently. *Your answer may contain either English, mathematical symbols, or both.*

**Solution**

(a) The point must lie on the curve and the tangent line is horizontal (i.e., \( \frac{dy}{dx} = 0 \)). So we must solve the following simultaneous set of equations for \( x \) and \( y \).

\[
\begin{align*}
3x^2 + 2xy + 4y^2 &= 132 \\
-3x - y &= 0 \\
\frac{x + 4y}{x + 4y} &= 0
\end{align*}
\]

(Note that the second equation is equivalent to \( y = -3x \).)

(b) Substituting \( y = -3x \) into the original equation gives \( 3x^2 - 6x^2 + 36x^2 = 132 \), or \( 33x^2 = 132 \). Hence \( x = -2 \) or \( x = 2 \). The \( x \)-coordinate of the rightmost point with a horizontal tangent is thus \( x = 2 \).

(c) We have \( y = -3x \), whence the \( y \)-coordinate of the rightmost point with a horizontal tangent is thus \( y = -6 \).

(d) A vertical tangent line has an undefined slope, so we replace the equation \( \frac{dy}{dx} = 0 \) with “denominator of \( \frac{dy}{dx} \) is 0”. That is, we must solve the following simultaneous set of equations:

\[
\begin{align*}
3x^2 + 2xy + 4y^2 &= 132 \\
x + 4y &= 0
\end{align*}
\]

5 pts 121. Each of the following limits is written in the form of a quotient. Which limits can be calculated using L’Hospital’s Rule directly, i.e., by applying L’Hospital’s Rule as the immediately next step without any other algebra or modification? Select all that apply.

(a) \( \lim_{x \to \pi} \left( \frac{\sin(7x)}{x} \right) \)

(b) \( \lim_{x \to 2} \left( \frac{x^3 + 3x - 14}{x^2 - 5x + 6} \right) \)

(c) \( \lim_{x \to -\infty} \left( \frac{x^{-1} + 5}{x^{-2} + 8} \right) \)

(d) \( \lim_{x \to 9} \left( \frac{x^{3/2} + x - 36}{x - \sqrt{x} - 6} \right) \)

(e) \( \lim_{x \to \infty} \left( \frac{e^x + 10}{e^x - 3} \right) \)

(f) \( \lim_{x \to -\infty} \left( \frac{e^x + 10}{e^x - 3} \right) \)
The only indeterminate quotients (for which L'Hospital's Rule is directly applicable) are \(0/0\) and \(\infty/\infty\). Hence the only limits above that can be computed with L'Hospital's Rule are: (b), (d), and (e).

**Solution**

122. Which of the following limits are equal to \(+\infty\)?

(a) \(\lim_{x \to 5^-} \left( \frac{x^2 + 25}{5 - x} \right)\)

(b) \(\lim_{x \to 5^+} \left( \frac{x^2 + 25}{5 - x} \right)\)

(c) \(\lim_{x \to -3^-} \left( \frac{x^3}{|x + 3|} \right)\)

(d) \(\lim_{x \to 0^-} \left( \frac{x^4 - 2x - 5}{\sin(x)} \right)\)

(e) \(\lim_{x \to 1^+} \left( \frac{x^6 - x^2}{x - 1} \right)\)

**Solution**

Direct substitution of each \(x\)-value gives non-zero \(\neq 0\) only for (a) - (d). A sign analysis of numerator and denominator then shows that only (a) and (d) are equal to \(+\infty\). As for (e), we apply L'Hospital's Rule and find

\[
\lim_{x \to 1^+} \left( \frac{x^6 - x^2}{x - 1} \right) \overset{H}{=} \lim_{x \to 1^+} \left( \frac{6x^5 - 2x}{1} \right) = 4
\]

Hence only (a) and (d) are correct choices.

123. Each of the following statements, some of which are false, describe how a certain rectangle may be changing over time. Select all of the true statements.

(a) If two opposite sides of the rectangle increase in length and if the area remains constant, then the other two opposite sides must decrease in length.

(b) If the area of the rectangle increases, then all sides of the rectangle must also increase in length.

(c) If the length of the rectangle remains the same, then the area and the width of the rectangle cannot change in opposite ways (i.e., one cannot increase while the other decreases).

(d) If two opposite sides of the rectangle increase in length and the other two opposite sides decrease in length, then the area of the rectangle must remain constant.

**Solution**
This problem can be answered by physical considerations alone. We may also use the equation $A = LW$, from which it follows:

$$\frac{dA}{dt} = \frac{dL}{dt}W + L\frac{dW}{dt}$$

Note that $L$ and $W$ must be positive numbers since they are lengths.

(a) True.
If $\frac{dL}{dt} > 0$ and $\frac{dA}{dt} = 0$, then $\frac{dW}{dt} = -\frac{W}{L} \frac{dL}{dt} < 0$.

(b) False.
If $\frac{dA}{dt} > 0$, it is possible for at least one of $\frac{dL}{dt}$ and $\frac{dW}{dt}$ to be negative. For instance, consider a rectangle with $L = W = 1$, $\frac{dL}{dt} = 2$, and $\frac{dW}{dt} = -1$.

(c) True.
If $\frac{dL}{dt} = 0$, then we must have $\frac{dA}{dt} = L\frac{dW}{dt}$. Since $L > 0$, we see that $\frac{dA}{dt}$ and $\frac{dW}{dt}$ must have the same sign.

(d) False.
If $\frac{dL}{dt} > 0$ and $\frac{dW}{dt} < 0$, it is possible to have $\frac{dA}{dt} \neq 0$. See part (b) for an example.

**10 pts 124.** Suppose the cost of manufacturing $x$ units is given by $C(x) = x^3 + 5x^2 + 12x + 50$.

(a) What is the exact cost of producing the 3rd unit?

(b) Using marginal analysis, estimate the cost of producing the 3rd unit.

**Solution**

(a) $C(3) - C(2) = 56$

(b) $C'(2) = (3x^2 + 10x + 12)|_{x=2} = 44$
125. Calculate the following limit or show that it does not exist. If the limit is infinite, your answer should be $+\infty$ or $-\infty$, not “does not exist”.

$$\lim_{x \to \infty} \left( \frac{5x^3 + 2x^2 + 8}{\ln(x)} \right)$$

**Solution**

Direct substitution of “$x \to \infty$” gives the indeterminate form $\infty^0$. So we use logarithms to write the limit as a quotient, and we use L’Hospital’s Rule.

$$L = \lim_{x \to \infty} \left( \frac{5x^3 + 2x^2 + 8}{\ln(x)} \right)$$

$$\ln(L) = \lim_{x \to \infty} \ln \left( \frac{5x^3 + 2x^2 + 8}{\ln(x)} \right)$$

$$\ln(L) = \lim_{x \to \infty} \left( \frac{\ln(5x^3 + 2x^2 + 8)}{\ln(x)} \cdot \frac{1}{5x^3 + 2x^2 + 8} \cdot (15x^2 + 4x) \cdot \frac{1}{x} \right)$$

$$\ln(L) \leq \lim_{x \to \infty} \left( \frac{15 + \frac{4}{x}}{5 + \frac{2}{x} + \frac{8}{x^2}} \right) = \frac{15 + 0 + 0}{5 + 0 + 0} = 3$$

So $\ln(L) = 3$, whence $L = e^3$. 

126. The volume of a cube is decreasing at the rate of 300 cm$^3$/sec at the moment its total surface area is 150 cm$^2$. What is the rate of change of the length of one edge of the cube at this moment?

**Solution**

Let $x$, $S$, and $V$ be the edge length, total surface area, and volume, respectively. Then $V = x^3$ and $S = 6x^2$. Differentiating these equations gives $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$ and $\frac{dS}{dt} = 12x \frac{dx}{dt}$.

At the moment described, we have $\frac{dV}{dt} = -300$ and $S = 150$, and we seek $\frac{dx}{dt}$. Substituting this information into our four equations gives the following.

$$V = x^3, \quad 150 = 6x^2, \quad -300 = 3x^2 \frac{dx}{dt}, \quad \frac{dS}{dt} = 12x \frac{dx}{dt}$$

The second equation above gives $x = 5$. Putting $x = 5$ into the third equation gives $\frac{dx}{dt} = -4$ cm/sec.

127. Use linear approximation or differentials to estimate the value of $(0.98)^3 - 5(0.98)^2 + 4(0.98) + 10$. 

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Solution
Put \( f(x) = x^3 - 5x^2 + 4x + 10 \). We use the tangent line to \( f \) at \( x = 1 \). Observe that \( f(1) = 10 \) and \( f'(1) = (3x^2 - 10x + 5)|_{x=1} = -2 \). Hence the desired tangent line is
\[
y = 10 - 2(x - 1)
\]
The desired approximation is then
\[
(0.98)^3 - 5(0.98)^2 + 4(0.98) + 10 \approx 10 - 2(0.98 - 1) = 10.04
\]

10 pts 128. Find an equation of the line tangent to the graph of
\[
\ln(xy + x - 7) = 2x + 4y - 30
\]
at the point \((1,7)\).

Solution
Using implicit differentiation with respect to \( x \) gives the following.
\[
\frac{1}{xy + x - 7} \cdot \left( x \frac{dy}{dx} + y + 1 \right) = 2 + 4 \frac{dy}{dx}
\]
Substituting \( x = 1 \) and \( y = 7 \) gives
\[
\frac{dy}{dx} + 8 = 2 + 4 \frac{dy}{dx}
\]
Hence the slope of the tangent line is \( \frac{dy}{dx} = 2 \), whence an equation of the tangent line is
\[
y = 7 + 2(x - 1)
\]

10 pts 129. Consider the function below.
\[
f(x) = \frac{x^3 + 2x^2 - 13x + 10}{x^2 - 1}
\]
Show that the line \( x = -1 \) is a vertical asymptote of \( f \), but the line \( x = 1 \) is not a vertical asymptote of \( f \).

Solution
For \( x = -1 \), direct substitution gives the form \( \frac{-24}{0} \), i.e., a nonzero divided by 0. Hence both one-sided limits of \( f \) at \( x = -1 \) are infinite, and so \( x = -1 \) is a vertical asymptote.
For \( x = 1 \), direct substitution gives the indeterminate form \( \frac{0}{0} \), which may indicate a vertical asymptote but not necessarily. So we use L’Hospital’s Rule.
\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \left( x^3 + 2x^2 - 13x + 10 \right) = \frac{3x^2 + 4x - 13}{2x} = \frac{-6}{2} = -3
\]
Since this limit exists and is not infinite, \( x = 1 \) is not a vertical asymptote.

**5 pts 130.** Suppose you want to calculate the limit of a product, i.e., a limit of the form

\[
L = \lim_{x \to a} (f(x)g(x))
\]

Suppose you have determined the following:

\[
\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = \infty
\]

You recall that you have learned that to calculate \( L \), you have to use L’Hospital’s Rule. What is the next step you must take before you are able to apply L’Hospital’s Rule directly to the limit \( L \)? *Your answer may contain either English, mathematical symbols, or both.*

**Solution**

Write the product \( f(x)g(x) \) as a quotient instead. For example, \( \frac{g(x)}{1/f(x)} \).

**5 pts 131.** Suppose the function \( f \) has domain \((-\infty, \infty)\). Give a brief explanation of how you would find all horizontal asymptotes of \( f \). Note that for this problem, \( f \) is unspecified; you should not assume it has any particular form. *Your answer may contain either English, mathematical symbols, or both.*

**Solution**

Compute the limit of \( f \) as \( x \to \infty \) and the limit of \( f \) as \( x \to -\infty \). If either (or both) of these limits is finite and, say, equal to \( L \), then the line \( y = L \) is a horizontal asymptote of \( f \). (Note that \( f \) can have zero, one, or two horizontal asymptotes.)

**10 pts 132.** A boat is pulled toward a dock by a rope through a ring on the dock 4 ft above the front of the boat. The rope is hauled in at the rate of 12 ft/sec.

(a) Which of the marked variables \((x, y, L, \text{ and } \theta)\) are changing over time?

(b) Write a mathematical equation that expresses the English sentence “The rope is hauled in at the rate of 12 ft/sec”.

(c) Is \( \cos(\theta) \) increasing, decreasing, or constant?
(d) Write a mathematical expression for “the rate at which the boat approaches the dock”.
(e) How fast in ft/sec is the boat approaching the dock when the rope is 5 ft long?

**Solution**

(a) The variables \( x, L, \) and \( \theta \) are changing over time.

(b) \( \frac{dL}{dt} = -12 \)

(c) Since \( \theta \) is decreasing (to 0), \( \cos(\theta) \) is increasing (to 1). Alternatively, note that \( \cos(\theta) = \frac{4}{L} \). Since \( L \) is decreasing, the fraction \( \frac{4}{L} \) is increasing.

(d) \( \frac{dx}{dt} \)

(e) Pythagorean theorem gives \( x^2 + 16 = L^2 \), whence \( 2x \frac{dx}{dt} = 2L \frac{dL}{dt} \). At the moment when \( L = 5 \), we have \( x = 3 \). Substituting these values and \( \frac{dL}{dt} = -12 \) into the second equation then gives \( \frac{dx}{dt} = -20 \) ft/sec.

(Since the question asks for the rate at which the boat is approaching the dock, we must report this as a positive number. So the boat approaches the dock at a rate of 20 ft/sec.)

133.10 pts Consider the curve described by the equation

\[
5x^2 - 4xy + y^2 = 8
\]

At any point on this curve, we have

\[
\frac{dy}{dx} = \frac{-5x + 2y}{-2x + y}
\]

Your task is to find the \( x \)-coordinate of each point on the curve where the tangent line is parallel to the line \( y = x \).

(a) Explain the steps you should take to the find the \( x \)-coordinate of each point on the curve where the tangent line is parallel to the line \( y = x \). Your answer may contain either English, mathematical symbols, or both.

(b) What is the \( x \)-coordinate of the leftmost (i.e., least \( x \)-coordinate) point on the curve where the tangent line is horizontal?

(c) What is the \( y \)-coordinate of the leftmost (i.e., least \( x \)-coordinate) point on the curve where the tangent line is horizontal?

(d) Explains how this problem would change if instead you wanted to find the points where the tangent line is perpendicular to the line \( y = 4 \). You do not have to solve the problem again, but only describe generally what you would do differently. Your answer may contain either English, mathematical symbols, or both.

**Solution**
(a) The point must lie on the curve and the tangent line has slope 1 (i.e., \( \frac{dy}{dx} = 1 \)). So we must solve the following simultaneous set of equations for \( x \) and \( y \).

\[
5x^2 - 4xy + y^2 = 8 \\
-5x + 2y \\
-2x + y = 1
\]

(Note that the second equation is equivalent to \( y = 3x \).)

(b) Substituting \( y = 3x \) into the original equation gives \( 5x^2 - 4x(3x) + (3x)^2 = 8 \), or \( 2x^2 = 8 \). Hence \( x = -2 \) or \( x = 2 \). The \( x \)-coordinate of the leftmost point with a tangent line parallel to \( y = x \) is \( x = -2 \).

(c) We have \( y = 3x \), whence the \( y \)-coordinate of the leftmost point with a tangent line parallel to \( y = x \) is \( y = -6 \).

(d) The line \( y = 4 \) is horizontal, so a perpendicular line is vertical. A vertical tangent line has an undefined slope, so we replace the equation \( \frac{dy}{dx} = 0 \) with “denominator of \( \frac{dy}{dx} \) is 0”. That is, we must solve the following simultaneous set of equations:

\[
5x^2 - 4xy + y^2 = 8 \\
-2x + y = 0
\]

4 pts 134. Which of the following are indeterminate form? Recall that in this course, we have learned that limits with indeterminate forms may often be computed using L’Hospital’s Rule.

(a) \( \frac{0}{0} \)  
(b) \( 0 \cdot \infty \)  
(c) \( \frac{\infty}{-\infty} \)  
(d) \( \frac{0}{\infty} \)  
(e) \( 2^\infty \)  
(f) \( 3 \cdot (-\infty) \)  
(g) \( \infty \cdot (-\infty) \)  
(h) \( \infty^0 \)  
(i) \( \infty \cdot \infty \)

**Solution**

The only indeterminate forms are (a), (b), (c), and (h). The other choices are, respectively: 0, \( \infty \), \(-\infty \), \( -\infty \), and \( \infty \).

4 pts 135. Determine which of the following limits are equal to \( -\infty \), or determine that no limit below is equal to \( -\infty \).

(a) \( \lim_{x \to -\infty} \left( \frac{x^2 - 5x - 6}{x - 6} \right) \)

(b) \( \lim_{x \to -\infty} \left( \frac{x^2 - 5x - 6}{x^2 - 12x + 36} \right) \)

(c) \( \lim_{x \to \infty} \left( \frac{x^2 - 5x - 6}{x - 6} \right) \)

(d) \( \lim_{x \to \infty} \left( \frac{x^2 - 5x - 6}{x^2 - 12x + 36} \right) \)
Solution
Choice (b) only.

(a) Direct substitution gives \(0 \div 0\), so use L'Hopital's Rule.

\[
\lim_{x \to 6^-} \left( \frac{x^2 - 5x - 6}{x - 6} \right) \overset{H}{=} \lim_{x \to 6^-} \left( \frac{2x - 5}{1} \right) = 7
\]

(b) Direct substitution gives \(0 \div 0\), so use L'Hopital’s Rule.

\[
\lim_{x \to 6^-} \left( \frac{x^2 - 5x - 6}{x^2 - 12x + 36} \right) \overset{H}{=} \lim_{x \to 6^-} \left( \frac{2x - 5}{2x - 12} \right) = \frac{7}{0^-} = -\infty
\]

(c) Factor out the highest power of numerator and denominator.

\[
\lim_{x \to \infty} \left( \frac{x^2 - 5x - 6}{x - 6} \right) = \lim_{x \to \infty} \left( \frac{x^2}{x} \cdot \frac{1 - \frac{5}{x} - \frac{6}{x^2}}{1 - \frac{6}{x}} \right) = \lim_{x \to \infty} \left( x \cdot \frac{1 - \frac{5}{x} - \frac{6}{x^2}}{1 - \frac{6}{x}} \right) = \infty
\]

(d) Factor out the highest power of numerator and denominator.

\[
\lim_{x \to \infty} \left( \frac{x^2 - 5x - 6}{x^2 - 12x + 36} \right) = \lim_{x \to \infty} \left( \frac{1 - \frac{5}{x} - \frac{6}{x^2}}{1 - \frac{12}{x} + \frac{36}{x^2}} \right) = \frac{1 - 0 - 0}{1 - 0 + 0} = 1
\]

4 pts 136. The numbers \(a, b,\) and \(c\) (which are not necessarily positive) satisfy the formula \(a = \frac{b}{c}\). The choices below describe scenarios in which the numbers \(a, b,\) and \(c\) are changing over time. There is at most one true scenario among them. Select the true scenario or determine that all scenarios are false.

(a) Suppose \(a, b,\) and \(c\) are all positive numbers. If \(a\) and \(b\) are both increasing, then \(c\) must also be increasing.

(b) Suppose \(b\) is a positive number and \(c\) is a negative number. If \(b\) and \(c\) are both increasing, then \(a\) must be decreasing.

(c) Suppose \(a, b,\) and \(c\) are all positive numbers. If \(a\) is constant, then it is possible for \(b\) and \(c\) to change in opposite ways (i.e., one can increase while the other decreases).

(d) Suppose \(c\) is a positive number. If \(b\) is constant and \(c\) is increasing, then \(a\) must be decreasing.

Solution
Choice (b) is the only true scenario.

To solve this problem, we first use implicit differentiation with respect to time to obtain

\[
a' = \frac{cb' - bc'}{c^2}
\]

where the primes denote differentiation with respect to \(t\).
(a) False. Put \( b = c = 1, a' = 2, \) and \( b' = 1. \) Then we have \( 2 = 1 - c', \) whence \( c' = -1. \) So it is possible for \( c \) to be decreasing.

(b) True. We have \( b > 0, c < 0, b' > 0, \) and \( c' > 0. \) A sign analysis of \( a' \) gives:

\[
a' = \frac{\odot \oplus - \odot \oplus}{\oplus} = \frac{\odot - \oplus}{\oplus}
\]

Note that a negative number minus a positive number is a negative number. So the numerator above is negative, whence \( a' \) must be negative.

(c) False. If \( a \) is constant, then \( a' = 0, \) and we must have \( cb' = bc', \) or \( b'/c' = b/c. \) The right side of this equation is positive, whence \( b'/c' \) must also be positive. This means that \( b' \) and \( c' \) must both have the same sign, i.e., \( b \) and \( c \) cannot change in opposite ways.

(d) False. If \( b \) is constant, then \( b' = 0, \) and we must have \( a' = -\frac{bc'}{c^2}. \) Since \( c \) and \( c' \) are both positive, we may take \( c = c' = 1 \) and \( b = -1, \) whence \( a' = 1. \) So it is possible for \( a \) to be increasing.

8 pts 137. If \( x \) units are produced, the total cost is \( C(x) = x^2 + 15x + 24 \) and the selling price per unit is

\[
p(x) = \frac{156}{x^2 - 4x + 16}
\]

(a) What is the exact cost of producing the 3rd unit?

(b) Using marginal analysis, estimate the revenue from the 3rd unit sold.

Solution

(a) \( C(3) - C(2) = 20 \)

(b) The revenue is

\[
R(x) = xp(x) = \frac{156x}{x^2 - 4x + 16}
\]

So by marginal analysis, the revenue from the 3rd unit is approximately

\[
R'(2) = \left( \frac{156(16 - x^2)}{(x^2 - 4x + 16)^2} \right) \bigg|_{x=2} = 13
\]
Midterm Exam #3

1 pt 138. Sketch a curve with the following properties:

- \( f' < 0 \) and \( f'' < 0 \) for \( x < 13 \)
- \( f' < 0 \) and \( f'' > 0 \) for \( x > 13 \)

Solution

1 pt 139. Sketch the graph of a function \( f \) that has a local maximum at \( x = a \) where \( f'(a) = 0 \).

Solution

1 pt 140. Sketch the graph of a function \( f \) that has a local maximum at \( x = b \) where \( f'(b) \) is undefined.

Solution
141. Determine from the given graph whether the function has any absolute extreme values on 

\((a, b)\).

**Solution**

The function has an absolute maximum value at \(x = c\) but does not have an absolute minimum value on \((a, b)\).

142. Suppose \(f\) is a continuous function such that all of the following hold:

\[
\int_{-1}^{6} f(x) \, dx = -15, \quad \int_{6}^{9} f(x) \, dx = 14, \quad \int_{0}^{9} f(x) \, dx = 19
\]

Calculate the quantities below or determine there is not enough information.

(a) \(\int_{-1}^{9} f(x) \, dx\) 
(b) \(\int_{0}^{6} f(x) \, dx\) 
(c) \(\int_{-1}^{6} |f(x)| \, dx\) 
(d) \(\left|\int_{-1}^{6} f(x) \, dx\right|\) 
(e) \(\int_{-1}^{0} f(x) \, dx\) 
(f) \(\int_{6}^{9} (3f(x) + 4) \, dx\)

**Solution**

(a) \(\int_{-1}^{9} f(x) \, dx = \int_{-1}^{6} f(x) \, dx + \int_{6}^{9} f(x) \, dx = -15 + 14 = -1\)

(b) \(\int_{0}^{6} f(x) \, dx = \int_{0}^{9} f(x) \, dx - \int_{6}^{9} f(x) \, dx = 19 - 14 = 5\)

(c) not enough information

(d) \(\left|\int_{-1}^{6} f(x) \, dx\right| = |-15| = 15\)

(e) Use part (b).

\[\int_{-1}^{0} f(x) \, dx = \int_{-1}^{6} f(x) \, dx - \int_{0}^{6} f(x) \, dx = -15 - 5 = -20\]
(f) Use linearity.
\[
\int_6^9 (3f(x) + 4) \, dx = 3 \int_6^9 f(x) \, dx + \int_6^9 4 \, dx
\]
The second integral is the area of a rectangle of height 4 and width 3. So we have:
\[
\int_6^9 (3f(x) + 4) \, dx = 3 \cdot 14 + 3 \cdot 4 = 54
\]

12 pts 143. Use the graph of \( y = f(x) \) to calculate the integrals below.

![Graph of f(x)](image)

(a) \( \int_0^1 f(x) \, dx \)
(b) \( \int_1^6 f(x) \, dx \)
(c) \( \int_{-10}^{10} f(x) \, dx \)

Solution

(a) The integral is the area of a trapezoid with parallel bases of length 6 and 7, with height 1. Hence
\[
\int_0^1 f(x) \, dx = \frac{1}{2} (6 + 7) \cdot 1 = 6.5
\]

(b) The integral represents the net area of a region that consists of a triangle (base 3, height 4) and a rectangle (base 2, height 2). Note that both are below the \( x \)-axis, and so the net area is negative.
\[
\int_1^6 f(x) \, dx = -\left( \frac{1}{2} \cdot 3 \cdot 4 + 2 \cdot 2 \right) = -10
\]

(c) We have already computed most of this integral in parts (a) and (b). For the remaining parts we have one triangle below the \( x \)-axis, one triangle above the \( x \)-axis, and one rectangle above the \( x \)-axis.
\[
\int_{-10}^{-6} f(x) \, dx = -\frac{1}{2} \cdot 4 \cdot 4 = -8
\]
\[
\int_{-6}^{0} f(x) \, dx = \frac{1}{2} \cdot 6 \cdot 6 = 18
\]
\[
\int_{6}^{10} f(x) \, dx = 1 \cdot 4 = 4
\]
Putting everything together gives:

\[ \int_{-10}^{10} f(x) \, dx = -8 + 18 + 6.5 - 10 + 4 = 10.5 \]

14 pts 144. The first two derivatives of the function \( f \) are given below.

\[ f'(x) = \frac{x}{(x - 6)^2(x + 48)}, \quad f''(x) = \frac{-2(x + 12)^2}{(x - 6)^3(x + 48)^2} \]

Find where \( f \) is increasing, decreasing, concave up, and concave down. Use interval notation in your answers. Also, find all relative extrema and inflection points of \( f \). Write “NONE” as your answer, if appropriate.

**Solution**

<table>
<thead>
<tr>
<th>Increasing</th>
<th>((-\infty, -48), [0, 6), (6, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decreasing</td>
<td>((-48, 0])</td>
</tr>
<tr>
<td>Concave up</td>
<td>((-\infty, -48), (-48, 6))</td>
</tr>
<tr>
<td>Concave down</td>
<td>((6, \infty))</td>
</tr>
<tr>
<td>(x)-coordinates of relative maxima</td>
<td>NONE</td>
</tr>
<tr>
<td>(x)-coordinates of relative minima</td>
<td>(x = 0)</td>
</tr>
<tr>
<td>(x)-coordinates of inflection points</td>
<td>NONE</td>
</tr>
</tbody>
</table>

6 pts 145. Let \( x \) be the level of production for a certain commodity. Suppose the marginal cost is modeled by the function

\[ C'(x) = 3x^2 + 2x \]

and the market price is modeled by the function

\[ p(x) = 144 - 2x \]

Suppose that the cost of producing the 1st unit of the commodity is 70.

(a) What is the cost of producing the first 3 units of the commodity?
(b) What is the level of production that maximizes the total profit?

**Solution**

(a) The total cost must have the following form:

\[ C(x) = \int C'(x) \, dx = \int (3x^2 + 2x) \, dx = x^3 + x^2 + K \]
where $K$ is some constant. The condition $C(1) = 70$ gives $1 + 1 + K = 70$, whence $K = 68$. So the total cost function is $C(x) = x^3 + x^2 + 68$. Hence the cost of the first 3 units is $C(3) = 104$.

(b) The total revenue is $R(x) = xp(x) = 144x - 2x^2$. Total profit is maximized when $C'(x) = R'(x)$, or when $3x^2 + 2x = 144 - 4x$. The solutions to this equation are $x = -8$ and $x = -6$. Hence the total profit is maximized when $x = 6$ (production cannot be negative).

8 pts 146. Consider the function
\[ g(x) = \frac{3}{2}x^4 + 8x^3 - 36x^2 \]
on the interval $[-7, 3]$.

(a) Where does $g$ have a local minimum on $(-7, 3)$? local maximum?

(b) Where does $g$ have a global minimum on $[-7, 3]$? global maximum?

Solution

(a) We solve $g'(x) = 0$ to find the critical numbers of $g$.
\[ g'(x) = 6x^3 + 24x^2 - 72x = 6x(x - 2)(x + 6) = 0 \]
Thus the critical numbers are $x = -6$, $x = 0$, and $x = 2$. We will use the second derivative test to classify these critical numbers. Observe that
\[ g''(x) = 6(3x^2 + 8x - 12) \]
and we have the following values: $g''(-6) = 288 > 0$, $g''(0) = -72 < 0$, and $g''(2) = 96 > 0$. Hence $g$ has a local minimum at both $x = -6$ and $x = 2$, and $g$ has a local maximum at $x = 0$.

(b) The global extrema can occur only at the endpoints of the interval or at the critical numbers. We have the following values: $g(-7) = -906.5$, $g(-6) = -1080$, $g(0) = 0$, $g(2) = -56$, and $g(3) = 13.5$. Hence on the interval $[-7, 3]$, $g$ has a global minimum at $x = -6$ and a global maximum at $x = 3$.

8 pts 147. Find all critical numbers of the function
\[ f(x) = 2x^{4/3} - 16x^{2/3} + 24 \]
Note that $f$ is continuous on the interval $(-\infty, \infty)$.

Solution

First observe that $f$ is not differentiable at $x = 0$ (recall that $x^n$ is not differentiable at $x = 0$ if $n < 1$). Hence $x = 0$ is a critical number. The remaining critical numbers are
solutions of $f'(x) = 0$.

\[
\begin{align*}
f'(x) &= 0 \\
\frac{8}{3} x^{1/3} - \frac{32}{3} x^{-1/3} &= 0 \\
\frac{8}{3} x^{-1/3} (x^{2/3} - 4) &= 0 \\
x^{2/3} - 4 &= 0 \\
x &= 64 \\
x &= -8 \text{ or } x = 8
\end{align*}
\]

Hence the critical numbers of $f$ are $x = -8$, $x = 0$, and $x = 8$.

\textbf{18 pts} 148. Suppose the local post office has a policy that all packages must be shaped like a rectangular box with a sum of length, width, and height not exceeding 144 inches. You plan to construct such a package whose length is 2 times its width. Find the dimensions of the package with the largest volume. For this problem, let $L$, $W$, and $H$ be the length, width, and height of the package, respectively.

(a) What is the objective function for this problem in terms of $L$, $W$, and $H$?

(b) There are two constraints for this problem. In terms of $L$, $W$, and $H$, give the constraint equation which corresponds to...

(a) ...the policy set by the post office.

(b) ...your specific plan to construct such a package.

(c) Find the objective function in terms of $W$ only.

(d) What is the interval of interest for the objective function?

(e) Find the values of $L$, $W$, and $H$ that give the largest volume.

(f) Suppose the post office adds the additional requirement that the width $W$ of the package must be no smaller than 36 inches and no larger than 40 inches. With this additional policy, what is the width of the package with the largest volume?

\textbf{Solution}

(a) We wish to maximize the volume of the package, so the objective function is $g(L, W, H) = LWH$.

(b) (a) $L + W + H = 144$

(b) $L = 2W$

(c) We already have $L = 2W$. From the first constraint, we get $3W + H = 144$, whence $H = 144 - 3W$. Hence the objective function in terms of $W$ only is

\[
f(W) = g(2W, W, 144 - 3W) = 2W^2(144 - 3W) = 288W^2 - 6W^3
\]

(d) Each of $L$, $W$, and $H$ must be non-negative numbers. (We allow them to be 0, since this would correspond to a degenerate package with no volume. That is okay.) The condition $L \geq 0$ is equivalent to $W \geq 0$ since $L = 2W$. The condition $H \geq 0$ is
equivalent to $144 - 3W \geq 0$, or $W \leq 48$. Hence the interval of interest (possible values of $W$) is $[0, 48]$.

(e) The critical numbers of $f$ are solutions to $f'(W) = 576W - 18W^2 = 18W(32 - W) = 0$. Hence the two critical numbers are $W = 0$ (already included as an endpoint) and $W = 32$. Since we are working on a closed interval, we may verify that $W = 32$ is the global maximum simply by checking the endpoint and critical values. Since $f(0) = f(48) = 0$ and $f(32) > 0$, it is clear that $W = 32$ gives the global maximum.

Hence the dimensions of the package with the largest volume are $L = 64$, $W = 32$, and $H = 48$.

(f) None of our previous work has changed except that the interval of interest is now $[36, 40]$. We have already determined that $f(W)$ has a global maximum on $[0, 48]$ at $W = 32$. Hence $f$ is decreasing on the interval $[36, 40]$. Hence $f(36) > f(40)$, and so the package with the largest volume now has $W = 36$. 
Final Exam

3 pts 149. The graph of \( f \) is given below. Find all values of \( a \) in \((-4, 4)\) such that \( \lim_{x \to a} f(x) \) does not exist.

Solution

\(-3\) and 1 only

3 pts 150. Which statement is true about the graph of \( f(x) = |x| + 91 \) at the point \((0, 91)\)?

(a) The graph has a tangent line at \( y = 91 \).
(b) The graph has infinitely many tangent lines.
(c) The graph has no tangent line.
(d) The graph has two tangent lines: \( y = x + 91 \) and \( y = -x + 91 \).
(e) None of the above statements is true.

Solution

Choice C.

3 pts 151. Suppose the cost (in dollars) of manufacturing \( q \) units is given by

\[ C(q) = 6q^2 + 34q + 112 \]

Use marginal analysis to estimate the cost of producing the 5th unit.

Solution

The exact cost of the 5th unit is \( \Delta C = C(5) - C(4) \), which is approximately \( C'(4) \) by linear approximation. Hence

\[ \Delta C \approx C'(4) = (12q + 34)\big|_{q=4} = 82 \]
3 pts 152. Consider the following function, where \( k \) is an unspecified constant.

\[
f(x) = \begin{cases} 
38 + kx & , \ x < 3 \\
kx^2 + x - k & , \ x \geq 3 
\end{cases}
\]

Find the value of \( k \) that is necessary to make \( f \) continuous for all \( x \).

**Solution**

First we calculate the left-limit, right-limit, and function value at \( x = 3 \).

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (38 + kx) = 38 + 3k \\
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (kx^2 + x - k) = 8k + 3 \\
f(3) = 8k + 3
\]

To make \( f \) continuous at \( x = 3 \), the left-limit, right-limit, and function value at \( x = 3 \) must all be equal. Hence we must have

\[
38 + 3k = 8k + 3
\]

Hence \( k = 7 \).

3 pts 153. The figure below shows the area of regions bounded by the graph of \( f \) and the \( x \)-axis. You are also given that \( a = 4 \), \( b = 6 \), and \( c = 15 \). Evaluate the integral \( \int_a^c (11f(x) - 6) \, dx \).

**Solution**

Split up the integral using linearity properties.

\[
\int_a^c (11f(x) - 6) \, dx = 11 \int_a^c f(x) \, dx - 6 \int_a^c 1 \, dx = 11 \cdot (13 - 8) - 6 \cdot (15 - 4) = -11
\]

13 pts 154. Consider the function \( f \) and its derivatives below.

\[
f(x) = \frac{99e^x}{(x - 25)^47} + 98 \quad , \quad f'(x) = \frac{99e^x(x - 72)}{(x - 25)^48} \quad , \quad f''(x) = \frac{99e^x((x - 72)^2 + 47)}{(x - 25)^49}
\]

Find where \( f \) is increasing, decreasing, concave up, and concave down. Use interval notation in your answers. Also, find all horizontal and vertical asymptotes, relative extrema, and inflection points of \( f \). Write “NONE” as your answer, if appropriate.
### Solution

<table>
<thead>
<tr>
<th>Increasing</th>
<th>$[72, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decreasing</td>
<td>$(-\infty, 25), (25, 72]$</td>
</tr>
<tr>
<td>Concave up</td>
<td>$(25, \infty)$</td>
</tr>
<tr>
<td>Concave down</td>
<td>$(-\infty, 25)$</td>
</tr>
<tr>
<td>Equations of horizontal asymptotes</td>
<td>$y = 98$</td>
</tr>
<tr>
<td>Equations of vertical asymptotes</td>
<td>$x = 25$</td>
</tr>
<tr>
<td>$x$-coordinates of relative maxima</td>
<td>NONE</td>
</tr>
<tr>
<td>$x$-coordinates of relative minima</td>
<td>$x = 72$</td>
</tr>
<tr>
<td>$x$-coordinates of inflection points</td>
<td>NONE</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>6 pts 155. A student is asked to calculate the following limit using L'Hôpital's Rule and show all their work.</th>
</tr>
</thead>
</table>

$$L = \lim_{x \to 0} \left( \frac{\sin(2x) + 17x^2 + 2x}{4x^2 + \tan(x)} \right)$$

The student decides to cheat, so they find the solution online (shown below) and they submit the work as their own!

$$L = \lim_{x \to 0} \left( \frac{\sin(2x) + 17x^2 + 2x}{4x^2 + \tan(x)} \right)$$
$$= \lim_{x \to 0} \left( \frac{2 \cos(2x) + 34x + 2}{8x + \sec(x)^2} \right)$$
$$= \lim_{x \to 0} \left( \frac{-4 \sin(2x) + 34}{8 + 2 \sec(x)^2 \tan(x)} \right)$$
$$= \frac{-4 \sin(0) + 34}{8 + 2 \sec(0)^2 \tan(0)}$$
$$= \frac{0 + 34}{8 + 0}$$
$$= \frac{17}{4}$$

Unfortunately, this solution contains an error, and so the student lost all credit for the problem. The student was also later determined to be responsible for cheating, and so they earned a grade of 0 on the entire exam!

Your task is to find and correct the error(s). Answer the following questions.

(a) There may be several errors in this solution. Which line is the first incorrect line?
(b) Explain the error in the first incorrect line in your own words.
(c) Calculate the correct value of $L$ (the original limit).

**Solution**
(a) The first error is in the third line.
(b) In this line the student is not allowed to apply L’Hôpital’s Rule since the previous line does not have an indeterminate form.
(c) Substitution $x = 0$ in the second line gives the correct value of the limit: 4.

6 pts 156. Consider the integral below.
\[ \int_{-2}^{1} \sqrt{9-(x-1)^2} \, dx \]

(a) Explain in your own words how you would calculate this integral without using Riemann sums or the fundamental theorem of calculus. Be as specific as possible. *Hint: Try graphing the integrand!*

(b) Find the exact value of the integral.

**Solution**
(a) Observe that the graph of $y = \sqrt{9-(x-1)^2}$ is the top half of a circle with center $(1, 0)$ and radius 3. The leftmost point on the circle is $(-2, 0)$. Thus the integral is equal to the area of the left half of this semi-disc. That is, the region is congruent to a quarter-disc with radius 3.
(b) The area of the region is $\pi r^2/4$ with $r = 3$, hence the area is $9\pi/4$.

6 pts 157. Consider the curve described by the following equation.
\[ e^{12x+2y} = 6y - 3xy + 1 \]

(a) Find $\frac{dy}{dx}$ at a general point on this curve.

(b) Calculate the slope of the line tangent to the curve at $(2, -12)$.

(c) There is a point on the curve close to the origin with coordinates $(0.07, b)$, and the line tangent to the curve at the origin is given by $y = 3x$. Use linear approximate to estimate the value of $b$.

**Solution**
(a) Differentiating each side of the equation with respect to $x$ gives:
\[ e^{12x+2y} \cdot (12 + 2y') = 6y' - 3xy' - 3y \]
Solving algebraically for $y'$ gives:
\[ \frac{dy}{dx} = \frac{12e^{12x+2y} + 3y}{6 - 3x - 2e^{12x+2y}} \]
(b) Substituting $x = 2$ and $y = -12$ into the expression above gives $\frac{dy}{dx} = 12$. 
(c) The tangent line at the origin is a linear approximation of the curve near the origin. Hence the point \((0.07, b)\) lies approximately on this tangent line. Hence \(b \approx 3(0.07) = 0.21\).

6 pts 158. We are given that the derivative of \(f\) is \(f'(x) = 3x^2 - 6x - 9\) and that \(f(1) = 10\). Use this information to answer the following.

(a) Find an equation of the line tangent to the graph of \(y = f(x)\) at \(x = 1\).
(b) Find the critical numbers of \(f\).
(c) Where does \(f\) have a local minimum value? local maximum value?
(d) Calculate \(f(0)\).
(e) Calculate the absolute maximum value of \(f\) on the interval \([0, 6]\). At what \(x\)-value does it occur?

Solution
(a) We have \(f'(1) = 3 - 6 - 9 = -12\), whence an equation of the tangent line is \(y = 10 - 12(x - 1)\).
(b) Solving \(f'(x) = 0\), we find that the critical numbers of \(f\) are \(x = -1\) and \(x = 3\).
(c) A sign chart for \(f'(x)\) reveals that \(f'(x)\) is positive on the intervals \((-\infty, -1)\) and \((3, \infty)\); and \(f'(x)\) is negative on the interval \((-1, 3)\). Since \(f'\) changes from positive to negative at \(x = -1\), a local maximum occurs at \(x = -1\). Since \(f'\) changes from negative to positive to \(x = 3\), a local minimum occurs at \(x = 3\).
(d) We find \(f(x)\) by finding the most general antiderivative of \(f'(x)\).

\[
f(x) = \int f'(x) \, dx = x^3 - 3x^2 - 9x + C
\]

The initial condition \(f(1) = 10\) implies \(1 - 3 - 9 + C = 10\), or \(C = 21\). Hence

\[
f(x) = x^3 - 3x^2 - 9x + 21
\]

So \(f(0) = 21\).
(e) The absolute maximum of \(f\) on \([0, 6]\) can occur only at an endpoint (0 or 6) or a critical number (-1 or 3). Calculating the values of \(f\) at these \(x\)-values gives: \(f(0) = 21\), \(f(-1) = 26\), \(f(3) = -6\), and \(f(6) = 75\). Hence the absolute maximum of \(f\) on \([0, 6]\) is 75, occurring at \(x = 6\).

12 pts 159. A local park has hired you to construct a rectangular flower garden surrounded by a grass border that is 1 m wide on two sides and 2 m wide on the other two sides. (See the figure blow.) The area of the garden only (the small rectangle) must be 126 m\(^2\). Your primary task is to find the dimensions of the garden that give the smallest possible combined area of the garden and the grass border. For this problem, let \(W\) be the horizontal width of the garden and let \(H\) be the vertical height of the garden.
(a) What is the objective function for this problem in terms of $W$ and $H$?
(b) What is the constraint equation for this problem in terms of $W$ and $H$?
(c) Find the objective function in terms of $W$ only.
(d) What is the interval of interest for the objective function?
(e) Find the values of $W$ and $H$ that minimize the total combined area.
(f) What horizontal width $W$ of the garden will maximize the total area?

**Solution**

(a) The width of the combined area is $W + 4$ and the height of the combined area if $H + 2$. We wish to minimize the combined area, and so the objective function is $g(w, H) = (W + 4)(H + 2)$.

(b) The garden must have an area of 126, and so the constraint equation is $WH = 126$.

(c) Solving for $H$ in the constraint gives $H = 126/W$, and substituting this into the objective gives:

$$f(W) = g\left(W, \frac{126}{W}\right) = (W + 4)\left(\frac{126}{W} + 2\right) = 134 + 2W + \frac{504}{W}$$

(d) The width $W$ can be any positive length (note that a length of 0 is not allowed since the garden area must be positive). So the interval of interest is $(0, \infty)$.

(e) We solve $f'(W) = 0$ to find the critical numbers.

$$f'(W) = 2 - \frac{504}{w^2} = 0 \implies W = \sqrt{252} = 6\sqrt{7}$$

Observe that $f''(w) = \frac{1108}{W^3}$, which is positive for all $W > 0$. So by the second derivative test, $W = 6\sqrt{7}$ gives a local minimum. Since it gives the only local extreme value on $(0, \infty)$, $f$ has a global minimum value on $(0, \infty)$ at $W = 6\sqrt{7}$. The corresponding height is $H = \frac{126}{6\sqrt{7}} = 3\sqrt{7}$.

(f) None of our work above changes. However, we now note that $f(w) \to \infty$ as $W \to 0^+$ or as $W \to \infty$. Hence there is no maximum combined area. We may obtain an arbitrarily large combined area by simply taking the width $W$ to be either arbitrarily small or arbitrarily large.
A farmer’s tractor pulls a rope of length 12 m attached to a bale of hay through a pulley is 8 m above the ground. The vertical distance between the tractor and the pulley (the distance from $P$ to $Q$) is 7 m. The tractor is moving to the left at rate of 2 m/sec, which causes the bale of hay to rise off the ground. Use this information to answer the following.

(a) The rate of change (with respect to time) of which variable is equal to the speed of the tractor?

(b) Use the Pythagorean theorem to find an equation that holds for all time and involves only the variables $x$ and $z$.

(c) Use the fact that the length of the rope is constant to find an equation that holds for all time and involves only the variables $z$ and $y$.

(d) Use the fact that the height of the pulley is constant to find an equation that holds for all time and involves only the variables $h$ and $y$.

(e) Combining the equations from parts (b), (c), and (d), find an equation that holds for all time and involves only the variables $x$ and $h$.

(f) The rate of change (with respect to time) of which variable is equal to the rate at which the bale of hay is rising?

(g) Find the rate at which the bale of hay is rising off the ground when the horizontal distance between the tractor and the bale of hay is 8 m.

Solution

(a) $x$

(b) $x^2 + 7^2 = z^2$, or $x^2 + 49 = z^2$

(c) $y + z = 12$

(d) $y + h = 8$

(e) Subtracting the last two equations gives $z - h = 4$, or $z = h + 4$. Substituting this expression for $z$ in the first equation gives $x^2 + 49 = (h + 4)^2$. We will write this equation as:

$$h = \sqrt{x^2 + 49} - 4$$

(f) $h$
(g) Differentiating the equation in part (e) gives:

\[
\frac{dh}{dt} = \frac{x \frac{dx}{dt}}{\sqrt{x^2 + 49}}
\]

We are given that \( \frac{dx}{dt} = 2 \) (speed of the tractor) and that \( x = 8 \) (tractor is 8 m horizontally away from pulley). Hence we have:

\[
\frac{dh}{dt} = \frac{16}{\sqrt{113}} \approx 1.51
\]

So the bale of hay is rising at approximately 1.51 m/sec.

12 pts 161. The graph of \( y = f(x) \) is shown below. The graph consists of four line segments and one semicircle.

Define the function \( g(x) \) by

\[
g(x) = \int_0^x f(t) \, dt
\]

Note that \( f \) and \( g \) are different functions! Use this information to answer the following.

(a) Calculate \( f'(9) \).
(b) Calculate \( f'(6) \).
(c) Calculate \( g'(6) \).
(d) Calculate \( g(11) - g(8) \).
(e) Is the statement “\( g(4) > g(0) \)” true or false?
(f) Find the critical numbers of \( g \) in the interval \( (0, 12) \).

Solution

(a) Observe that \( f'(9) \) is simply the slope of the given graph at \( x = 9 \). Hence \( f'(9) = \frac{3-0}{10-8} = 1.5 \).

(b) Observe that \( f'(6) \) is the derivative of the given graph at \( x = 6 \), and \( f \) has a horizontal tangent line at \( x = 6 \). Hence \( f'(6) = 0 \).

(c) By the fundamental theorem of calculus, \( g'(x) = f(x) \). Hence \( g'(6) = f(6) = -2 \).

(d) By the additivity property of integrals, \( g(11) - g(8) = \int_8^{11} f(t) \, dt \). This is the area of the region below the graph of \( y = f(t) \) and above the interval \( [8, 11] \) on the \( t \)-axis. Note that this region is a triangle with base 3 and height 3. Hence \( g(11) - g(8) = \frac{1}{2} \cdot 3 \cdot 3 = 4.5 \).
(e) Note that \( g(0) = 0 \) by properties of integrals, and \( g(4) > 0 \) since \( g(4) \) is the area of a triangle that lies above the \( t \)-axis. Hence the given statement is true.

(f) The critical numbers of \( g \) are those \( x \)-values where either \( g'(x) = 0 \) or \( g'(x) \) does not exist. Recall from part (c) that \( g'(x) = f(x) \). Clearly \( f(x) \) is defined everywhere on \((0, 12)\). So the only critical numbers of \( g \) are the solutions to \( f'(x) = 0 \): \( x = 4, x = 8, \) and \( x = 11 \).

12 pts 162. For this problem, you will explore the substitution rule for two different integrals. Consider the first (definite) integral:

\[
J_1 = \int_{e^{-3}}^{e^2} \frac{2 \ln(x) - 3}{5x} \, dx
\]

Use the substitution \( u = 2 \ln(x) - 3 \) to compute this integral. After you do the the substitution and translate the integral from being in terms of \( x \) to being in terms of \( u \), you have an integral of the following form:

\[
J_1 = \int_a^b g(u) \, du
\]

where \( a < b \) and there is no number to the left of the integral sign.

(a) After the substitution, what is the integrand \( g(u) \)?

(b) After the substitution, what is the lower limit of integration? upper limit of integration?

Now use the fundamental theorem of calculate \( J_1 \), giving the following:

\[
J_1 = \int_a^b g(u) \, du = G(b) - G(a)
\]

(c) What is the relationship between \( g \) and \( G \)?

(d) Calculate \( J_1 \).

Now consider the second (indefinite) integral:

\[
J_2 = \int \frac{\ln(x)}{3x^2} \, dx
\]

Use the substitution \( u = \ln(x) \). After you do the the substitution and translate the integral from being in terms of \( x \) to being in terms of \( u \), you have an integral of the following form:

\[
J_2 = \int f(u) \, du
\]

where there is no number to the left of the integral sign.

(e) After the substitution, what is the integrand \( f(u) \)?

\textbf{Solution}

(a) We have \( u = 2 \ln(x) - 3 \), whence \( \frac{du}{dx} = \frac{2}{x} \), or \( dx = \frac{x}{2} \, du \). Hence we have:

\[
\frac{2 \ln(x) - 3}{5x} \, dx = \frac{u}{5x} \cdot \left( \frac{x}{2} \, du \right) = \frac{u}{10} \, du
\]

So the new integrand is \( g(u) = u/10 \).

(b) We find the new limits of integration by substituting the old limits of integration...
into our relation $u = 2 \ln(x) - 3$. Hence the new limits are:

\[
x = e^{-3} \implies u = 2 \cdot (-3) - 3 = -9
\]
\[
x = e^2 \implies u = 2 \cdot (2) - 3 = 1
\]

So the new lower and upper limits of integration are $-9$ and $1$, respectively.

(c) $g$ is the derivative of $G$ (equivalently, $G$ is an antiderivative of $g$).

(d) Combining all of the previous parts, we have:

\[
J_1 = \int_{-9}^{1} \frac{u}{10} \, du = \frac{u^2}{20}\bigg|_{-9}^{1} = \frac{1}{20} (1 - 81) = -4
\]

(e) We have $u = \ln(x)$, whence $\frac{du}{dx} = \frac{1}{x}$, or $dx = x \, du$. Hence we have:

\[
J_2 = \int \frac{u}{3x^2} \cdot (x \, du) = \int \frac{u}{3x} \, du
\]

We are still left with a factor of $x$, but the integrand must be only in terms of $u$. Since $u = \ln(x)$, we have $x = e^u$. Hence we have:

\[
J_2 = \int \frac{u}{3e^u} \, du = \int \frac{1}{3} u e^{-u} \, du
\]

So the new integrand is $f(u) = \frac{1}{3} u e^{-u}$. 

7 Summer 2020
Midterm Exam #1

5 pts 163. Use the graph to calculate \( \lim_{x \to 3} f(x) \) or determine that the limit does not exist.

[Graph of a function with points at x = 1, 2, 3, 4, 5, and y = 1, 2, 3, 4, 5.]

**Solution**

\( \lim_{x \to 3} f(x) \) does not exist.

7 pts 164. For the graph below, find all values of \( a \) such that both of the following conditions are true:

- \( \lim_{x \to a} f(x) \) exists
- this limit is not equal to \( f(a) \)

[Graph of a function with points at x = 1, 2, 3, 4, 5, and y = 1, 2, 3, 4, 5.]

**Solution**

\( a = 1 \) only.

12 pts 165. Calculate \( \lim_{x \to 3} \left( \frac{x^3 + 2x^2 - 15x}{x^2 - 9} \right) \).

**Solution**
Factor and cancel.
\[
\lim_{x \to 3} \left( \frac{x^3 + 2x^2 - 15x}{x^2 - 9} \right) = \lim_{x \to 3} \left( \frac{x(x + 5)(x - 3)}{(x + 3)(x - 3)} \right) = \lim_{x \to 3} \left( \frac{x(x + 5)}{x + 3} \right) = \frac{3 \cdot 8}{6} = 4
\]

12 pts 166. Calculate \( \lim_{x \to 0} \left( \frac{\sin(6x)^2}{x^2 \cos(2x)} \right) \).

**Solution**
First we regroup terms and add factors of 6 to use the special trigonometric limit.
\[
\lim_{x \to 0} \left( \frac{\sin(6x)^2}{x^2 \cos(2x)} \right) = \lim_{x \to 0} \left( \frac{\sin(6x)}{6x} \cdot \frac{\sin(6x)}{6x} \cdot \frac{36}{\cos(2x)} \right)
\]
Now we compute the limit of each factor and use the special limit \( \lim_{x \to 0} \left( \frac{\sin(6x)}{6x} \right) = 1. \)
\[
\lim_{x \to 0} \left( \frac{\sin(6x)}{6x}, \frac{\sin(6x)}{6x}, \frac{36}{\cos(2x)} \right) = 1 \cdot 1 \cdot \frac{36}{1} = 36
\]

12 pts 167. The parts of this problem are related.
(a) Suppose \( x < 3 \). Write an algebraic expression that is equivalent to \( |x - 3| \) but without absolute value bars.
(b) Calculate \( \lim_{x \to 2} \left( \frac{|x - 3| - 1}{x - 2} \right) \). Explain why your work to part (a) is relevant here and precisely where you use it.

**Solution**
(a) If \( x < 3 \), then \( x - 3 < 0 \), whence \( |x - 3| = -(x - 3) = 3 - x \).
(b) Part (a) is relevant here since we want to calculate a limit as \( x \to 2 \) and \( x = 2 \) satisfies the inequality \( x < 3 \). Hence, for all \( x \) sufficiently close to 2 (on both sides), we have \( |x - 3| = 3 - x \). Now we may compute the limit.
\[
\lim_{x \to 2} \left( \frac{|x - 3| - 1}{x - 2} \right) = \lim_{x \to 2} \left( \frac{(3 - x) - 1}{x - 2} \right) = \lim_{x \to 2} \left( \frac{2 - x}{x - 2} \right) = \lim_{x \to 2} (-1) = -1
\]

13 pts 168. The parts of this problem are related.
(a) Consider the function below.
\[
f(x) = \begin{cases} 
\frac{x - 1}{3 - \sqrt{10} - x} , & x \neq 1 \\
\frac{-6}{x} , & x = 1 
\end{cases}
\]
Show that \( \lim_{x \to 1} f(x) \neq f(1) \).
(b) Now consider the similar function below.

\[
g(x) = \begin{cases} 
  \frac{x - 1}{3 - \sqrt{10 - x}}, & x \neq 1 \\
  b, & x = 1
\end{cases}
\]

where \( b \) is an unspecified constant. You are asked to determine whether it is true that
\[
\lim_{x \to 1} g(x) \neq g(1).
\]

How does your work for part (a) change, if at all, to answer this question? Explain your answer.

Solution

(a) Rationalize the denominator.

\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \left( \frac{x - 1}{3 - \sqrt{10 - x}} \cdot \frac{3 + \sqrt{10 - x}}{3 + \sqrt{10 - x}} \right) = \lim_{x \to 1} \left( \frac{(x - 1)(3 + \sqrt{10 - x})}{9 - (10 - x)} \right) = \lim_{x \to 1} \left( \frac{(x - 1)(3 + \sqrt{10 - x})}{x - 1} \right) = \lim_{x \to 1} (3 + \sqrt{10 - x}) = 3 + \sqrt{10 - 1} = 6
\]

Observe that 6 \( \neq -6 = f(1) \).

(b) The function value \( g(1) \) has no effect on our calculation of \( \lim_{x \to 1} g(x) \), which is equal to \( \lim_{x \to 1} f(x) = 6 \). Hence our work from part (a) does not change – we need only check whether \( b = 6 \).

13 pts 169. Let \( f(x) = 8 - \frac{1}{5x} \). Fully simplify the difference quotient \( \frac{f(x + h) - f(x)}{h} \) with \( h \neq 0 \). In your work, make clear where you the assumption \( h \neq 0 \).

Solution

We have the following.

\[
\frac{f(x + h) - f(x)}{h} = \frac{\left( 8 - \frac{1}{5(x+h)} \right) - \left( 8 - \frac{1}{5x} \right)}{h} = -\frac{1}{5(x+h)} + \frac{1}{5x} = -\frac{1}{5x} + \frac{1}{5x} - \frac{1}{5(x+h)} + \frac{1}{5x} = -\frac{x}{5x(x+h)} + \frac{h}{5hx(x+h)} = \frac{h}{5hx(x+h)}
\]

At this point, since \( h \neq 0 \), we may cancel the common factor of \( h \) and obtain our final answer.

\[
\frac{f(x + h) - f(x)}{h} = \frac{1}{5x(x+h)}
\]

14 pts 170. For both parts of this problem, consider the following inequality.

\[
\frac{(x - 3)(x - 6)}{x - 5} < 0
\]

Your goal is to identify an error in a false solution of this inequality, and then to solve the inequality yourself.
(a) A student submits the following work for solving this equality.

“First we multiple both sides by \((x - 5)\). On the left side, this factor cancels, and on the right side we get 0. So we have \((x - 3)(x - 6) < 0\). The graph of \(y = (x - 3)(x - 6)\) is a parabola that opens upward and crosses the \(x\)-axis at \(x = 3\) and \(x = 6\). This means that the graph is below the \(x\)-axis between these two \(x\)-values. So the solution to \((x - 3)(x - 6) < 0\) is the interval \((3, 6)\). But since the original inequality was undefined at \(x = 5\), we also have to exclude 5. So the final answer is \((3, 5) \cup (5, 6)\).”

The student’s teacher does not give full credit for this solution, simply noting that \(x = 4\) is included in the student’s answer, but \(x = 4\) does not satisfy the original inequality. So the final answer must be wrong.

What is the student’s error? Be as specific as possible and explain why this is an error.

To explain why the given solution is wrong, it is not enough to simply write the correct solution and observe that the two solutions are different.

(b) Solve the original inequality. Write your answer using interval notation.

Solution

(a) The quantity \((x - 5)\) may take negative values (i.e., if \(x < 5\)), and in that case, multiplying both sides of the inequality by \((x - 5)\) would reverse the direction of the inequality. The student’s work implicitly assumes that \((x - 5)\) is positive throughout, evident by the student’s not reversing the direction of the inequality. The student’s primary error is then never properly considering the case in which \((x - 5) < 0\).

(b) We will use a number line or sign chart. The cut points for our number line are \(x = 3\), \(x = 5\), and \(x = 6\). For each of the resulting subintervals on our number line, we test the truth of the inequality.

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign of (\frac{(x-3)(x-6)}{(x-5)})</th>
<th>truth of inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 3))</td>
<td>(x = 0)</td>
<td>(\ominus)</td>
<td>true</td>
</tr>
<tr>
<td>((3, 5))</td>
<td>(x = 4)</td>
<td>(\oplus)</td>
<td>false</td>
</tr>
<tr>
<td>((5, 6))</td>
<td>(x = 5.5)</td>
<td>(\ominus)</td>
<td>true</td>
</tr>
<tr>
<td>((6, \infty))</td>
<td>(x = 7)</td>
<td>(\oplus)</td>
<td>false</td>
</tr>
</tbody>
</table>

Hence the solution to the inequality is \((-\infty, 3) \cup (5, 6)\).
12 pts 171. Consider the function below.

\[ f(x) = \begin{cases} 
  x^2 + 4x - 1 & , \quad x < 2 \\
  11 & , \quad x = 2 \\
  19 - x^3 & , \quad x > 2 
\end{cases} \]

A student correctly calculates that \( \lim_{x \to 2} f(x) = 11 \) and enters this as their final answer on an online exam, initial getting full credit. However, after inspecting the student’s work, the teacher overrides this score and gives no credit. The teacher writes the comment “you have not correctly justified your answer.” The student wrote the following:

“Since \( f(x) \) is defined for all \( x \) and \( f(2) = 11 \), the answer is \( \lim_{x \to 2} f(x) = 11 \).”

(a) Why is the student’s justification incorrect?
(b) Write a complete and correct justification for the statement \( \lim_{x \to 2} f(x) = 11 \).

**Solution**

(a) Even though the student’s final answer is correct, the value of a function at \( x = a \) is irrelevant to the calculation of \( \lim_{x \to a} f(x) \). (For instance, it’s possible for \( f(a) \) and \( \lim_{x \to a} f(x) \) to be different.) So the justification is incorrect.

(b) Note that \( x = 2 \) is the transition point of the piecewise-defined function \( f(x) \). So we will justify the statement \( \lim_{x \to 2} f(x) = 11 \) using one-sided limits.

\[
\begin{align*}
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (x^2 + 4x - 1) = 4 + 8 - 1 = 11 \\
\lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (19 - x^3) = 19 - 8 = 11
\end{align*}
\]

Since the left-limit and right-limit are both equal to 11, we conclude that \( \lim_{x \to 2} f(x) = 11 \).
Midterm Exam #2

172. Let \( h(x) = \frac{f(x)}{g(x)} \) and suppose the following:

- \( f \) and \( g \) are continuous for all \( x \)
- \( \lim_{x \to a} g(x) = 0 \)

Is the following statement true or false?

The line \( x = a \) is necessarily a vertical asymptote of \( h(x) \).

You must justify your answer. This means that if your answer is “true”, you should explain why the above statement is always true. If your answer is “false”, you should give an example to show that the above statement is sometimes false.

**Solution**

False. The issue here is that if \( \lim_{x \to a} f(x) = 0 \) also, then \( h \) may or may not have a vertical asymptote at \( x = a \).

For an explicit example, let \( f(x) = g(x) = x \). Then \( f \) and \( g \) are continuous for all \( x \) and \( \lim_{x \to 0} g(x) = 0 \), but \( \frac{f(x)}{g(x)} \) does not have a vertical asymptote at \( x = 0 \) since \( \lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \).

173. Suppose that as \( x \) increases to 1, the values of \( f(x) \) get larger and larger, and the values stay positive. Is the following statement true or false?

Therefore, \( \lim_{x \to 1^-} f(x) = +\infty \).

You must justify your answer. This means that if your answer is “true”, you should explain why the above statement is always true. If your answer is “false”, you should give an example to show that the above statement is sometimes false.

**Solution**

False. The issue here is that the phrase “larger and larger” does not imply “arbitrarily large”, which is the more accurate description of what it means for a limit to be infinite.

For an explicit example, let \( f(x) = x \). Then the values of \( f(x) \) get larger and larger (i.e., increase) as \( x \) increases to 1. But \( \lim_{x \to 1^-} f(x) = 1 \).

174. In a certain parking garage, the cost of parking is $20 per hour or fraction thereof. For example, if you are in the garage for two hours and one minute, you pay $60. Let \( P(t) \) be the cost of parking for \( t \) hours, where \( t \) is any non-negative real number. Is the following statement true or false?

\( P(t) \) is a continuous function of \( t \).

You must justify your answer.
Solution
False. The function $P(t)$ has a jump discontinuity at each non-negative integer (i.e., at $x = 0, x = 1, x = 2, \text{etc.}$).

For instance, consider the example given in the problem itself. The cost of parking is $40 up to and including a period of 2 hours. However, as soon as you are in the garage one moment past 2 hours, the price jumps to $60. Mathematically, this means all of the following: $\lim_{t \to 2^{-}} P(t) = 40$, $\lim_{t \to 2^{+}} P(t) = 60$, and $P(2) = 40$. Hence $P(t)$ is not continuous at $t = 2$. (A similar argument holds for at any other non-negative integer value of $t$.)

175. Consider the following function, where $a$ and $b$ are unknown parameters.

$$f(x) = \begin{cases} 
3, & x \leq -1 \\
ax^2 + 2x + b, & -1 < x \leq 2 \\
14 - ax, & x > 2
\end{cases}$$

Find the values of $a$ and $b$ for which $f$ is continuous for all $x$, or determine that no such values exist. **You must use calculus methods to solve this problem. You must use proper notation and coherent logic.**

Solution
Each piece of $f$ is continuous for all $x$, so we need only force continuity at the transition points, $x = -1$ and $x = 2$. At each of these $x$-values, to have continuity, the left-limit, right-limit, and function value must all be equal. For $x = -1$, we must have:

$$\lim_{x \to -1^{-}} f(x) = \lim_{x \to -1^{+}} f(x) = f(-1)$$

$$\lim_{x \to -1^{-}} (3) = \lim_{x \to -1^{+}} (ax^2 + 2x + b) = (3)|_{x=-1}$$

$$3 = a - 2 + b = 3$$

Hence we obtain $a + b = 5$. Now for $x = 2$, we must have:

$$\lim_{x \to 2^{-}} f(x) = \lim_{x \to 2^{+}} f(x) = f(2)$$

$$\lim_{x \to 2^{-}} (ax^2 + 2x + b) = \lim_{x \to 2^{+}} (14 - ax) = (ax^2 + 2x + b)|_{x=2}$$

$$4a + 4 + b = 14 - 2a = 4a + 4 + b$$

Hence we obtain $6a + b = 10$.

To find $a$ and $b$ we solve the simultaneous system of equations:

$$a + b = 5$$

$$6a + b = 10$$

Subtracting the first equation from the second gives $5a = 5$, whence $a = 1$. Back-substitution then gives $b = 4$. 

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Let \( f(x) = \frac{9x - x^3}{x^2 + x - 6} \).

(a) Calculate all vertical asymptotes of \( f \). Justify your answer.
(b) Where is \( f \) discontinuous? Write your answer using interval notation.
(c) For each point at which \( f \) is discontinuous, determine what value should be assigned to \( f \), if possible, to guarantee that \( f \) will be continuous there.

Solution
(a) Putting the denominator to 0 gives \( x^2 + x - 6 = 0 \), with solutions \( x = -3 \) or \( x = 2 \).

Direct substitution of \( x = 2 \) into \( f \) gives the (undefined) expression \( \frac{10}{0} \) (i.e., a non-zero number divided by zero). Hence \( x = 2 \) is a vertical asymptote. However, for \( x = -3 \), we observe the following.

\[
\lim_{x \to -3} \left( \frac{9x - x^3}{x^2 + x - 6} \right) = \lim_{x \to -3} \left( \frac{x(3-x)(3+x)}{(x-2)(x+3)} \right) = \lim_{x \to -3} \left( \frac{x(3-x)}{x-2} \right) = \frac{18}{5}
\]

Since this limit is not infinite, the line \( x = -3 \) is not a vertical asymptote. The only vertical asymptote is \( x = 2 \).

(b) Since \( f \) is a ratio two continuous functions, \( f \) is discontinuous only where its denominator is 0. Hence \( f \) is discontinuous only at \( x = 2 \) and \( x = -3 \).

(c) From our work in part (a), we know that \( x = 2 \) is a vertical asymptote. Thus it is impossible to redefine \( f(2) \) to make \( f \) continuous at \( x = 2 \). (Why? The limit \( \lim_{x \to 2} f(x) \) does not exist.)

However, for \( x = -3 \), we have \( \lim_{x \to -3} f(x) = \frac{18}{5} \). Hence if we redefine \( f(-3) \) to be \( \frac{18}{5} \), then \( f \) becomes continuous at \( x = -3 \).

Let \( f(x) = \frac{(x-3)(2x+1)}{(5x+2)(3x-10)} \). Calculate all horizontal asymptotes of \( f \). You must show all your work.

Solution
We must calculate the limits of \( f \) at infinity. First we assume \( x \neq 0 \) and factor out the highest power of numerator and denominator separately to prepare the calculation of those limits. In particular, we factor out \( x \) from each term.

\[
\frac{(x-3)(2x+1)}{(5x+2)(3x-10)} = \frac{x^2}{x^2} \cdot \frac{(1 - \frac{3}{x})(2 + \frac{1}{x})}{(5 + \frac{2}{x})(3 - \frac{10}{x})} = \frac{(1 - \frac{3}{x})(2 + \frac{1}{x})}{(5 + \frac{2}{x})(3 - \frac{10}{x})}
\]

Now we note that \( \lim_{x \to -\infty} \frac{1}{x} = \lim_{x \to +\infty} \frac{1}{x} = 0 \). Hence we have

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left( \frac{(1 - \frac{3}{x})(2 + \frac{1}{x})}{(5 + \frac{2}{x})(3 - \frac{10}{x})} \right) = \frac{(1 - 0)(2 + 0)}{(5 + 0)(3 - 0)} = \frac{2}{15}
\]
Hence \( f \) has a single horizontal asymptote: \( y = \frac{2}{15} \).

24 pts 178. Let \( f(x) = \frac{3 + 7e^{2x}}{1 - e^x} \). Calculate each of the following limits.

(a) \( \lim_{x \to -\infty} f(x) \)
(b) \( \lim_{x \to +\infty} f(x) \)
(c) \( \lim_{x \to 0^-} f(x) \)

Hint: A graph of \( y = e^x \) will be very helpful.

Solution
(a) We recall that \( \lim_{x \to -\infty} (e^x) = 0 \), whence \( \lim_{x \to -\infty} (e^{2x}) = 0 \) also since \( e^{2x} = (e^x)^2 \). So we immediately have:
\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left( \frac{3 + 7e^{2x}}{1 - e^x} \right) = \frac{3 + 7 \cdot 0}{1 - 0} = 3
\]

(b) We recall that \( \lim_{x \to +\infty} (e^x) = +\infty \), whence \( \lim_{x \to +\infty} (e^{2x}) = +\infty \) also since \( e^{2x} = (e^x)^2 \). This would give the indeterminate form \( \frac{\infty}{-\infty} \) in our limit, so we instead factor out the “highest power” (or dominant term) as \( x \to +\infty \) of the numerator and denominator separately. For the numerator, the dominant term is \( e^{2x} \). For the denominator, the dominant term is \( e^x \). So now we have:
\[
\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left( \frac{e^{2x}}{e^x} \cdot \frac{3e^{-2x} + 7}{e^{-x} - 1} \right) = \lim_{x \to +\infty} \left( e^x \cdot \frac{3e^{-2x} + 7}{e^{-x} - 1} \right)
\]
Now we recall that \( \lim_{x \to +\infty} (e^{-x}) = 0 \), whence \( \lim_{x \to +\infty} (e^{-2x}) = 0 \) also since \( e^{2x} = (e^x)^2 \). So our limit is:
\[
\lim_{x \to +\infty} \left( e^x \cdot \frac{3e^{-2x} + 7}{e^{-x} - 1} \right) = \lim_{x \to +\infty} (e^x) \cdot \lim_{x \to +\infty} \left( \frac{3e^{-2x} + 7}{e^{-x} - 1} \right) = (+\infty) \cdot \frac{0 + 7}{0 - 1} = -\infty
\]

(c) Direct substitution of \( x = 0 \) into \( f(x) \) gives the (undefined) expression \( \frac{10}{0} \), which means that both one-sided limits at \( x = 0 \) are infinite. So we perform a sign analysis to determine whether the limit is positive or negative infinity.

As \( x \to 0^- \) the numerator \( (3 + 7e^{2x}) \to 10 \), which is positive. For the denominator, however, we note that \( e^x \) is an increasing function for all \( x \). Hence \( 1 = e^0 > e^x \) (or \( 1 - e^x > 0 \)) for all \( x < 0 \). (We can deduce this from a simple graph of \( y = e^x \). Alternatively, a test point shows that \( 1 - e^x > 0 \) for all \( x \) sufficiently close to and
less than 0.) Hence the denominator is positive as $x \to 0^-$. Putting this altogether gives the following:

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left( \frac{3 + 7e^{2x}}{1 - e^x} \right) = \frac{10}{0^+} \cdot \infty = \oplus \cdot \infty = +\infty
\]
10 pts 179. Explain the relationship between $f'(3)$ and the line tangent to the graph of $y = f(x)$ at $x = 3$.

**Solution**
The slope of the tangent line at $x = 3$ is $f'(3)$.

10 pts 180. Suppose $f'(7)$ exists. What can be said about the limit $\lim_{x \to 7} f(x)$?

**Solution**
Since $f$ is differentiable at $x = 7$, $f$ must also be continuous at $x = 7$. Hence $\lim_{x \to 7} f(x)$ exists and is equal to $f(7)$.

15 pts 181. Consider the following limit.

$$\lim_{h \to 0} \left( \frac{(4 + h)^{3/2} - 8}{h} \right)$$

Use the limit definition of derivative to identify this limit as the derivative of some function $f(x)$ at the point $x = a$. Then calculate the value of the limit.

**Solution**
Let $f(x) = x^{3/2}$ and $a = 4$. Then, by the definition of derivative, the given limit is equal to $f'(4)$. To compute the limit, we use the power rule to note that $f'(x) = \frac{3}{2} x^{1/2}$. So $f'(4)$, and hence the given limit, is equal to $\frac{3}{2} \cdot 4^{1/2} = 3$.

182. For each part, calculate the derivative by any valid method.

8 pts (a) $f(x) = x^2 \cos(3x) + \frac{1}{5x}$

8 pts (b) $f(x) = \sqrt{\sin \left( \frac{e^x}{x+1} \right)}$

**Solution**
(a) Write the second term as $\frac{1}{5x} = \frac{1}{5} x^{-1}$. Then use product rule and power rule.

$$f'(x) = 2x \cos(3x) - 3x^2 \sin(3x) - \frac{1}{5} x^{-2}$$

(b) Use chain rule twice. For the second application of chain rule, use quotient rule.

$$f'(x) = \frac{1}{2} \left( \sin \left( \frac{e^x}{x+1} \right) \right)^{-1/2} \cos \left( \frac{e^x}{x+1} \right) \cdot \frac{e^x(x+1) - e^x \cdot 1}{(x+1)^2}$$

16 pts 183. Let $f(x) = x^{15} e^{2-5x}$. Find the $x$-value of each point where the tangent line is horizontal.
Solution
The tangent line is horizontal wherever \( f'(x) = 0 \). We find the derivative using the product rule and chain rule.

\[
f'(x) = 15x^{14}e^{2-5x} - 5x^{15}e^{2-5x} = 5x^{14}(3 - x)e^{2-5x}
\]

Solving \( f'(x) = 0 \), we find that there is a horizontal tangent line at \( x = 0 \) and \( x = 3 \).

15 pts Let \( f(x) = 3x^5 - 2x^3 + 7x - 16 \). Find an equation of the tangent line at \( x = -1 \).

Solution
The tangent line passes through the point \((-1, f(-1)) = (-1, -24)\). Now observe that \( f'(x) = 15x^4 - 6x^2 + 7 \), whence the slope of the tangent line is \( f'(-1) = 16 \). So an equation of the tangent line is:

\[
y = -24 + 16(x + 1)
\]

18 pts Use the graph of \( y = f(x) \) below to answer the following questions.

(a) In the interval \((-6, 10)\), where is \( f \) not differentiable?
(b) Calculate a reasonable estimate of \( f'(0) \). Explain your reasoning.
(c) In the interval \((-6, 10)\), where is \( f'(x) = 0 \)?
(d) In the interval \((-6, 10)\), where is \( f'(x) < 0 \)?
(e) In the interval \((-6, 10)\), where is \( f'(x) > 0 \)?

Solution
(a) \( x = -4, x = 3, x = 5, \) and \( x = 7 \)
(b) We use the secant line through the points \((-2, 3)\) and \((2, 1)\) to estimate \( f'(0) \). The slope of this secant line is \( m = \frac{1-3}{2-(-2)} = -\frac{1}{2} \). Hence we estimate \( f'(0) \approx -\frac{1}{2} \).
(c) \( x = -2, x = 2, \) and the interval \((3, 5)\)
(d) \((-2, 2) \cup (5, 7)\)
(e) \((-6, -4) \cup (-4, -2) \cup (2, 3) \cup (7, 10)\)
Midterm Exam #4

18 pts 186. If $x$ units of a certain commodity are sold, the selling price per unit is $p(x) = \frac{5000}{x^2 + 64}$.

(a) Calculate the revenue function.
(b) Calculate the exact revenue derived from the 7th unit.
(c) Using marginal analysis, estimate the revenue derived from the 7th unit.

Solution
(a) $R(x) = xp(x) = \frac{5000x}{x^2 + 64}$
(b) The exact revenue is
$$R(7) - R(6) = \frac{35000}{113} - \frac{30000}{100} = \frac{1100}{113} \approx 9.735$$
(c) The approximate revenue is
$$R'(6) = \left(\frac{5000(64 - x^2)}{(x^2 + 64)^2}\right)\bigg|_{x=6} = \frac{5000 \cdot 28}{100^2} = 14$$

20 pts 187. Consider the curve described by the equation
$$x^4 - x^2y + y^4 = 1$$

(a) Find $\frac{dy}{dx}$ at a general point on the curve.
(b) Find an equation of the line tangent to the curve at the point $(-1, 1)$.

Solution
(a) Use implicit differentiation.
$$4x^3 - 2xy - x^2 \cdot \frac{dy}{dx} + 4y^3 \cdot \frac{dy}{dx} = 0$$
Solving algebraically for $\frac{dy}{dx}$ gives:
$$\frac{dy}{dx} = \frac{2xy - 4x^3}{4y^3 - x^2}$$
(b) The slope of the tangent line is
$$m = \frac{dy}{dx}\bigg|_{(x,y)=(-1,1)} = \left(\frac{xy - 2x^3}{4y^3 - x^2}\right)\bigg|_{(x,y)=(-1,1)} = \frac{2}{3}$$
Hence an equation of the tangent line is

\[ y = 1 + \frac{2}{3}(x + 1) \]

28 pts. 188. In a right triangle, the base is decreasing in length by 3 cm/sec and the area is increasing by 15 cm\(^2\)/sec. (The triangle always remains a right triangle.) At the time when the base is 15 cm in length and the height is 20 cm in length...

(a) ... at what rate is the height changing? (Give a number only.)

(b) ... at what rate is the length of the hypotenuse changing? (Give a number only.)

(c) What are the units of your answer in part (a)?

(d) In part (b), is the length of the hypotenuse increasing, decreasing, or staying constant?

Solution

(a) Let \( b \), \( h \), and \( A \) be the base, height, and the area of the triangle, respectively. Then we have \( A = \frac{1}{2}bh \). Differentiating with respect to \( t \) gives:

\[ \frac{dA}{dt} = \frac{1}{2} \frac{db}{dt} h + \frac{1}{2} b \frac{dh}{dt} \]

Now we substitute the given information: \( \frac{db}{dt} = -3 \), \( \frac{dA}{dt} = 15 \), \( b = 15 \), and \( h = 20 \).

\[ 15 = \frac{1}{2} \cdot (-3) \cdot 20 + \frac{1}{2} \cdot 15 \cdot \frac{dh}{dt} \]

Solving for \( \frac{dh}{dt} \) gives \( \frac{dh}{dt} = 6 \).

(b) Let the length of the hypotenuse be \( L \). Then \( b^2 + h^2 = L^2 \). Differentiating with respect to \( t \) gives:

\[ 2b \frac{db}{dt} + 2h \frac{dh}{dt} = 2L \frac{dL}{dt} \]

When \( b = 15 \) and \( h = 20 \), we have \( L = 25 \). Now we also substitute the given information and our work from part (a).

\[ 2 \cdot 15 \cdot (-3) + 2 \cdot 20 \cdot 6 = 2 \cdot 25 \cdot \frac{dL}{dt} \]

Solving for \( \frac{dL}{dt} \) gives \( \frac{dL}{dt} = 3 \).

(c) The units of \( \frac{dh}{dt} \) are cm/sec.

(d) Since \( \frac{dL}{dt} > 0 \), the length of the hypotenuse is increasing.
The total number of gallons in a water tank at \( t \) hours is given by \( N(t) = 40t^{2/5} \). Use a linear approximation to estimate the number of gallons added to the water between \( t = 32 \) and \( t = 35 \).

**Solution**

We use the tangent line to \( N(t) \) at \( t = 32 \). The tangent line passes through the point \((32, N(32))\). The slope of the tangent line is

\[
N'(32) = \left. \frac{2}{5} \cdot 40t^{-3/5} \right|_{t=32} = 16 \cdot \frac{1}{8} = 2
\]

Hence the equation of the tangent line is

\[
y = N(32) + 2(t - 32)
\]

This means that if \( t \) is near 32, then \( N(t) \approx N(32) + 2(t - 32) \). Hence the approximate number of gallons added in the described interval is

\[
\delta N = N(35) - N(32) \approx N(32) + 2(35 - 32) - N(32) = 6
\]

On an online exam, a student uses logarithmic differentiation to find the first derivative of

\[
f(x) = (3 + \sin(x))^{2+x^2}
\]

They type the following two lines for their work.

\[
y = (3 + \sin(x))^{2+x^2}
\]

\[
\ln(y) = \ln (\cdots)
\]

Unfortunately, the student runs out of time and is unable to submit the rest of their work. Oh no! Find \( f'(x) \) by completing the student’s work. (You must include the first lines of work above in your own work.)

**Solution**

We take logs of both sides, use logarithm laws, and then use implicit differentiation.

\[
y = (3 + \sin(x))^{2+x^2}
\]

\[
\ln(y) = \ln \left( (3 + \sin(x))^{2+x^2} \right)
\]

\[
\ln(y) = (2 + x^2) \ln (3 + \sin(x))
\]

\[
\frac{1}{y} \cdot \frac{dy}{dx} = 2x \cdot \ln (3 + \sin(x)) + (2 + x^2) \cdot \frac{1}{3 + \sin(x)} \cdot \cos(x)
\]

Now solve for \( \frac{dy}{dx} \) and replace \( y \) with \( f(x) \).

\[
f'(x) = (3 + \sin(x))^{2+x^2} \cdot \left( 2x \ln (3 + \sin(x)) + \frac{(2 + x^2) \cos(x)}{3 + \sin(x)} \right)
\]
Use the graph to answer the following questions. Read each question carefully. Some questions ask about \( f \) and others ask about \( f' \).

(a) Find the absolute maximum of \( f'(x) \) on \((0, 10)\) or determine that it does not exist.

(b) Find the absolute minimum of \( f'(x) \) on \((0, 10)\) or determine that it does not exist.

(c) Find all critical numbers of \( f(x) \) in \((0, 10)\).

Solution

(a) Since \( f' \) takes on the value 4 and no value larger, 4 is the absolute maximum.

(b) The range of \( f' \) is \((10, 4]\), and so there is no absolute minimum.

(c) The critical numbers of \( f \) are \( x = 4 \) (because \( f'(4) \) does not exist) and \( x = 9 \) (because \( f'(9) = 0 \)).

14 pts 192. If \( x \) units of a certain commodity are produced, then the marginal cost is

\[
\frac{dC}{dx} = 9x^2 + 4x + 15x^{1/4} + 10
\]

Suppose the total cost of producing the 1st unit is 100. Calculate the total cost of producing the first 16 units.

Solution

Antidifferentiation gives us the total cost function.

\[
C(x) = \int \left(9x^2 + 4x + 15x^{1/4} + 10\right) \, dx = 3x^3 + 2x^2 + 12x^{5/4} + 10x + K
\]

We are given that \( C(1) = 100 \), whence \( 3 + 2 + 12 + 10 + K = 100 \), and so \( K = 73 \). So then the total cost of producing 16 units is

\[
C(16) = \left(3x^3 + 2x^2 + 12x^{5/4} + 10x + 73\right) \bigg|_{x=16} = 13,417
\]
193. Let \( f(x) = \frac{1-2x}{6+x^2} \). Find the absolute extrema of \( f \) and the \( x \)-values at which they occur on \([-3, 2]\).

**Solution**

The function \( f \) is differentiable for all \( x \), and so the only critical numbers are solutions to \( f'(x) = 0 \). We have

\[
 f'(x) = \frac{(-2)(6+x^2) - 2x(1-2x)}{(6+x^2)^2} = \frac{2(x-3)(x+2)}{(6+x^2)^2}
\]

Hence the critical numbers are \( x = 3 \) (not in the interval \([-3, 2]\)) and \( x = -2 \). We now compare critical and endpoint values:

\[
 f(-3) = \frac{7}{15}, \quad f(-2) = \frac{1}{2}, \quad f(2) = -\frac{3}{10}
\]

Hence the absolute maximum is \( \frac{1}{2} \) at \( x = -2 \) and the absolute minimum is \( -\frac{3}{10} \) at \( x = 2 \).

194. Calculate the following limit or show that it does not exist.

\[
 \lim_{x \to 1} \left( \frac{\tan(\pi x)}{\sqrt{2 + x^3 - \sqrt{2 + x}}} \right)
\]

**Solution**

Direct substitution of \( x = 1 \) gives the indeterminate form \( \frac{0}{0} \). So we use l’Hospital’s rule.

\[
 \lim_{x \to 1} \left( \frac{\tan(\pi x)}{\sqrt{2 + x^3 - \sqrt{2 + x}}} \right) \overset{H}{=} \lim_{x \to 1} \left( \frac{\pi \sec(\pi x)^2}{\frac{3x^2}{2\sqrt{2+x}} - \frac{1}{2\sqrt{2+x}}} \right) = \frac{\pi \cdot 1^2}{\frac{3}{2\sqrt{3}} - \frac{1}{2\sqrt{3}}} = \sqrt{3}\pi
\]

195. Let \( f(x) = x^{1/3}(x - 16)^{1/5} \). Find all critical numbers of \( f \). Be sure to make clear why each number you find really is a critical number. (Note: the domain of \( f \) is all real numbers.)

**Solution**

Recall that \( g(x) = x^n \) is not differentiable at \( x = 0 \) if \( 0 < n < 1 \). Hence \( f \) is not differentiable at \( x = 0 \) and \( x = 16 \), and so both of these \( x \)-values are critical numbers. Now we solve the equation \( f'(x) = 0 \):

\[
 f'(x) = \frac{1}{3}x^{-2/3}(x - 16)^{1/5} + x^{1/3} \cdot \frac{1}{5}(x - 16)^{-4/5} = \frac{8(x - 10)}{15x^{2/3}(x - 16)^{4/5}}
\]

Hence the only other critical number is \( x = 10 \).
20 pts 196. Consider the following limit.

\[ L = \lim_{x \to -3} (4 + x)^{\frac{7}{(6 + 2x)}} \]

(a) What indeterminate form does this limit have?
(b) Explain why l’Hospital’s rule cannot be used on this limit in its current form.
(c) Calculate the value of \( L \).

Solution
(a) Direct substitution of \( x = -3 \) gives “1/0”, which is equivalent to the indeterminate form “\( 1^\infty \)”.
(b) L’Hospital’s rule cannot be used because the limit is not in the form of a quotient.
(c) We take logarithms and then use l’Hospital’s rule.

\[
\ln(L) = \lim_{x \to -3} \ln \left( (4 + x)^{\frac{7}{(6 + 2x)}} \right) = \lim_{x \to -3} \left( \frac{7}{6 + 2x} \cdot \ln(4 + x) \right)
\]

\[
= \lim_{x \to -3} \left( \frac{7 \ln(4 + x)}{6 + 2x} \right) = \lim_{x \to -3} \left( \frac{7 \cdot \frac{1}{4 + x}}{2} \right) = \frac{7}{2}
\]

So \( \ln(L) = \frac{7}{2} \), whence \( L = e^{7/2} \).
Midterm Exam #6

25 pts 197. Suppose $f$ is continuous for all $x$ and its first derivative is given by $f'(x) = (x - 4)^2(x + 2)$.

(a) Where is $f$ decreasing?

(b) A student writes “since $f'(4) = 0$, there is a local extremum (either min or max) at $x = 4$”. Is the student correct? Explain.

(c) Where is $f$ concave up?

(d) Find the $x$-coordinate of each inflection point of $f$.

Solution

(a) We see that $f'(x) = 0$ when $x = 4$ and when $x = -2$. So we have two cut points for our sign chart for $f'$.

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -2)$</td>
<td>$f'(-3)$</td>
<td>$\oplus\ominus$</td>
<td>decreasing</td>
</tr>
<tr>
<td>$(-2, 4)$</td>
<td>$f'(0)$</td>
<td>$\ominus\ominus$</td>
<td>increasing</td>
</tr>
<tr>
<td>$(4, \infty)$</td>
<td>$f'(5)$</td>
<td>$\ominus\ominus$</td>
<td>increasing</td>
</tr>
</tbody>
</table>

Thus $f$ is decreasing on the interval $(-\infty, -2]$.

(b) The student is incorrect. In general, the vanishing of the derivative at $x = a$ is not sufficient for there to be a local extremum at $x = a$. There must also be a sign change in the derivative at $x = a$. Indeed, in this case we see that $f$ is increasing on the interval $[-2, \infty)$, whence there is no local extremum at $x = 4$.

(c) The second derivative of $f$ is given by:

$$f''(x) = 2(x - 4) \cdot 1 \cdot (x + 2) + (x - 4)^2 \cdot 1 = 3x(x - 4)$$

Hence $f''(x) = 0$ at $x = 0$ and $x = 4$. So we have two cut points for our sign chart for $f''$.

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$f''(-1)$</td>
<td>$\ominus\ominus$</td>
<td>concave up</td>
</tr>
<tr>
<td>$(0, 4)$</td>
<td>$f''(1)$</td>
<td>$\ominus\ominus$</td>
<td>concave down</td>
</tr>
<tr>
<td>$(4, \infty)$</td>
<td>$f''(5)$</td>
<td>$\ominus\ominus$</td>
<td>concave up</td>
</tr>
</tbody>
</table>

Thus $f$ is concave up on the intervals $(-\infty, 0]$ and $[4, \infty)$. 
(d) There is an inflection point at both $x = 0$ and $x = 4$ (concavity change and $f$ is continuous there).

25 pts 198. Farmer Brown wants to create a rectangular pen that must enclose exactly 1800 ft$^2$. The fencing along the two horizontal sides costs $10/ft and the fencing along the two vertical sides costs $5/ft. (The cost is different because some parts of the fence have to be taller than other parts.) Let $x$ denote the length of the horizontal sides and let $y$ denote the length of the vertical sides.

(a) What are the dimensions of the cheapest pen?
(b) What is the total cost of the cheapest pen?
(c) Justify that your answer really does give the cheapest pen.

Solution

(a) The total cost of the fence is $F(x, y) = 20x + 10y$. We wish to maximize $F$ subject to the constraint $xy = 1800$. Hence our objective function is $f(x) = 20x + \frac{18000}{x}$, and our interval of interest is $(0, \infty)$. Observe that

$$f'(x) = 20 - \frac{18000}{x^2}.$$ 

Solving $f'(x) = 0$ gives us the only critical number in our interval: $x = 30$. Hence the optimal dimensions of the fence are $x = 30$ ft and $y = \frac{1800}{30} = 60$ ft. (We will finish justifying this conclusion in part (c).)

(b) The cost of the cheapest pen is $F(30, 60) = 20 \cdot 30 + 10 \cdot 60 = 1200$ dollars.

(c) Observe that $f''(x) = \frac{36000}{x^3}$, and so $f''(30) > 0$. Hence by the second derivative test, $x = 30$ gives a local minimum of $f(x)$. Since $x = 30$ is the only critical number of $f$ on $(0, 30)$, we conclude that this local minimum is also an absolute minimum.

25 pts 199. Suppose $f$ satisfies the following properties.

- $f$ is continuous and differentiable on $(-\infty, 3) \cup (3, \infty)$
- $x = 3$ is a vertical asymptote of $f$
- $\lim_{x \to \infty} f(x) = 1$
- the only $x$-values for which $f'(x) = 0$ are $x = 0$ and $x = 5$
- the only $x$-values for which $f''(x) = 0$ are $x = 0$ and $x = 7$

A sign chart for the first and second derivatives of $f$ are given below.

![Sign chart for $f''(x)$]
Do not attempt to find an algebraic formula for $f$. Use the above information to answer the following questions about $f$.

(a) Where is $f$ increasing?
(b) Where is $f$ concave down?
(c) At which $x$-value(s) does $f$ have a local minimum?
(d) At which $x$-value(s) does $f$ have a local maximum?
(e) Calculate $\lim_{x \to 3^+} f(x)$ or determine there is not enough information to do so.
(f) Calculate $\lim_{x \to -\infty} f(x)$ or determine there is not enough information to do so.
(g) Sketch a possible graph of $y = f(x)$. Make sure to clearly mark and label all of the following: local minima, local maxima, inflection points, vertical asymptotes, horizontal asymptotes. Your graph does not have to be to scale, but the shape must be correct.

Solution

(a) On the sign chart for $f'$ we look for intervals where $f'$ is non-negative. Hence $f$ is increasing on $(3, 5]$.
(b) On the sign chart for $f''$ we look for intervals where $f''$ is non-positive. Hence $f$ is concave down on $[0, 3)$ and $(3, 7]$.
(c) The first derivative of $f$ never transitions from negative to positive at a point of continuity ($f$ is discontinuous at $x = 3$). So there is no local minimum.
(d) The first derivative of $f$ transitions from positive to negative at $x = 5$ (and $f$ is continuous there). So there is a local maximum at $x = 5$.
(e) Since $x = 3$ is a vertical asymptote, we know that $\lim_{x \to 3^+} f(x)$ is infinite. Since $f$ is increasing on $(3, 5]$, we must have $\lim_{x \to 3^+} f(x) = -\infty$. (This is also consistent with the negative concavity of $f$ on $(3, 7]$.
(f) If $\lim_{x \to -\infty} f(x) = L$ for some finite $L$, then there are three possibilities, all of which are inconsistent with the given information:

- **The graph of $f$ approaches the asymptote $y = L$ from above.** Since $f$ is differentiable this would imply that $f$ would be increasing on an interval of the form $(-\infty, a]$. But $f$ is decreasing on $(-\infty, 0]$.
- **The graph of $f$ approaches the asymptote $y = L$ from below.** Since $f$ is differentiable this would imply that $f$ would have negative concavity on an interval of the form $(-\infty, a]$. But $f$ is concave up on $(-\infty, 0]$.
- **The graph of $f$ oscillates about the asymptote $y = L$.** Since $f$ is differentiable, this would imply that $f$ would have infinitely many local extrema in the interval $(-\infty, 0]$. But the only local extremum is at $x = 5$.

Since $f$ is decreasing on $(-\infty, 0]$, it is also not possible that $\lim_{x \to -\infty} f(x) = -\infty$. 
Thus the only possibility left that is consistent with all of the given information is \( \lim_{x \to -\infty} f(x) = \infty \).

(g) One possibility is shown below.

![Graph](image)

25 pts \[200\] In a certain video game, the player may adjust the values of their character’s *Intelligence* (denoted by \( x \)) and *Dexterity* (denoted by \( y \)). These power values must be non-negative can be any real number (they need not be whole numbers). The player cannot arbitrarily adjust their power, but rather these values must satisfy the equation \( x^2 + y^2 = 100 \). The total damage done (denoted by \( D \)) by the spell *Thunderbolt* is given by \( D = x + 3y \).

(a) How should the player adjust their power so that *Thunderbolt* does the most possible damage? Be sure to fully justify your answer.

(b) What is the minimum possible damage that *Thunderbolt* will do, regardless of how the player adjusts their character’s power? How should a player adjust these power values to achieve the minimum possible damage? Explain your answer.

**Solution**

(a) We wish to maximize the function \( D(x, y) = x + 3y \) subject to the constraint \( x^2 + y^2 = 100 \) (with \( x \) and \( y \) non-negative). Hence our objective function is \( f(x) = x + 3\sqrt{100 - x^2} \), and our interval of interest is \([0, 10]\). Observe that

\[
f'(x) = 1 - \frac{3x}{\sqrt{100 - x^2}}
\]

Solving \( f'(x) = 0 \) gives us the only critical number in our interval: \( x = \sqrt{10} \).

Since \( f \) is continuous on \([0, 10]\), we know, by the extreme value theorem, that the absolute extrema of \( f \) exist. Hence we make a table values: \( f(0) = 30 \), \( f(10) = 10 \), and \( f(\sqrt{10}) = 10\sqrt{10} \approx 31 \). Hence \( x = \sqrt{10} \) gives the maximum possible value of \( f \). This corresponds to an *Intelligence* value of \( \sqrt{10} \) and a *Dexterity* value of \( y = \sqrt{100 - x^2} = 3\sqrt{10} \).

(b) From our previous work, we see that the absolute minimum of \( D \) is 10, occurring when \( x = 10 \) (and \( y = 0 \)).
20. Let \( f(x) = 5 + \int_{-3}^{x} t^2 e^t \, dt \). Find an equation of the tangent line to \( f \) at \( x = -3 \).

**Solution**

Note that \( f(-3) = 5 + 0 = 5 \) and, by the fundamental theorem of calculus, \( f'(x) = x^2 e^x \). Hence \( f'(-3) = 9e^{-3} \), and an equation of our tangent line is

\[
y = 5 + 9e^{-3}(x + 3)
\]

20. Suppose \( f \) is continuous on \([0, 8]\) and has the following integrals:

\[
\int_{0}^{3} f(x) \, dx = 2 \quad \int_{3}^{5} f(x) \, dx = 7 \quad \int_{0}^{8} f(x) \, dx = 15
\]

For each part, calculate the integral using this information or determine there is not enough information to do so.

(a) \( \int_{0}^{5} f(x) \, dx \)  
(b) \( \int_{5}^{3} f(x) \, dx \)  
(c) \( \int_{5}^{8} f(x) \, dx \)  
(d) \( \int_{3}^{8} (2f(x) - 6) \, dx \)

**Solution**

(a) \( \int_{0}^{5} f(x) \, dx = \int_{0}^{3} f(x) \, dx + \int_{3}^{5} f(x) \, dx = 2 + 7 = 9 \)

(b) \( \int_{5}^{3} f(x) \, dx = -\int_{3}^{5} f(x) \, dx = -7 \)

(c) \( \int_{5}^{8} f(x) \, dx = \int_{0}^{8} f(x) \, dx - \int_{0}^{5} f(x) \, dx = 15 - 9 = 6 \)

(d) First observe:

\[
\int_{3}^{8} (2f(x) - 6) \, dx = 2 \cdot \int_{3}^{8} f(x) \, dx - \int_{3}^{8} 6 \, dx
\]

For the second integral on the right side, we note that it gives the area of a rectangle with length \( 8 - 3 = 5 \) and height 6. Hence

\[
\int_{3}^{8} 6 \, dx = 5 \cdot 6 = 30
\]

For the other integral, we have the following:

\[
\int_{3}^{8} f(x) \, dx = \int_{0}^{8} f(x) \, dx - \int_{0}^{3} f(x) \, dx = 15 - 2 = 13
\]
Putting this altogether gives us our final answer:

\[
\int_3^8 (2f(x) - 6) \, dx = 2 \int_3^8 f(x) \, dx - \int_3^8 6 \, dx = 2 \cdot 13 - 30 = -4
\]
integral, which is calculated using the fundamental theorem of calculus below.

\[
\int_{2}^{5} (25 - x^2) \, dx = \left[ (25x - \frac{1}{3}x^3) \right]_{2}^{5} = \left( 125 - \frac{1}{3} \cdot 125 \right) - \left( 50 - \frac{8}{3} \right) = 36
\]

\[\text{20 pts} \quad 205. \text{ Find the unique positive value of } a \text{ such that } \int_{0}^{a} \frac{x}{x^2 + 1} \, dx = 3.\]

**Solution**

We use substitution rule with \( u = x^2 + 1 \) to calculate the integral. Note that with this choice of \( u \), we have \( \frac{du}{dx} = 2x \), or \( dx = \frac{du}{2x} \). The limits of integration change from \( x = 0 \) and \( x = a \) to \( u = 1 \) and \( u = a^2 + 1 \), respectively. Hence we have the following:

\[
\int_{0}^{a} \frac{x}{x^2 + 1} \, dx = \int_{1}^{a^2+1} \frac{1}{2u} \, du = \frac{1}{2} \ln(u) \bigg|_{1}^{a^2+1} = \frac{1}{2} \ln(a^2 + 1) - 0 = \frac{1}{2} \ln(a^2 + 1)
\]

We now solve the equation \( \frac{1}{2} \ln(a^2 + 1) = 3 \) to find that \( a = \sqrt{e^6 - 1} \) (we have kept only the positive root).
8 Fall 2020
Midterm Exam #1 (Version A)

6 pts 206. Find the difference quotient \( f(x + h) - f(x) \) for \( f(x) = \sqrt{x + 2} \). Write your answer without square roots or fractional exponents in the numerator.

Solution
We have the following.

\[
\frac{f(x + h) - f(x)}{h} = \frac{\sqrt{x + h + 2} - \sqrt{x + 2}}{h} = \frac{(x + h + 2) - (x + 2)}{h(\sqrt{x + h + 2} + \sqrt{x + 2})} = \frac{1}{\sqrt{x + h + 2} + \sqrt{x + 2}}
\]

6 pts 207. Calculate the following limit or determine it does not exist. Show all work.

\[
\lim_{x \to 5} \left( \frac{25 - x^2}{x - 5} \right)
\]

Solution
Factor and cancel.

\[
\lim_{x \to 5} \left( \frac{25 - x^2}{x - 5} \right) = \lim_{x \to 5} \left( \frac{(5 - x)(5 + x)}{x - 5} \right) = \lim_{x \to 5} \left( -(5 + x) \right) = -10
\]

6 pts 208. Calculate the following limit or determine it does not exist. Show all work.

\[
\lim_{x \to 4} \left( \frac{\frac{1}{x} - \frac{1}{4}}{4 - x} \right)
\]

Solution
Simplify, factor, and cancel.

\[
\lim_{x \to 4} \left( \frac{\frac{1}{x} - \frac{1}{4}}{4 - x} \right) = \lim_{x \to 4} \left( \frac{4 - x}{4x(4 - x)} \right) = \lim_{x \to 4} \left( \frac{1}{4x} \right) = \frac{1}{16}
\]
2 pts 209. The graph of a function $y = g(x)$ is given below.

Find the domain of $g(x)$. Write your answer in interval notation.

**Solution**
The domain is the set of allowed $x$-values for $g(x)$. Hence the domain is $(0, 5]$.

2 pts 210. The graph of a function $y = g(x)$ is given below.

Find $g(g(4))$.

**Solution**
$g(g(4)) = g(3) = 2.$
The graph of a function \( y = g(x) \) is given below.

As \( x \) approaches 2, which of the limits exist?

(a) the lefthand limit exists, the righthand limit does not
(b) the righthand limit exists, the lefthand limit does not
(c) both the lefthand and righthand limits exist
(d) neither of the lefthand and righthand limits exist

Solution
Choice (c).

A student is asked to solve a certain limit and determines the limit does not exist. (This may or may not be the correct answer.) They write the following for their justification:

I used the direct substitution property to evaluate the limit. I noticed the denominator gives me a zero, therefore the limit does not exist.

Explain why the student’s justification is incorrect.

Solution
No. If direct substitution property gives \( \frac{0}{0} \) this only means that the limit cannot be computed by direct substitution (since \( \frac{0}{0} \) is undefined. Additionally, we also know that there are many limits which arise in this manner that actually do exist. For example, see the limit in question #3 or #4 of this exam.

True or false? “For the function \( f(x) \) below, \( \lim_{x \to 0} f(x) \) exists.”

\[
f(x) = \begin{cases} 
3e^x - 7 & , \ x < 0 \\
4 + \sin(x) & , \ x \geq 0
\end{cases}
\]

Justify your response.
Solution

False.

Observe the following.

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (3e^x - 7) = 3 - 7 = -4 , \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4 + \sin(x)) = 4 + 0 = 4
\]

Since the left- and right-limits are not equal, \( \lim_{x \to 0} f(x) \) does not exist.

6 pts 214. A student is asked to calculate the following limit:

\[
\lim_{x \to 0} \left( \frac{x \cos x}{\sin(3x)} \right)
\]

Analyze their work below. (Note that the correct answer is \( \frac{1}{3} \).)

\[
\lim_{x \to 0} \left( \frac{x \cos x}{\sin(3x)} \right) = \lim_{x \to 0} \left( \frac{x \cos x}{3 \sin x} \right) \tag{1}
\]

\[
= \left[ \lim_{x \to 0} \left( \frac{1}{3} \right) \right] \left[ \lim_{x \to 0} \left( \frac{x}{\sin x} \right) \right] \left[ \lim_{x \to 0} (\cos x) \right] \tag{2}
\]

\[
= \left( \frac{1}{3} \right) (1)(0) \tag{3}
\]

\[
= 0 \tag{4}
\]

There are two distinct errors in the student’s solution. Identify the lines in which they occur and describe each error.

Solution

Line (1) has an error: \( \sin(3x) \neq 3 \sin x \).

Line (3) has an error: \( \lim_{x \to 0} \cos x \neq 0 \) (this limit is 1)

9 pts 215. While solving the logarithmic equation

\[
\log_2(3x + 1) = 3
\]

a student wrote the following steps:

\[
\log_2(3x) + \log_2(1) = 3 \tag{5}
\]

\[
\log_2(3x) + 0 = 3 \tag{6}
\]

\[
3x = 3^2 \tag{7}
\]

\[
x = 3 \tag{8}
\]

Analyze the student’s work and answer the following questions.

(a) There are two distinct errors in the student’s solution. Identify the lines in which they occur and describe each error.

(b) What is the correct solution to the original equation?
Solution

(a) Line (5) has an error: logarithms do not distribute over sums.

Line (7) has an error: the right side should be $2^3$ since $\log_b(y) = x$ is equivalent to $y = b^x$.

(b) Exponentiating the equation $\log_2(3x + 1) = 3$ immediately gives $3x + 1 = 2^3 = 8$.
Then solving for $x$ gives $x = \frac{7}{3}$. 
Midterm Exam #1 (Version B)

216. Sketch the graph of a function with all of the given properties. Do not attempt to find a formula for the function.

- \( g(1) = 0 \)
- \( g(2) = 1 \)
- \( g(3) = -2 \)
- \( \lim_{x \to 1^-} f(x) \) does not exist
- \( \lim_{x \to 2} f(x) = 0 \)
- \( \lim_{x \to 3^-} f(x) = -1 \) and \( \lim_{x \to 3^+} f(x) = -2 \)

Solution
There are many possibilities. Here is one.

217. Find the difference quotient \( \frac{f(3 + h) - f(3)}{h} \) for \( f(x) = \frac{6}{9 - 2x} \). Assume \( h \neq 0 \) and simplify as much as possible. Your answer cannot contain a complex fraction (fraction within a fraction).

Solution
We have the following.

\[
\frac{f(3 + h) - f(3)}{h} = \frac{\frac{6}{9 - 2(3 + h)} - 2}{h} = \frac{\frac{6}{9 - 2h} - 2}{h} = \frac{6 - 2(3 - 2h)}{h(3 - 2h)} = \frac{4h}{h(3 - 2h)} = \frac{4}{3 - 2h}
\]

218. Consider the following function.

\[
f(x) = \begin{cases} 
4 & , \ x \leq 0 \\
\frac{x - 4}{\frac{1}{4} - \frac{1}{x}} & , \ 0 < x < 4 \\
16 & , \ x = 4 \\
g(x) & , \ x > 4
\end{cases}
\]

where \( g(x) \) is an unspecified function with domain \([4, \infty)\).
(a) Show that \( \lim_{x \to 4^-} f(x) = f(4) \).

(b) Suppose \( g(4) = 16 \). Is it necessarily true that \( \lim_{x \to 4} f(x) \) exists? Justify your response.

**Solution**

(a) Use the “second piece” of \( f \) to compute the limit.

\[
\lim_{x \to 4^-} \left( \frac{x - 4}{\frac{1}{4} - \frac{1}{x}} \right) = \lim_{x \to 4^-} \left( \frac{4x(x - 4)}{x - 4} \right) = \lim_{x \to 4^-} (4x) = 4 \cdot 4 = 16
\]

Since \( f(4) = 16 \), we have shown the desired statement.

(b) No. For instance, let \( g \) be the following function:

\[
g(x) = \begin{cases} 
16 & x = 4 \\
0 & x > 4 
\end{cases}
\]

Then \( g(4) = 16 \), but \( \lim_{x \to 4} f(x) \) does not exist because \( \lim_{x \to 4^+} f(x) = \lim_{x \to 4^-} g(x) = 0 \), which is not equal to \( \lim_{x \to 4^-} f(x) = 16 \).

The main issue here is that we really need the right limit, not the function value, of \( g \) at \( x = 4 \) to be equal to 16.

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4 pts 219. A student is asked to solve a certain limit and determines the limit does not exist. (This may or may not be the correct answer.) They write the following for their justification:

I used the direct substitution property to evaluate the limit. I obtained the expression “\( \frac{0}{0} \)”, which is undefined. Therefore the limit does not exist.

Is the student’s justification correct? Explain.

**Note:** To answer this question, it is not necessary to be given the actual limit the student was asked to compute.

**Solution**

No. If direct substitution property gives “\( \frac{0}{0} \)” this only means that the limit cannot be computed by direct substitution (since \( \frac{0}{0} \) is undefined. Additionally, we also know that there are many limits which arise in this manner that actually do exist. For example, see the limit in question #4a of this exam.

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220. Consider the limit \( \lim_{x \to 3} \left( \frac{(5x - c)(x + 4)}{x - 3} \right) \), where \( c \) is an unspecified constant.

(a) For what value(s) of \( c \) does this limit exist? Explain.

(b) Suppose the limit exists. What is its value? Show all work.
Solution

(a) Since direct substitution of \( x = 3 \) gives 0 in the denominator, the only hope we have of this limit existing is if we get cancellation. That is, there must be a common factor in numerator and denominator to cancel. (Alternatively, we must have a \( \frac{0}{0} \) form upon substitution of \( x = 3 \).) This means that the numerator must be 0 if \( x = 3 \).

\[
0 = (5 \cdot 3 - c)(3 + 4) = (15 - c) \cdot 7 \implies c = 15
\]

(b) If the limit exists, then we must have \( c = 15 \), in which case we have:

\[
\lim_{x \to 3} \left( \frac{5x - 15}{x - 3} \right) = \lim_{x \to 3} \left( \frac{5(x - 3)}{x - 3} \right) = \lim_{x \to 3} (5(x + 4)) = 35
\]

221. Suppose we have all of the following:

\[\log_3(x) = A, \quad \log_3(y) = B, \quad \log_5(z) = C\]

Write each of the following in terms of \( A, B, \) and \( C \). Show all work. Your final answer cannot contain “log”.

6 pts (a) \( \log_3 \left( \frac{\sqrt{x}}{9y^4} \right) \)

2 pts (b) \( \log_5(z) \)

Solution

(a) We use basic properties of logs.

\[
\log_3 \left( \frac{\sqrt{x}}{9y^4} \right) = \log_3(\sqrt{x}) - \log_3(9) - \log_3(y^4) = \frac{1}{2} \log_3(x) - 2 - 4 \log_3(y) = \frac{1}{2} A - 2 - 4B
\]

(b) By definition of the logarithm, we have:

\[
\log_5(z) = C \iff z = (b^5)^C
\]

Hence \( z = b^{5C} \). Now taking the log (base \( b \)) of both sides gives \( \log_b(z) = 5C \).

4 pts 222. The graph of a function \( y = g(x) \) is given below.
Find the range of $g(x)$. Write your answer in interval notation.

Solution
The range is the set of values of $g(x)$ (i.e., the $y$-values). Hence the range is $(0, 4]$.

4 pts 223. The graph of a function $y = g(x)$ is given below.

Find $g(g(1.5))$.

Solution
$g(g(1.5)) = g(2.5) = 4.$
Midterm Exam #2

8 pts 224. Let \( h(x) = \frac{\sqrt{3x^2 + x + 10}}{2 - 5x} \). Calculate all horizontal asymptotes of \( h \). Show all work.

Solution
For \( x \neq 0 \), we have the following algebra:

\[
\frac{\sqrt{3x^2 + x + 10}}{2 - 5x} = \frac{\sqrt{x^2 \left(3 + \frac{1}{x} + \frac{10}{x^2}\right)}}{x \left(\frac{2}{x} - 5\right)} = \frac{|x| \sqrt{3 + \frac{1}{x} + \frac{10}{x^2}}}{x \left(\frac{2}{x} - 5\right)}
\]

We have the fact that \( \sqrt{x^2} = |x| \). To compute the horizontal asymptotes, we compute the limits at infinity. For \( x \to \infty \), we may assume that \( x > 0 \), and so \( |x| = x \).

\[
\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \left( \frac{|x|}{x} \cdot \frac{\sqrt{3 + \frac{1}{x} + \frac{10}{x^2}}}{\frac{2}{x} - 5} \right) = \lim_{x \to \infty} \left( \frac{x}{x} \cdot \frac{\sqrt{3 + \frac{1}{x} + \frac{10}{x^2}}}{\frac{2}{x} - 5} \right) = \lim_{x \to \infty} \left( \frac{\sqrt{3 + \frac{1}{x} + \frac{10}{x^2}}}{\frac{2}{x} - 5} \right) = \frac{\sqrt{3 + 0 + 0}}{0 - 5} = -\frac{\sqrt{3}}{5}
\]

For \( x \to -\infty \), we may assume that \( x < 0 \), and so \( |x| = -x \).

\[
\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \left( \frac{|x|}{x} \cdot \frac{\sqrt{3 + \frac{1}{x} + \frac{10}{x^2}}}{\frac{2}{x} - 5} \right) = \lim_{x \to -\infty} \left( \frac{-x}{x} \cdot \frac{\sqrt{3 + \frac{1}{x} + \frac{10}{x^2}}}{\frac{2}{x} - 5} \right) = \lim_{x \to -\infty} \left( \frac{-\sqrt{3 + 0 + 0}}{0 - 5} \right) = \frac{-\sqrt{3}}{5}
\]

Hence the two horizontal asymptotes are \( y = -\frac{\sqrt{3}}{5} \) (as \( x \to \infty \)) and \( y = \frac{\sqrt{3}}{5} \) (as \( x \to -\infty \)).

8 pts 225. Determine where the following function is continuous. Write your answer using interval notation. You must use proper notation and calculus methods to justify your answer.

\[
f(x) = \begin{cases} 
\frac{x^2 - 9}{x - 3} & x < 3 \\
0 & x = 3 \\
5x - 9 & 3 < x < 4 \\
11 & x = 4 \\
27 - x^2 & x > 4 
\end{cases}
\]

Solution
Each piece of \( f \) is a rational function (actually, a polynomial) on their respective domains. So each piece is continuous. Hence we need only check continuity at \( x = 3 \) and \( x = 4 \). For
$x = 3$, we have the following:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5x - 9) = 6, \quad f(3) = 0$$

Since the right-limit and function value are not equal at $x = 3$, $f$ is not continuous at $x = 3$. (Note: we don’t even have to consider the left-limit here. However, the left-limit is 6.)

For $x = 4$, we have the following:

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (5x - 9) = 11, \quad \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (27 - x^2) = 11, \quad f(4) = 11$$

Since the left-limit, right-limit, and function value at $x = 4$ are all equal, $f$ is continuous at $x = 4$. Hence $f$ is continuous on $(-\infty, 3) \cup (3, \infty)$.

226. Consider the function $f$ below, where $A$, $B$, and $C$ are unspecified constants.

$$f(x) = \begin{cases} 
2x^3 + Ax & x < -1 \\
C & x = -1 \\
Bx^2 + 4 & x > -1 
\end{cases}$$

(a) Calculate $\lim_{x \to -1^-} f(x)$. No work required.

(b) Calculate $\lim_{x \to -1^+} f(x)$. No work required.

(c) How must $A$ and $B$ be related if $\lim_{x \to -1} f(x)$ exists? No work required.

(d) Suppose $C = 10$ and $f$ is continuous for all $x$. Find the values of $A$ and $B$. Explain your reasoning briefly or show all work.

Solution

(a) $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (2x^3 + Ax) = -2 - A$

(b) $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (Bx^2 + 4) = B + 4$

(c) The left- and right-limits must be equal, so we must have that $-2 - A = B + 4$.

(d) To have continuity at $x = -1$, we must have the left-limit, right-limit, and function value all equal. That is, we must have

$$-2 - A = B + 4 = 10$$

Solving for $A$ and $B$ then gives $A = -12$ and $B = 6$.

227. Consider the function $f(x) = \frac{(ax - 6)(x + 1)}{x - 2}$ where $a$ is an unspecified constant. For which value(s) of $a$ does the graph of $f$ have a vertical asymptote? What is the equation of this vertical asymptote?
Solution
The function $f$ has a vertical asymptote at $x = 2$ (where denominator is 0), as long as the denominator is not also a factor of the numerator. (Recall that if this happens, then the common factors would cancel and we would have a removable discontinuity, not a vertical asymptote.) Hence the numerator of $f$ must be nonzero if we substitute $x = 2$.

$$(2a - 6)(2 + 1) \neq 0 \implies a \neq 3$$

So $f$ has a vertical asymptote at $x = 2$ if and only if $a \neq 3$.

5 pts 228. Consider the function $f(x) = \frac{(ax - 6)(x + 1)}{x - 2}$ where $a$ is an unspecified constant. For which of the following values of $a$ does $f$ have a horizontal asymptote?

(a) $a = 0$ only
(b) $a = 6$ only
(c) $a = 2$ only
(d) $a = -1$ only
(e) $f$ has a horizontal asymptote for no value of $a$
(f) $f$ has a horizontal asymptote for all values of $a$
(g) there is not enough information to determine
(h) none of the above

Solution
Choice (a).
If $a \neq 0$, we have the following:

$$\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \left( \frac{x^2}{x} \cdot \frac{(a - \frac{6}{x})(1 + \frac{1}{x})}{1 - \frac{2}{x}} \right) = \lim_{x \to \pm \infty} \left( x \cdot \frac{(a - 0)(1 + 0)}{1 - 0} \right) = \pm \infty \cdot a = \pm \infty$$

(or the signs are reversed if $a < 0$). So there is no horizontal asymptote if $a \neq 0$. Also note that if $a \neq 0$, the numerator of $f$ has degree 2 and the denominator of $f$ has degree 1. From precalculus, you may have learned that this implies $f$ has no horizontal asymptote. However, if $a = 0$, then we have

$$\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \left( \frac{-6(x + 1)}{x - 2} \right) = -6$$

So there is a horizontal asymptote at $y = 6$. (Note that in the case $a = 0$, the numerator and denominator both have degree 1, whence there must be a horizontal asymptote.)
5 pts 229. For which value(s) of \( n \) is the following statement true?

\[
\lim_{x \to 2^-} (2 - x)^n = -\infty
\]

(a) \( n = -3 \)
(b) \( n = -6 \)
(c) \( n = 2 \)
(d) \( n = 8 \)
(e) \( n = 0 \)
(f) there is no value of \( n \) for which the statement is true
(g) the statement is true for all values of \( n \)
(h) none of the above

Solution
Choice (f).
If \( x \to 2^- \), then this means \( x \) is close to 2 and \( x < 2 \), whence \( 2 - x \) is a small positive number. But then \((2 - x)^n\) is positive for any value of \( n \), including \( n = 0 \) (since \((2-x)^0 = 1\)). So the given limit can never be \(-\infty\). The correct answer is “there is no value of \( n \) for which the statement is true”.

5 pts 230. Suppose the line \( y = 3 \) is a horizontal asymptote for \( f \). Which of the following statements MUST be true? Select all that apply.

(a) \( f(x) \neq 3 \) for all \( x \) in the domain of \( f \)
(b) \( f(3) \) is undefined
(c) \( \lim_{x \to 3} f(x) = \infty \)
(d) \( \lim_{x \to \infty} f(x) = 3 \)
(e) none of the above

Solution
Choice (e).
For choices (a), (b), and (c), consider \( f(x) = 3 \) (constant function). Then \( f \) has a horizontal asymptote at \( y = 3 \), but none of (a), (b), and (c) is true.

For choice (d), consider \( f(x) = e^x + 3 \). Then \( f \) has a horizontal asymptote at \( y = 3 \) because \( \lim_{x \to \infty} f(x) = 3 \), but choice (d) is false since \( \lim_{x \to \infty} f(x) = \infty \).

Hence choice (e) must be correct.
5 pts 231. Choose the equation from the following that expresses the fact that a function $f$ is continuous at 6.

(a) $\lim_{x \to 6} f(x) = f(6)$
(b) $\lim_{x \to 6} f(x) = 6$
(c) $\lim_{x \to 6} f(x) = f(6)$
(d) $\lim_{x \to 6} f(x) = 6$
(e) $\lim_{x \to 6} f(x) = 0$
(f) $\lim_{x \to 6} f(x) = \infty$
(g) $\lim_{x \to \infty} f(x) = f(6)$
(h) $\lim_{x \to \infty} f(x) = \infty$

Solution
Choice (c). (Definition of continuity.)
Midterm Exam #3

232. Consider the graph of \( y = f(x) \) below.

(a) For which values of \( x \) is \( f'(x) \geq 0? \) Choose from \( x_0, x_1, x_2, x_3, x_4, \) and \( x_5. \) Select all that apply.

(b) For which values of \( x \) does \( f'(x) \) not exist? Choose from \( x_0, x_1, x_2, x_3, x_4, \) and \( x_5. \) Select all that apply.

(c) Give a brief, 1-sentence explanation of your answer to part (b).

Solution

(a) \( x_2, x_4, x_5 \)

(b) \( x_0, x_1 \)

(c) At \( x_0, f \) is not continuous. At \( x_1, \) the graph of \( f \) has a sharp corner.

233. Consider the function \( f(x) = x^3 - 6x + c, \) where \( c \) is an unspecified constant. Suppose the line \( 102x - y = 609 \) is tangent to the graph of \( y = f(x) \) at the point \( P \) in the first quadrant.

(a) What is the value of \( f'(x) \) at the point \( P? \) Give a brief, 1-sentence explanation.

(b) Find the \( x \)-coordinate of \( P. \)

(c) Find the \( y \)-coordinate of \( P. \)

(d) Find the value of \( c. \)

Solution

(a) The slope of the tangent line at \( P \) is 102, hence \( f'(x) = 102 \) at \( P. \)

(b) We solve the equation \( f'(x) = 102. \)

\[
3x^2 - 6 = 102 \implies x^2 = 36 \implies x = 6
\]

(We reject the solution \( x = -6 \) since \( P \) is in the first quadrant.)

(c) The tangent line and graph of \( f \) coincide at the point of tangency. So substituting \( x = 6 \) into the equation of the tangent line gives \( y = 102 \cdot 6 - 609 = 3. \)
(d) We have \( f(6) = 6^3 - 6 \cdot 6 + c = 180 + c \). On the other hand, from part (c), \( f(6) = 3 \). Hence \( 180 + c = 3 \), and so \( c = -177 \).

**234.** Let \( f(x) = \frac{8e^x}{x - 3} \). Find the equation of each horizontal tangent line of \( f \). Show all work.

**Solution**
A horizontal tangent line occurs at points where \( f'(x) = 0 \).

\[
f'(x) = \frac{8e^x(x - 3) - 8e^x \cdot 1}{(x - 3)^2} = \frac{8e^x(x - 4)}{(x - 3)^2}
\]

Solving \( f'(x) = 0 \) gives \( x = 4 \) (whence \( f(4) = 8e^4 \)). Hence the only horizontal tangent line is \( y = 8e^4 \).

**235.** Consider the following limit.

\[
\lim_{x \to \pi/8} \left( \frac{\tan(2x) - 1}{x - \pi/8} \right)
\]

(a) Use the limit definition of derivative to identify this limit as the derivative of some function \( f(x) \) at the point \( x = a \). You must explicitly identify \( f \) and \( a \).

(b) Use your identifications in part (a) to calculate the given limit. Show all work.

**Solution**
(a) The limit definition of derivative is

\[
f'(a) = \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right)
\]

Comparing this to the given limit, we find that \( a = \pi/8 \) and \( f(x) = \tan(2x) \).

(b) By part (a), the given limit is \( f'((\pi/8)) \). Chain rule gives \( f'((\pi/8)) = 2\sec(2\pi/8)^2 \), whence the value of the limit is \( f'((\pi/8)) = 2 \cdot (\sqrt{2})^2 = 4 \).

**236.** Suppose \( f(1) = -8 \) and \( f'(1) = 12 \). Let \( F(x) = x^3f(x) + 10 \). Find an equation of the tangent line to \( F \) at \( x = 1 \). Show all work.

**Solution**
Observe that \( F(1) = f(1) + 10 = 2 \). Hence the point of tangency is \((1, 2)\). Using product rule, we have

\[
F'(x) = 3x^2 f(x) + x^3 f'(x)
\]

Hence the slope of the tangent line is \( F'(1) = 3f(1) + f'(1) = -12 \). So the equation of the desired tangent line is

\[
y = 2 - 12(x - 1)
\]
Midterm Exam #4

3 pts. Suppose $f$ is differentiable on $(-\infty, \infty)$, $f(5) = 3$, and $f'(5) = -7$. Use linear approximation to estimate $f(5.1)$.

Fill in the blank. Your answer must be a number rounded to two decimal places. Enter “999” if the answer is “not enough information”. No work is required.

Solution
The tangent line to $f$ at $x = 5$ is $y = 3 - 7(x - 5)$. Hence $f(5.1) \approx 3 - 7(5.1 - 5) = 2.3$.

19 pts. At a certain moment, a race official is watching a race car approach the finish line along a straight track at some constant, positive speed. Suppose the official is sitting still at the finish line, 20 m from the point where the car will cross.

For parts (a)–(e), the allowed answers are “positive”, “negative”, “zero”, or “not enough information”.

(a) At the moment described, what is the sign of $\frac{dx}{dt}$?
(b) At the moment described, what is the sign of $\frac{dy}{dt}$?
(c) At the moment described, what is the sign of $\frac{dL}{dt}$?
(d) At the moment described, what is the sign of $\frac{d(\cos(\theta))}{dt}$?
(e) At the moment described, what is the sign of $\frac{d^2x}{dt^2}$?
(f) Suppose the speed of the car is 70 m/sec. At what rate is the distance between the car and the race official changing when the car is 60 m from the finish line? Your answer must have the correct units. Your answer must be exact. No decimal approximations.

Solution
(a) negative ($x$ is decreasing)
(b) zero ($y$ is constant)
(c) negative ($L$ is decreasing)
(d) negative ($\theta$ is increasing to 90-degrees, whence $\cos(\theta)$ is decreasing to 0)
(e) zero (the speed of the car is constant, whence $\frac{dx}{dt}$ is constant)
(f) Observe that $x^2 + 400 = L^2$, and so $x\frac{dx}{dt} = L\frac{dL}{dt}$. Substituting $\frac{dx}{dt} = -70$ and $x = 60$
gives the equations:

\[ 3600 + 400 = L^2, \quad -4200 = \frac{dL}{dt} \]

The first equation gives \( L = \sqrt{4000} \), whence the second equation gives

\[ \frac{dL}{dt} = -\frac{4200}{\sqrt{4000}} = -21\sqrt{10} \]

The distance between the car and the official decreases at a rate of \( 21\sqrt{10} \) m/sec.

12 pts 239. Use linear approximation to estimate \( \sqrt[3]{29} - \sqrt[3]{27} \). Show all work. Your final answer must be exact and may not contain any radicals.

**Solution**

We use the tangent line to \( f(x) = x^{1/3} \) at \( x = 27 \) to estimate \( \sqrt[3]{29} \). Observe that \( f(27) = 3 \) and \( f'(27) = \frac{1}{3}x^{-2/3} \bigg|_{x=27} = \frac{1}{27} \). The desired tangent line is thus \( y = 3 + \frac{1}{27}(x - 27) \), which gives:

\[ \sqrt[3]{29} - \sqrt[3]{27} \approx 3 + \frac{1}{27}(29 - 27) - 3 = \frac{2}{27} \]

16 pts 240. Consider the curve defined by

\[ ax^2y - 3xy^2 + 4x = b \]

where \( a \) and \( b \) are unspecified constants.

(a) Show that \( \frac{dy}{dx} = \frac{3y^2 - 2axy - 4}{ax^2 - 6xy} \). You must show all steps.

(b) Suppose the tangent line to the curve at the point \((1,1)\) is \( y = 1 + 5(x - 1) \). Use part (a) to find the value of \( a \).

(c) Use your answer to part (b) to find the value of \( b \).

**Solution**

(a) Differentiate both sides of the equation with respect to \( x \), using product rule and chain rule on each of the first two terms.

\[ 2axy + ax^2 \frac{dy}{dx} - 3y^2 - 6xy \frac{dy}{dx} + 4 = 0 \]

Collecting like terms and factoring gives:

\[ (ax^2 - 6xy) \frac{dy}{dx} + (2axy - 3y^2 + 4) = 0 \]

Elementary algebra then gives the desired result.

(b) The slope of the tangent line at \((1,1)\) is 5, whence

\[ 5 = \left. \frac{dy}{dx} \right|_{(x,y)=(1,1)} = \left. \left( \frac{3y^2 - 2axy - 4}{ax^2 - 6xy} \right) \right|_{(x,y)=(1,1)} = \frac{-2a - 1}{a - 6} \]
Solving for $a$ gives $a = \frac{29}{7}$.

(c) The point $(1, 1)$ lies on the curve, i.e., the point $(1, 1)$ satisfies the original equation. This implies $a + 1 = b$, and so $b = \frac{36}{7}$.
Midterm Exam #5

9 pts 241. Use the graph of \( f \) below to answer the questions. Assume that the domain of \( f \) is \((−∞, ∞)\).

(a) At which \( x \)-value does a local minimum of \( f \) occur? (Choose one from \( x = 0 \) through \( x = 5 \).)
(b) Select all of the critical points of \( f \). (Select all from 0 through 5.)
(c) Estimate the absolute maximum of \( f \) on \([0, 3]\). (Enter your answer as a number, rounded to one decimal place. Enter “999” if there is no absolute maximum on \([0, 3]\).)

Solution
(a) \( x = 2 \)
(b) 1, 2, and 5
(c) 4

9 pts 242. Consider the limit \( \lim_{x \to 2^-} ((x - 2) \ln(2 - x)) \).

(a) Does this limit have an indeterminate form? If so, which indeterminate form?
(b) Explain why l’Hospital’s rule cannot be used on this limit in its current form.
(c) Write the expression in an equivalent form to which l’Hospital’s rule may be applied. (You do not have to retype “\( \lim \) \( x \to 2^- \)” in your answer.)

Note: Do not attempt to calculate the limit. You are not required to calculate the limit.

Solution
(a) Yes, the form \( 0 \cdot (−∞) \).
(b) The expression is not written as an indeterminate quotient.
(c) One possibility is \( \lim_{x \to 2^-} \left( \frac{\ln(2 - x)}{1/(x - 2)} \right) \).

10 pts 243. Suppose \( f'(x) \) is continuous with \( f(3) = 2 \) and \( f'(3) = -8 \). Calculate the following limit or determine that it does not exist. Show all work and justify all steps.

\[
\lim_{x \to 1} \left( \frac{2x^4 - f(3x^{1/4})}{x^2 - 4x + 3} \right)
\]
Solution

Direct substitution of \( x = 1 \) gives \( \frac{0}{0} \), and so we use l'Hospital’s rule, followed by direct substitution.

\[
\lim_{x \to 1} \left( \frac{2x^4 - f(3x^{1/4})}{x^2 - 4x + 3} \right) = \lim_{x \to 1} \left( \frac{8x^3 - f'(3x^{1/4}) \cdot \frac{3}{4}x^{-3/4}}{2x - 4} \right) = \frac{8 - (-8) \cdot \frac{3}{4}}{-2} = -7
\]

11 pts 244. Let \( V(t) \) denote the volume of water, measured in gallons, in a tank at time \( t \). The tank is initially filled with 5 gallons of water. At \( t = 0 \), water flows in at a rate in gal/min given by \( V'(t) = 0.5(196 - t^2) \) for \( 0 \leq t \leq 10 \). Find the total amount of water in the tank after 4 minutes.

Solution

Computing the antiderivative of \( V'(t) \) immediately gives \( V(t) = 0.5(196t - \frac{1}{3}t^3) + C \) for some constant \( C \). The condition \( V(0) = 5 \) implies \( C = 5 \), whence \( V(t) = 98t - \frac{1}{6}t^2 + 5 \). The volume of water in the tank after 4 minutes is \( V(4) = \frac{1183}{3} \approx 394.333 \) gallons.

11 pts 245. (You will need a basic calculator for this problem.)

Consider the function

\[
f(t) = \frac{a}{t^2 - 3t + 25}
\]

where \( a \) is an unspecified positive constant. Suppose the absolute minimum of \( f \) on \( [0, 6] \) is 3. Answer the following. Show all work and justify your conclusions.

(a) Find the value of \( a \).

(b) Calculate the absolute maximum of \( f \) on \( [0, 6] \).

Hint: First find the absolute extrema of \( f \) on \( [0, 6] \), in terms of \( a \).

Solution

(a) We first find the absolute extrema of \( f \) on \( [0, 6] \) in terms of \( a \). Since \( f \) is differentiable for all \( t \), the only critical points are solutions to \( f'(t) = 0 \).

\[
f'(t) = \frac{a(2t - 3)}{(t^2 - 3t + 25)^2} = 0 \implies t = 1.5
\]

We now make a table that includes any critical values and endpoint values. Observe:

\[
f(0) = \frac{a}{25} \quad , \quad f(1.5) = \frac{4a}{91} \quad , \quad f(6) = \frac{a}{43}
\]

Since \( a \) is positive, we see that the largest of these values is \( f(1.5) \) and the smallest of these values is \( f(6) \). We are given that the absolute minimum is 3, and so \( f(6) = \frac{a}{43} = 3 \), whence \( a = 129 \).

(b) From our previous work, the absolute maximum is \( f(1.5) = \frac{4a}{91} \). With \( a = 129 \), we see that the absolute maximum is \( \frac{516}{91} \approx 5.670 \).
3 pts 246. The figure below shows the graphs of $f$, $f'$, and $f''$. Identify which graph is that of $f''$.

Solution

$f''(x) = C(x)$.

It is clear that $A'(x) = B(x)$ and $B'(x) = C(x)$ by observing the locations of relative extrema and zeros. For instance, $B(x)$ has a zero wherever $A(x)$ has a relative extremum, and $C(x)$ has a zero wherever $B(x)$ has a relative extremum. (Strictly speaking, this is not enough to conclude $A'(x) = B(x)$ and $B'(x) = C(x)$. However, we can also observe intervals of increase. For instance, $A(x)$ is decreasing wherever $B(x)$ is negative and $A(x)$ is increasing wherever $B(x)$ is positive. The same observation holds for $B(x)$ and $C(x)$.)

It follows that $A''(x) = C(x)$, and so $f'' = C$.

4 pts 247. The figure below shows the graphs of $f$, $f'$, and $f''$. Identify which graph is which.
Solution
It is clear that the only choice for \( B(x) \) is \( f(x) \) since \( B \) has a removable discontinuity at \( x = 2 \) but \( A(x) \) and \( C(x) \) do not. Now we simply observe the behavior near \( x = 2 \). Note that \( B(x) \) is increasing on \((2 - \epsilon, 2)\) and decreasing on \((2, 2 + \epsilon)\) for some small \( \epsilon > 0 \). Hence \( B'(x) > 0 \) on \((2 - \epsilon, 2)\) and \( B'(x) < 0 \) on \((2, 2 + \epsilon)\). The only function with these signs is \( A(x) \), whence \( B'(x) = A(x) \). That leaves only \( A'(x) = C(x) \), which we can again verify by a similar argument.

Hence \( f = B, f' = A, \) and \( f'' = C \).

12 pts. Sketch the graph of a function \( f(x) \) that satisfies all of the following:

- \( \lim_{x \to -\infty} f(x) = 1 \)
- \( \lim_{x \to \infty} f(x) = 6 \)
- \( x = -3 \) is a vertical asymptote for \( f \)
- \( f'(2) = 0 \)
- \( f''(5) = 0 \)
- \( f'(x) > 0 \) on \((2, \infty)\)
- \( f'(x) < 0 \) on \((-\infty, -3)\) and \((-3, 2)\)
- \( f''(x) > 0 \) on \((-3, 5)\)
- \( f''(x) < 0 \) on \((-\infty, -3)\) and \((5, \infty)\)

You must label all relative extrema, inflection points, vertical asymptotes and horizontal asymptotes. Your graph is does not have to be to scale, but the shape and labels must be correct.

Solution
There is one relative minimum at \( x = 2 \) and one inflection point at \( x = 5 \). The lines \( y = 1 \) and \( y = 6 \) are both horizontal asymptotes. Here is one possibility for the graph.
The first and second derivative of \( f \) are given below.

\[
f'(x) = \frac{(x+2)^{1/5}}{(x-25)^2}, \quad f''(x) = \frac{-9(x+5)}{5(x-25)^3(x+2)^{4/5}}
\]

You may assume that \( f \) has a vertical asymptote at \( x = 25 \), but do not attempt to calculate \( f \) explicitly. Answer all of the following. You must show all work. Intervals should be as inclusive as possible.

(a) Where is \( f \) decreasing?
(b) Where is \( f \) increasing?
(c) At which \( x \)-values, if any, does \( f \) have a local minimum?
(d) At which \( x \)-values, if any, does \( f \) have a local maximum?
(e) Where is \( f \) concave down?
(f) Where is \( f \) concave up?
(g) At which \( x \)-values, if any, does \( f \) have an inflection point?

Solution

- Parts (a)–(d)

We see that \( f'(x) = 0 \) only when \( x = -2 \). Since \( x = 25 \) is a vertical asymptote for \( f \) (or we observe that \( f' \) is not continuous at \( x = 25 \)), the cut points for our sign chart for \( f' \) are \( x = -2 \) and \( x = 25 \).
interval test point sign

$(-\infty, -2) \quad f'(-3) = \bigcirc \bigcirc = \bigcirc$

$(-2, 25) \quad f'(0) = \bigcirc \bigcirc = \bigcirc$

$(25, \infty) \quad f'(30) \bigcirc \bigcirc = \bigcirc$

Hence $f$ is decreasing on $(-\infty, -2]$; $f$ is increasing on $[-2, 25)$ and $(25, \infty)$; $f$ has a local minimum at $x = -2$; $f$ has no local maximum.

- Parts (e)–(g)

We see that $f''(x) = 0$ only when $x = -5$ and $f''(x)$ does not exist when $x = -2$. Since $x = 25$ is a vertical asymptote for $f$ (or we observe that $f''$ is not continuous at $x = 25$), the cut points for our sign chart for $f''$ are $x = -5, x = -2,$ and $x = 25$.

$\begin{array}{|c|c|c|}
\hline
\text{interval} & \text{test point} & \text{sign} \\
\hline
(-\infty, -5) & f''(-6) = \bigcirc \bigcirc \bigcirc = \bigcirc \\
(-5, -2) & f''(-3) = \bigcirc \bigcirc \bigcirc = \bigcirc \\
(-2, 25) & f''(0) = \bigcirc \bigcirc \bigcirc = \bigcirc \\
(25, \infty) & f''(30) \bigcirc \bigcirc \bigcirc = \bigcirc \\
\hline
\end{array}$

Hence $f$ is concave down on $(-\infty, -5]$ and $(25, \infty)$; $f$ is concave up on $[-5, -25)$; $f$ has an inflection point at $x = -5$.

17 pts 250. A rectangular box with a square base and no top is being constructed to hold a volume of 150 cm$^3$. The material for the base of the container costs $6/cm^2$ and the material for the sides of the container costs $2/cm^2$. Find the dimensions of the cheapest possible container.

You must use calculus methods taught in this course to solve this problem. You must show all work. You must fully justify why your answer really does give the cheapest container. Give exact answers only.

Solution

Let $x$ be the length of the square base and let $y$ be the height of the box, both measured in cm. Our objective function is the total cost of the box, which is given by:

$$C(x, y) = 6x^2 + 2 \cdot 4xy$$
Our constraint is that the volume must be 150 cm$^3$, whence $x^2y = 150$, or $y = 150/x^2$. Hence our objective function in terms of $x$ only is

$$f(x) = C \left( x, \frac{150}{x^2} \right) = 6x^2 + \frac{1200}{x}$$

We seek an absolute minimum of $f$ on the interval of interest $(0, \infty)$. We have:

$$f'(x) = 12x - \frac{1200}{x^2}$$

The only positive solution to $f'(x) = 0$, and thus our only critical point, is $x = 100^{1/3}$. Observe that $f''(x) = 12 + \frac{2400}{x^3} > 0$ for all $x > 0$. Hence $f$ is concave up on $(0, \infty)$, whence $x = 100^{1/3}$ gives a local minimum of $f$. Since this is the only critical point, it must also give the absolute minimum.

The dimensions of the cheapest box are $x = 100^{1/3}$ and $y = \frac{150}{100^{2/3}} = 1.5 \cdot 100^{1/3}$.
9 Spring 2021
Midterm Exam #1

251. Use the graph of \( y = f(x) \) below to answer the following questions.

(a) Calculate \( f(f(2)) \).
(b) Find where \( f(x) = 0 \).
(c) State the domain of \( f \) in interval notation.
(d) State the range of \( f \) in interval notation.
(e) For each limit, determine its value or determine that it does not exist. (If the limit is infinite, enter \(+\infty\) or \(-\infty\) as your answer, as appropriate.)

\[
\begin{align*}
\lim_{x \to 0^-} f(x) & \quad \lim_{x \to 0^+} f(x) & \quad \lim_{x \to 0} f(x)
\end{align*}
\]

(f) For each limit, determine its value or determine that it does not exist. (If the limit is infinite, enter \(+\infty\) or \(-\infty\) as your answer, as appropriate.)

\[
\begin{align*}
\lim_{x \to 3^-} f(x) & \quad \lim_{x \to 3^+} f(x)
\end{align*}
\]

Solution

(a) \( f(f(2)) = f(0) = 4 \)
(b) \( x = -2, x = 2, x = 4, x = 6 \)
(c) \((-3, 3) \cup (3, 7]\)
(d) \((-6, \infty)\)
(e) \( \lim_{x \to 0^-} f(x) = 4, \lim_{x \to 0^+} f(x) = -6, \lim_{x \to 0} f(x) \) does not exist
(f) \( \lim_{x \to 3^-} f(x) = \infty, \lim_{x \to 3^+} f(x) = 4 \)
12 pts 252. Suppose you have exactly 840 ft of fencing that will be used to build an enclosure that consists of two identical rectangular pens that share a common fence. Let $x$ be the (vertical) length of each pen and let $y$ be the (horizontal) width of each pen. See the figure below.

(a) Find an expression for $F(x)$, the area of one individual pen, as a function of $x$.
(b) Now suppose that, for each of the two pens, the sum of the length and width must not exceed 250 ft. In the context of this problem, what is the domain of $F$? Write your answer in interval notation.

Solution
(a) Since we have 840 total fencing, we have that $3x + 4y = 840$, or $y = \frac{1}{4}(840 - 3x)$. The area of one individual pen is $xy$. Hence $F(x) = \frac{1}{4}x(840 - 3x)$.

(b) For one pen we are given that $x + y \leq 250$, or $x + \frac{1}{4}(840 - 3x) \leq 250$. Rearranging this inequality gives $x \leq 160$. Of course, the length $x$ must be non-negative (so $x \geq 0$) and the width $y$ must also be non-negative (so $\frac{1}{4}(840 - 3x) \geq 0$, or $x \leq 280$). Putting these restrictions altogether gives the domain of $F$ as $[0, 160]$ (or $0 \leq x \leq 160$).

8 pts 253. Suppose $\log_3(x) = A$ and $\log_3(y) = B$. Rewrite the expression below in terms of $A$ and $B$. Your final answer may not contain any logarithms.

$$\log_3 \left( \frac{27\sqrt{x}}{y^4} \right)$$

Solution
We have the following:

$$\log_3 \left( \frac{27\sqrt{x}}{y^4} \right) = \log_3(27) + \log_3(\sqrt{x}) - \log_3(y^4) = 3 + \frac{1}{2} \log_3(x) - 4 \log_3(y) = 3 + \frac{1}{2}A - 4B$$

10 pts 254. Consider the following statement.

“Suppose $f$ and $g$ are functions with $g(3) = 1$. Put $H(x) = \frac{f(x)}{g(x) - 1}$. Then $H$ must have a vertical asymptote at $x = 3$.”

Is the above statement true or false? Explain your answer in 1 or 2 sentences. Your answer should contain English with few mathematical symbols.
Solution
False. Let \( f(x) = x - 3 \) and \( g(x) = x - 2 \). Then \( g(3) = 1 \) but \( H(x) = \frac{f(x)}{g(x) - 1} = \frac{x - 3}{x - 3} \) does not have a vertical asymptote at \( x = 3 \) since \( \lim_{x \to 3} H(x) = 1 \) (i.e., the limit exists and is finite).

Other acceptable explanations:
- “Since the limit of \( f \) and \( g \) (and hence the limit of \( H \)) as \( x \to 3 \) does not depend on the function values \( f(3) \) and \( g(3) \), we cannot say for sure whether \( H \) has a vertical asymptote at \( x = 3 \). There is not enough information.”
- “If \( f(3) = 0 \), then direct substitution of \( x = 3 \) into \( H \) gives the indeterminate form \( \frac{0}{0} \), which does not necessarily indicate a vertical asymptote. There may be some algebraic cancellation that allows the limit \( \lim_{x \to 3} H(x) \) to exist.”

255. Suppose \( \lim_{x \to 0} f(x) = 4 \). Evaluate \( \lim_{x \to 0} \left( \frac{xf(x)}{\sin(5x)} \right) \). Choose the best answer below.

(i) \( \lim_{x \to 0} \left( \frac{xf(x)}{\sin(5x)} \right) \) exists and is equal to _____.
(ii) \( \lim_{x \to 0} \left( \frac{xf(x)}{\sin(5x)} \right) \) does not exist.
(iii) We do not have enough information to determine the value of \( \lim_{x \to 0} \left( \frac{xf(x)}{\sin(5x)} \right) \).

Solution
Choice (i). We have
\[
\lim_{x \to 0} \left( \frac{xf(x)}{\sin(5x)} \right) = \lim_{x \to 0} \left( \frac{1}{5} \cdot \frac{5x}{\sin(5x)} \right) \cdot \lim_{x \to 0} f(x) = \frac{1}{5} \cdot 1 \cdot 4 = \frac{4}{5}
\]

256. Consider the following limit, where \( a \) is an unspecified constant.
\[
\lim_{x \to -3} \left( \frac{x^2 - a}{x^3 + x^2 - 6x} \right)
\]

(a) Find the value of \( a \) for which this limit exists.
(b) For this value of \( a \), calculate the value of the limit.

Solution
(a) Direct substitution of \( x = -3 \) gives the undefined expression \( \frac{9-a}{0} \). If the given limit exists, then the only possibility is that this undefined expression is, in fact \( \frac{9}{0} \). (If the expression were \( \frac{\text{nonzero}}{0} \), we would have a vertical asymptote at \( x = -3 \) instead.) Hence \( 9 - a = 0 \), and so \( a = 9 \).
(b) With \( a = 9 \), we have the following.

\[
\lim_{x \to -3} \left( \frac{x^2 - 9}{x^2 + x^2 - 6x} \right) = \lim_{x \to -3} \left( \frac{(x - 3)(x + 3)}{x(x - 2)(x + 3)} \right) = \lim_{x \to -3} \left( \frac{x - 3}{x(x - 2)} \right) = -\frac{2}{5}
\]

16 pts 257. Consider the following function, where \( k \) is an unspecified constant.

\[
g(x) = \begin{cases} 
xe^{x+4} - 7 \ln(x + 5) & x < -4 \\
-4 \cos(\pi x) & -4 < x < 5 \\
10 & x = 5 \\
\sqrt{2x - 5} + k & 5 < x 
\end{cases}
\]

Note that \( g(-4) \) is undefined.

(a) Does \( \lim_{x \to -4} g(x) \) exist? Choose the best answer below.

(i) Yes, \( \lim_{x \to -4} g(x) \) exists and is equal to _____.

(ii) Yes, \( \lim_{x \to -4} g(x) \) exists but we cannot determine its value with the given information.

(iii) No, \( \lim_{x \to -4} g(x) \) does not exist because the corresponding one-sided limits are not equal.

(iv) No, \( \lim_{x \to -4} g(x) \) does not exist because \( g(-4) \) does not exist.

(v) No, \( \lim_{x \to -4} g(x) \) does not exist because the limit is infinite.

(b) Calculate the following limits. Your answer may contain \( k \).

\[
\begin{align*}
\lim_{x \to 5^-} g(x) & \quad \lim_{x \to 5^+} g(x)
\end{align*}
\]

(c) Is it possible to choose a value of \( k \) so that \( \lim_{x \to 5} g(x) \) exists? Choose the best answer below.

(i) Yes, \( \lim_{x \to 5} g(x) \) exists if we choose \( k = _____. \)

(ii) No. There is no value of \( k \) for which \( \lim_{x \to 5} g(x) \) exists.

Solution
(a) Choice (i). Note the following:

\[
\begin{align*}
\lim_{x \to -4} g(x) &= \lim_{x \to -4} (xe^{x+4} - 7 \ln(x + 5)) = -4 \cdot 1 - 7 \cdot 0 = -4 \\
\lim_{x \to -4} g(x) &= \lim_{x \to -4} (-4 \cos(\pi x)) = -4 \cdot \cos(-4\pi) = -4
\end{align*}
\]

The left- and right-limits at \( x = -4 \) are both equal to \(-4\), hence \( \lim_{x \to -4} g(x) = -4 \).

(Note that the function value \( g(-4) \), which is undefined, is irrelevant.)

(b) We have \( \lim_{x \to 5^-} g(x) = -4 \cos(5\pi) = 4 \) and \( \lim_{x \to 5^+} g(x) = \lim_{x \to 5^+} (\sqrt{2x - 5} + k) = \sqrt{5} + k \).
(c) Choice (i). From part (b), we need $4 = \sqrt{5} + k$, or $k = 4 - \sqrt{5}$. (Again, the function value $g(5)$, which is 10, is irrelevant.)

16 pts 258. Let $f(x) = \frac{(x + a)(x - 3)}{(x - 2)(x + 1)}$, where $a$ is an unspecified, positive constant. For each part, calculate the limit in terms of $a$ or determine the limit does not exist. (If the limit is infinite, enter $+\infty$ or $-\infty$ as your answer, as appropriate.)

(a) $\lim_{x \to 0} f(x)$
(b) $\lim_{x \to 2^-} f(x)$
(c) $\lim_{x \to 2^+} f(x)$
(d) $\lim_{x \to 2} f(x)$

Solution

(a) Use direct substitution.

$$\lim_{x \to 0} f(x) = \frac{(0 + a)(0 - 3)}{(0 - 2)(0 + 1)} = \frac{3a}{2}$$

(b) Substitution of $x = 2$ gives $\frac{(2 + a)}{0}$. Since $a > 0$, this expression is $\frac{\text{nonzero}}{0}$, which means $x = 2$ is a vertical asymptote of $f$. So we must perform a sign analysis.

We have $-(2 + a) < 0$, and so the numerator is negative as $x \to 2$. For the denominator, we note that since $x \to 2^-$ (i.e., $x < 2$), we have $x + 1 > 0$ and $x - 2 < 0$. Hence the entire expression for $f(x)$ is positive as $x \to 2^-$. Hence $\lim_{x \to 2^-} f(x) = \infty$.

(c) As in part (c), we perform a sign analysis. However, since $x \to 2^+$, we have $x - 2 > 0$ now. Hence $\lim_{x \to 2^+} f(x) = -\infty$.

(d) The limits in parts (b) and (c) are not equal, so $\lim_{x \to 2} f(x)$ does not exist.
Midterm Exam #2

259. The following limit represents the derivative of a function \( f \) at a point \( a \).

\[
f'(a) = \lim_{h \to 0} \left( \frac{5 \ln(e^4 + h) - 20}{h} \right)
\]

(a) Find a possible function \( f(x) \).
(b) For your choice of \( f \) in part (a), find a possible value of \( a \).
(c) Calculate the value of the limit. Explain your calculation briefly in one sentence.

Solution

We compare the limit to the definition of the derivative.

\[
f'(a) = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \right)
\]

(a) \( f(x) = 5 \ln(x) \)
(b) \( a = e^4 \) (note that \( f(e^4) = 5 \cdot 4 = 20 \))
(c) We have \( f'(e^4) = \left. \frac{5}{x} \right|_{x=e^4} = \frac{5}{e^4} \).

260. Use the graph of \( f \) below to answer the following questions. Dashed lines indicate the location of asymptotes.

(a) Calculate \( \lim_{x \to \infty} f(x) \).
(b) Calculate \( \lim_{x \to -\infty} f(x) \).
(c) List the values of \( x \) where \( f \) is not continuous.
(d) List the values of \( x \) where \( f \) is not differentiable.
(e) What is the sign of \( f'(-1) \)? (choices: positive, negative, zero, does not exist)
(f) What is the sign of \( f'(0.5) \)? (choices: positive, negative, zero, does not exist)
Solution
(a) \( \lim_{x \to \infty} f(x) = -4 \)
(b) \( \lim_{x \to -\infty} f(x) = 3 \)
(c) \( x = 0, x = 4, x = 5 \)
(d) \( x = 0, x = 1, x = 4, x = 5 \)
(e) negative
(f) positive

261. Consider the function \( g \) below, where \( a \) and \( b \) are unspecified constants.

\[
g(x) = \begin{cases} 
be^x + a + 1 & x \leq 0 \\
ax^2 + b(x + 3) & 0 < x \leq 1 \\
a \cos(\pi x) + 7bx & 1 < x 
\end{cases}
\]

Assume that \( g \) is continuous for all \( x \).

(a) What relation must hold between \( a \) and \( b \) for \( g \) to be continuous at \( x = 0 \)? Your answer should be an equation involving \( a \) and \( b \).

(b) What relation must hold between \( a \) and \( b \) for \( g \) to be continuous at \( x = 1 \)? Your answer should be an equation involving \( a \) and \( b \).

(c) Calculate the value of \( a \).

(d) Calculate the value of \( b \).

Solution
(a) We must have the following:

\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x)
\]

\[
\lim_{x \to 0^-} (be^x + a + 1) = \lim_{x \to 0^+} (ax^2 + b(x + 3))
\]

\[
b + a + 1 = 3b
\]

(b) We must have the following:

\[
\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x)
\]

\[
\lim_{x \to 1^-} (ax^2 + b(x + 3)) = \lim_{x \to 1^+} (a \cos(\pi x) + 7bx)
\]

\[
a + 4b = -a + 7b
\]

(c) The equations from parts (a) and (b) must be true simultaneously. Part (b) implies \( a = 1.5b \), whence part (a) gives \( b + 1.5b + 1 = 3b \), or \( b = 2 \). Hence \( a = 3 \).

(d) See part (c).
For each statement, determine whether it is true or false.

(a) If \( \lim_{x \to a} f(x) \) can be evaluated by direct substitution, then \( f \) must be continuous at \( x = a \).
(b) The value of \( \lim_{x \to a} f(x) \), if it exists, is found by calculating \( f(a) \).
(c) If \( f \) is not differentiable at \( x = a \), then \( f \) is also not continuous at \( x = a \).

Solution
(a) True. This statement is equivalent to \( \lim_{x \to a} f(x) = f(a) \) which is the definition of continuity (of \( f \) at \( x = a \)).
(b) False. The limit \( \lim_{x \to a} f(x) = f(a) \) is independent of \( f(a) \). (Indeed, the latter need not even exist for the limit to exist.)
(c) False. The function \( f(x) = |x| \) is not differentiable at \( x = 0 \) but continuous for all \( x \).

Suppose that an equation of the tangent line to \( f \) at \( x = 5 \) is \( y = 3x - 8 \). Let \( g(x) = \frac{f(x)}{x^2 + 10} \).

(a) Calculate \( f(5) \).
(b) Calculate \( f'(5) \).
(c) Calculate \( g(5) \).
(d) Calculate \( g'(5) \).
(e) Write down an equation of the tangent line to \( g \) at \( x = 5 \).

Solution
(a) The tangent line to \( f \) at \( x = 5 \) passes through \((5, 7)\) whence \( f(5) = 7 \).
(b) The slope of the line \( y = 3x - 8 \), which is tangent to \( f \) at \( x = 5 \), is 3. Hence \( f'(5) = 3 \).
(c) We have \( g(5) = \frac{f(5)}{35} = \frac{1}{5} \).
(d) We use quotient rule to find \( g'(x) \).

\[
g'(x) = \frac{f'(x) \cdot (x^2 + 10) - f(x) \cdot 2x}{(x^2 + 10)^2}
\]

Hence \( g'(5) = \frac{3 \cdot 35 - 7 \cdot 10}{35^2} = \frac{1}{35} \).
(e) The tangent to \( g \) at \( x = 5 \) is \( y = \frac{1}{5} + \frac{1}{35}(x - 5) \).

Suppose \( f(2) = -7 \) and \( f'(2) = 3 \).

(a) Let \( g(x) = \cos(x)f(x) \). Calculate \( g'(2) \). Your answer must be exact.
(b) Let \( h(x) = e^{2f(x)+3} \). Calculate \( h'(2) \). Your answer must be exact.
Math 135 Midterm Exam #2

Solution
(a) We use product rule.

\[ g'(x) = -\sin(x)f(x) + \cos(x)f'(x) \]

Hence \( g'(2) = 7\sin(2) + 3\cos(2) \).

(b) We use chain rule.

\[ h'(x) = e^{2f(x) + 3} \cdot 2f'(x) \]

Hence \( h'(2) = 6e^{-11} \).

16 pts 265. Let \( f(x) = x^2 + bx + c \), where \( b \) and \( c \) are unspecified constants. An equation of the tangent line to \( f \) at \( x = 3 \) is \( 12x + y = 10 \).

(a) Calculate \( f(3) \). Your answer must be a number; it cannot contain the letters \( b \) or \( c \).
(b) Calculate \( f'(3) \). Your answer must be a number; it cannot contain the letters \( b \) or \( c \).
(c) Calculate the value of \( b \).
(d) Calculate the value of \( c \).

Solution
(a) The tangent to \( f \) at \( x = 3 \) is \( 12x + y = 10 \), which passes through the point \( (3, -26) \). Hence \( f(3) = -26 \).

(b) The tangent to \( f \) at \( x = 3 \) is \( 12x + y = 10 \), which has slope \(-12\). Hence \( f'(3) = -12 \).

(c) We have \( f'(x) = 2x + b \), whence \( f'(3) = 6 + b \). From part (b), we must have \( 6 + b = -12 \), whence \( b = -18 \).

(d) We have \( f(x) = x^2 - 18x + c \), whence \( f(3) = -45 + c \). From part (a), we must have \( -45 + c = -26 \), whence \( c = 19 \).
A local gym has two cylindrical swimming pools. The larger pool has radius 20 meters and is filled with water. The smaller pool has radius 12 meters and is empty. Water is drained from the large pool and immediately emptied into the small pool. The height of the water in the small pool increases at a rate of 0.3 m/min.

Let $V_L$, $V_S$, $h_L$, and $h_S$ refer to the volume of the large pool, volume of the small pool, height of the large pool and height of the small pool, respectively.

(a) How are $\frac{dV_L}{dt}$ and $\frac{dV_S}{dt}$ related?

(b) What is the sign of $\frac{dh_L}{dt}$?

(c) Find $\frac{dV_S}{dt}$.

(d) Find $\frac{dh_L}{dt}$.

**Solution**

(a) The water in the two pools change at the same absolute rate. But the large pool drains while the small pool fills. Hence $\frac{dV_L}{dt} = -\frac{dV_S}{dt}$.

(b) Water drains from the larger pool, whence $\frac{dh_L}{dt}$ is negative.

(c) We have $V_S = 144\pi h_S$, whence $\frac{dV_S}{dt} = 144\pi \frac{dh_S}{dt}$. Given that $\frac{dh_S}{dt} = 0.3$, we find $\frac{dV_S}{dt} = 28.8\pi$ m$^3$/min.

(d) We have $V_L = 400\pi h_L$, whence $\frac{dV_L}{dt} = 400\pi \frac{dh_L}{dt}$. Using parts (a) and (c), we have:

$$-28.8\pi = -\frac{dV_S}{dt} = \frac{dV_L}{dt} = 400\pi \frac{dh_L}{dt}$$

Hence $\frac{dh_L}{dt} = -0.072$ m/min.

Use the identity $4^2 + \sqrt{4} = 18$ and linear approximation to estimate $(3.81)^2 + \sqrt{3.81}$.

**Solution**

Put $f(x) = x^2 + \sqrt{x}$. Note that $f(4) = 18$ and $f'(4) = \left(2x + \frac{1}{2\sqrt{x}}\right)\bigg|_{x=16} = 8 + \frac{1}{4} = \frac{35}{4}$.

Hence the tangent line to $f$ at $x = 4$ is

$$y = 18 + \frac{35}{4}(x - 4)$$

Use the identity $4^2 + \sqrt{4} = 18$ and linear approximation to estimate $(3.81)^2 + \sqrt{3.81}$.
Since \( x = 3.81 \) is near the point of tangency \( (x = 4) \), we have
\[
(3.81)^2 + \sqrt{3.81} = f(3.81) \approx 18 + \frac{35}{4} (3.81 - 4) = 18 + \frac{35}{4} \cdot -\frac{19}{100} = \frac{1307}{80} = 16.3375
\]

15 pts 268. A manufacturer models the total cost (in dollars) of producing \( x \) items by the function \( C(x) = x^2 + 4x + 3 \), and the price per item (in dollars) is \( p(x) = \frac{98x + 49}{x + 3} \).

(a) Calculate the exact cost of producing the 5th item.
(b) Using marginal analysis, estimate the revenue derived from producing the 5th item.

Solution
(a) \( C(5) - C(4) = 48 - 35 = 13 \).

(b) The revenue is \( R(x) = xp(x) = \frac{98x^2 + 49x}{x + 3} \). Hence the desired marginal revenue is
\[
R'(4) = \left. \left( \frac{49(2x^2 + 12x + 3)}{(x + 3)^2} \right) \right|_{x=4} = 83
\]

20 pts 269. Consider the curve defined by
\[
\sqrt{xy} = ay^3 + b
\]
where \( a \) and \( b \) are unspecified constants. Suppose the equation of the tangent line to the curve at the point \((3, 3)\) is \( y = 3 + 4(x - 3) \).

(a) What is the value of \( \frac{dy}{dx} \) at \((3, 3)\)?
(b) Calculate \( a \) and \( b \).

Solution
(a) The slope of the tangent is line is 4, hence \( \frac{dy}{dx} = 4 \) at \((3, 3)\).

(b) We first use implicit differentiation on the equation of the curve.
\[
\frac{1}{2} (xy)^{-1/2} \cdot \left( x \frac{dy}{dx} + y \right) = 3ay^2 \cdot \frac{dy}{dx}
\]

We now substitute \( x = 3, y = 3, \) and \( \frac{dy}{dx} = 4 \), which gives us \( \frac{15}{6} = 108a \), whence \( a = \frac{5}{216} \). We now substitute \( x = 3, y = 3, \) and \( a = \frac{5}{216} \) into the equation for the curve, which gives us \( 3 = \frac{135}{216} + b \), whence \( b = \frac{19}{8} \).

15 pts 270. Suppose \( f'' \) is continuous. You are also given the following values:
\[
f \left( \frac{1}{8} \right) = 20, \quad f' \left( \frac{1}{8} \right) = -22
\]

Calculate the following limit.
\[
\lim_{x \to \frac{1}{8}} \left( \frac{20 - f \left( \frac{1}{x} \right)}{x^2 + x - 72} \right)
\]
Solution
Since \( f \) is continuous, we may substitute \( x = 8 \) to obtain the indeterminate form \( \frac{0}{0} \). So we may use L’Hospital’s Rule.

\[
\lim_{x \to 8} \left( \frac{20 - f\left( \frac{1}{x} \right)}{x^2 + x - 72} \right) = \lim_{x \to 8} \left( \frac{-f'(x^{-1}) \cdot (-x^{-2})}{2x + 1} \right)
\]

Since \( f' \) is continuous, we substitute \( x = 8 \), and we find the limit is \( \frac{-(22) \cdot (-8^{-2})}{17} = -\frac{11}{544} \).

20 pts 271. The length \( L \) (measured in meters) of a certain fish depends on time \( t \) (measured in years since birth) and is modeled by the function \( f \).

\[
L = f(t) = 4t^{2.99}
\]

The mass \( m \) (measured in kilograms) of the fish depends on the length \( L \) and is modeled by the function \( g \).

\[
m = g(L)
\]

The function \( g \) is not explicitly given.

(a) Describe in one English sentence, as precisely and specifically as you can, what the quantity \( Q = f(4) - f(0) \) represents in the context of this problem.

(b) Describe in one English sentence, as precisely and specifically as you can, what the quantity \( f'(1) \) represents in the context of this problem.

(c) What are the units of \( g'(4.87) \)?

(d) Suppose that \( \frac{dm}{dL} = 7 \) when \( L = 4 \). (Note that \( L = 4 \) when \( t = 1 \).) At what rate (measured in kg/yr) is the mass of the fish changing with respect to time exactly 1 year after its birth?

Solution
(a) The quantity \( Q \) is the length the fish grows in the first 4 years of its life.

(b) The quantity \( f'(1) \) is the rate at which the length of the fish is changing with respect to time exactly 1 year after its birth.

(c) The number \( g'(4.87) \) is the same as \( \frac{dm}{dL} \) evaluated at \( L = 4.87 \), whence \( g'(4.87) \) has units of kg/m (kilograms per meter).

(d) Note that \( m = g(L) = g(f(t)) \). By the chain rule, we have

\[
\frac{dm}{dt} = g'(f(t)) \cdot f'(t)
\]

We put \( t = 1 \) and note that \( f(1) = 4 \) and \( f'(1) = (11.96t^{1.99}) \bigg|_{t=1} = 11.96 \). Hence

\[
\frac{dm}{dt} \bigg|_{t=1} = g'(f(1)) \cdot f'(1) = g'(4) \cdot f'(1) = 7 \cdot 11.96 = 83.72
\]
Midterm Exam #4

16 pts 272. Consider the function below, where $A$ is an unspecified positive constant.

$$f(x) = \frac{A}{x - 8\sqrt{x} + 60}$$

For parts (c) and (d) only, assume the absolute minimum of $f$ on $[0, 21]$ is 8.

(a) List all $x$-values that must be tested to find the absolute extrema of $f$ on $[0, 21]$.
(b) At which $x$-value does the absolute minimum of $f$ occur on $[0, 21]$?
(c) Find the value of $A$.
(d) Find the absolute maximum of $f$ on $[0, 21]$ and all $x$-values at which it occurs.

Solution

(a) We must test the endpoints of the interval ($x = 0$ and $x = 21$), as well as any critical numbers. Note that $f$ is differentiable on $(0, 21)$, so the only critical numbers are solutions to $f'(x) = 0$.

$$f'(x) = \frac{-A \left(1 - \frac{4}{\sqrt{x}}\right)}{(x - 8\sqrt{x} + 60)^2}$$

Hence the only critical number (and only other number we must test) is $x = 16$.

(b) We test the $x$-values in part (a). Observe the following: $f(0) = \frac{A}{60}$, $f(16) = \frac{A}{44}$, and $f(21) = \frac{A}{81-8\sqrt{21}} \approx \frac{A}{44.3}$. Hence the minimum of $f$ on $[0, 21]$ occurs at $x = 0$.

(c) We are given that the minimum is 8, and so part (b) implies $f(0) = \frac{A}{60} = 8$. Hence $A = 480$.

(d) From part (b), the absolute maximum is $f(16) = \frac{A}{44} = \frac{480}{44} = \frac{120}{11}$ (occurring only at $x = 16$).

18 pts 273. An airline policy states that all baggage must be shaped like a rectangular box with the sum of the length, width, and height not exceeding 122 inches. You plan to purchase a bag from a company that makes customized bagged whose height must be 3 times its width. Find the dimensions of the baggage with the largest volume. (Let $L$, $W$, and $H$ be the length, width, and height of the baggage, respectively.)

(a) Before considering any constraints particular to this problem, find the objective function in terms of $L$, $W$, and $H$.

(b) There are two constraints for this problem. One constraint is from the airline and the other is from the baggage company. Find these constraints.

(c) Write the objective function in terms of $W$ only.

(d) Find the interval of interest for the objective function in part (c).

(e) Find the dimensions of the baggage with the largest volume.
Solution
(a) We seek the largest volume, whence the objective is \( F(L, W, H) = LWH \).
(b) The airline gives the constraint \( L + W + H = 122 \) and the baggage company gives the constraint \( H = 3W \).
(c) From part (b), we have \( L = 122 - W - H = 122 - 4W \), and so the objective in terms of \( W \) only is
\[
f(W) = f(122 - 4W, W, 3W) = 366W^2 - 12W^3
\]
(d) All measurements must be non-negative. So we must have \( L \geq 0 \) (equivalent to \( W \leq \frac{122}{4} = \frac{61}{2} \)), \( W \geq 0 \), and \( H \geq 0 \) (equivalent to \( W \geq 0 \)). Hence the interval of interest for \( W \) is \([0, \frac{61}{2}]\).
(e) Observe that \( f'(W) = 732W - 36W^2 = 12W(61 - 3W) \), hence the only critical number of \( f \) is \( W = \frac{61}{3} \). To verify this gives us a maximum volume, we note that \( f(0) = f(\frac{61}{3}) = 0 \) (testing endpoints). Since \( f(\frac{61}{3}) \) is clearly positive, we must have an absolute maximum of \( f \) on the interval at \( W = \frac{61}{3} \). The desired dimensions are thus:
\[
L = \frac{122}{3} , \quad W = \frac{61}{3} , \quad H = 61
\]

14 pts 274. Consider the function
\[
f''(x) = \frac{(x-2)^2(x-5)^3}{(x-9)^5}
\]
You may assume the domain of \( f \) is \((-\infty, 9) \cup (9, \infty)\).
(a) Find where \( f \) is concave up.
(b) Find where \( f \) is concave down.
(c) Find where \( f \) has an inflection point.

Solution
We determine a sign chat for \( f''(x) \). The cut points are \( x = 2 \), \( x = 5 \), and \( x = 9 \).

<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign</th>
<th>shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 2))</td>
<td>(f''(0) = \frac{\Theta}{\Theta} )</td>
<td>(\oplus)</td>
<td>concave up</td>
</tr>
<tr>
<td>((2, 5))</td>
<td>(f''(3) = \frac{\Theta}{\Theta} )</td>
<td>(\oplus)</td>
<td>concave up</td>
</tr>
<tr>
<td>((5, 9))</td>
<td>(f''(6) = \frac{\Theta}{\Theta} )</td>
<td>(\ominus)</td>
<td>concave down</td>
</tr>
<tr>
<td>((9, \infty))</td>
<td>(f''(10) = \frac{\Theta}{\Theta} )</td>
<td>(\oplus)</td>
<td>concave up</td>
</tr>
</tbody>
</table>

(a) \( f \) is concave up on \((-\infty, 5] \) and \((9, \infty)\).
(b) \( f \) is concave down on \([5, 9)\).
(c) $f$ has an inflection point only at $x = 5$. (Note that $x = 9$ is not in the domain of $f$.)

**14 pts 275.** A particle travels along the $x$-axis in such a way that its velocity (measured in ft/sec) at any time $t$ (measures in sec) is

$$v(t) = 4t^3 - 2t + 2$$

The particle is at $x = 3$ when $t = 2$.

(a) Find the position of the particle at any time $t$.
(b) Find the position of the particle at time $t = 4$.
(c) Find the acceleration of the particle when $t = 4$.

**Solution**

(a) To find the position, we find the antiderivative of $v(t)$ first.

$$x(t) = \int v(t) \, dt = \int (4t^3 - 2t + 2) \, dt = t^4 - t^2 + 2t + C$$

We are given $x = 3$ when $t = 2$, whence $3 = 16 - 4 + 4 + C$, and so $C = -13$. the position of the particle at any time $t$ is

$$x(t) = t^4 - t^2 + 2t - 13$$

(b) We have $x(4) = 256 - 16 + 8 - 13 = 235$.
(c) The acceleration is the derivative of velocity, so $a(4) = v'(4) = (12t^2 - 2)|_{t=4} = 190$.

**8 pts 276.** Use the graph of $y = f(x)$ on $[0, 14]$ below to answer the questions.

(a) List the critical points of $f$ in $(0, 14)$.
(b) How many local extrema does $f$ have on $(0, 14)$?
(c) Find the absolute maximum of $f$ and the $x$-value at which it occurs.
(d) Find the absolute minimum of $f$ and the $x$-value at which it occurs.

Solution
(a) Critical points are where $f' = 0$ or where $f'$ does not exist. The critical points are $x = 2$, $x = 8$, and $x = 12$.
(b) There are three local extrema (at the three critical points in part (a)).
(c) The absolute maximum of $f$ is 10 at $x = 2$.
(d) The absolute minimum of $f$ is $-7.3$ at $x = 14$.

Use the graph of $y = f'(x)$ below to answer the questions. Note that this is a graph of the first derivative of $f$, not a graph of $f$. You may assume that $f'(x)$ has a vertical asymptote at $x = 14$ and that the domain of $f$ is $(0, 14) \cup (14, 20)$.

(a) Find the critical points of $f$.
(b) Where is $f$ increasing?
(c) Where is $f$ decreasing?
(d) Where does $f$ have a local minimum?
(e) Where does $f$ have a local maximum?

Solution
(a) The critical points of $f$ are $x$-values in the interior of the domain of $f$ such that either $f' = 0$ or $f'$ does not exist. Hence the critical points of $f$ are $x = 5$, $x = 12$, and $x = 16$.
(b) We make a sign chart for $f'(x)$.
Hence \( f \) is increasing on \((0, 5)\) and \([16, 20)\).

(c) From part (b), we see \( f \) is decreasing on \([5, 14)\) and \((14, 16]\).

(d) \( x = 16 \): \( f' \) changes sign from negative to positive here

(e) \( x = 5 \): \( f' \) changes sign from positive to negative here

12 pts. The figure below shows the graphs of two functions. One function is \( f(x) \) and the other is \( f'(x) \), but you are not told which is which.

(a) Which graph is that of \( y = f(x) \)?

(b) Explain your answer to part (a) based on the behavior of the graphs at \( x = 4 \) only.

(c) Explain your answer to part (a) based on the behavior of the graphs near \( x = 3.5 \) only.

Solution

(a) The dashed orange curve is the graph of \( y = f(x) \).

(b) The dashed orange curve has a local maximum at \( x = 4 \), whereas the blue solid graph cross the \( x \)-axis from above to below (positive to negative values) at \( x = 4 \). This is consistent only if the dashed orange curve is the graph of \( y = f(x) \).

(c) The dashed orange curve is increasing (so its derivative should be positive) and concave down (so its derivative should be decreasing) at \( x = 3.5 \). This is consistent
only if the blue solid graph is, indeed, the graph of \( y = f'(x) \).
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Midterm Exam #1

14 pts 279. The graph of \( y = f(x) \) is given below. (Note that \( f \) is piecewise linear.)

An explicit formula for \( f(x) \) can be written in the following form.

\[
f(x) = \begin{cases} 
  y_1(x) & \text{if } -8 \leq x < A \\
  y_2(x) & \text{if } B \leq x \leq 8 
\end{cases}
\]

Calculate each of \( A \), \( B \), \( y_1(x) \), and \( y_2(x) \).

Solution
We see that the graph of \( f \) consists of two line segments, one valid for \(-8 \leq x < 6\) (hence \( A = 6 \)) and the other valid for \( 6 \leq x \leq 8 \) (hence \( B = 6 \)). Using point-slope form, we find that the equations of these line segments are given by the following:

\[
y_1(x) = 1 + \frac{-6 - 1}{6 - (-8)}(x - (-8)) = 1 - \frac{1}{2}(x + 8)
\]

\[
y_2(x) = -6 + \frac{-6 - 3}{8 - 6}(x - 8) = -6 - \frac{9}{2}(x - 8)
\]

10 pts 280. Use the graph of \( y = f(x) \) below to answer the following questions.
(a) Calculate $f(f(2))$.
(b) State the domain of $f$ in interval notation.
(c) State the range of $f$ in interval notation.

**Solution**

(a) Since $f$ is piecewise linear, we can use point-slope form to find an equation for $f$ valid for $0 \leq x < 3$.

$$f(x) = 2 + \frac{0 - 2}{3 - 0}(x - 0) = 2 - \frac{2}{3}x$$

Hence we find $f(2) = 2 - \frac{2}{3} \cdot 2 = \frac{2}{3}$, whence $f(f(2)) = f(\frac{2}{3}) = 2 - \frac{2}{3} \cdot 23 = \frac{14}{9}$.

(b) The domain of $f$ is $[0, 5)$.

(c) The range of $f$ is $(0, 5)$.

---

**9 pts 281.** Suppose $\log_3(x) = A$ and $\log_3(y) = B$. Rewrite the expression below in terms of $A$ and $B$. Your final answer may not contain any logarithms.

$$\log_3 \left( \frac{27\sqrt{x}}{y^4} \right)$$

**Solution**

We have the following:

$$\log_3 \left( \frac{27\sqrt{x}}{y^4} \right) = \log_3(27) + \log_3(\sqrt{x}) - \log_3(y^4) = 3 + \frac{1}{2} \log_3(x) - 4 \log_3(y) = 3 + \frac{1}{2}A - 4B$$

---

**9 pts 282.** Rewrite the expression below as a single logarithm. Assume $x$ and $y$ are positive.

$$\frac{1}{2} (\log_5(x) - 7 \log_5(y)) + 3 \log_5(x - 1)$$
Solution
We have the following:
\[ \frac{1}{2} (\log_5(x) - 7 \log_5(y)) + 3 \log_5(x - 1) = \frac{1}{2} \log_5 \left( \frac{x}{y^7} \right) + \log_5 \left( (x - 1)^3 \right) \]
\[ = \log_5 \left( \frac{x^{1/2}}{y^{7/2}} \right) + \log_5 \left( (x - 1)^3 \right) = \log_5 \left( \frac{x^{1/2}(x - 1)^3}{y^{7/2}} \right) \]

11 pts 283. Suppose \( \cos(\theta) = A/7 \) with \( 0 < A < 7 \) and \( \sin(\theta) < 0 \). Calculate \( \sec(\theta) \), \( \sin(\theta) \), and \( \tan(\theta) \) in terms of \( A \).

Solution
By definition of secant,
\[ \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{7}{A} \]
Using the Pythagorean identity \( \cos^2(\theta) + \sin^2(\theta) = 1 \) and recalling that \( \sin(\theta) < 0 \), we have
\[ \sin(\theta) = -\sqrt{1 - \cos^2(\theta)} = -\sqrt{1 - \frac{A^2}{49}} = -\frac{\sqrt{49 - A^2}}{7} \]
Again by definition of tangent,
\[ \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = -\frac{\sqrt{1 - \frac{A^2}{49}}}{A/7} = -\frac{\sqrt{49 - A^2}}{A} \]

11 pts 284. A bacteria colony has an initial population of 3500. The population grows exponentially and triples every 7 hours. Recall that this means the population \( P \) at time \( t \) satisfies \( P(t) = P_0 e^{kt} \) for some constants \( A \) and \( k \).

(a) Find the exact value of the growth constant \( k \).
(b) Find the population after 25 hours. Round your answer to the nearest whole number.
(c) Find the time (in hours) when the population will be 12,600. Round your answer to 2 decimal places.

Solution
(a) We are given that \( P(7) = 3P(0) \), or \( e^{7k} = 3 \). Hence \( k = \frac{1}{7} \ln(3) \).
(b) \( P(25) = 3500e^{25k} = 3500 \cdot 3^{25/7} \approx 177040 \).
(c) We have to solve the equation \( 12600 = 3500e^{kt} \) for \( t \). Dividing by 3500 and taking logarithms gives \( t = 7 \cdot \frac{\ln(18/5)}{\ln(3)} \approx 8.16 \).
A rectangular box is constructed according to the following rules.

- the length of the box is twice its width
- the height of the box is 5 feet more than three times the length

Let $\ell$, $w$, and $h$ denote the length, width, and height of the box, respectively, measured in feet.

(a) Write the height of the box in terms of $w$.
(b) Write an expression for $V(w)$, the volume of the box measured in cubic feet, as a function of its width.
(c) Suppose the rules also require that the sum of the box’s width and height to be less than 26 feet. Under this condition, what is the domain of the function $V(w)$?

**Solution**

(a) The first condition gives $\ell = 2w$, and the second condition gives $h = 3\ell + 5$. Hence $h = 3(2w) + 5 = 6w + 5$.

(b) The volume of the box is $V(w) = \ell \cdot w \cdot h = 2w \cdot w \cdot (6w + 5)$.

(c) We are given that $w + h < 26$, or $w + 6w + 5 < 26$. Solving for $w$ gives $w < 3$. Since width must also be non-negative, we find that the domain of $V(w)$ is $0 \leq w < 3$, or $w \in [0, 3)$ in interval notation.

**286.** Let $f(x) = \frac{2}{3x}$ and assume $h \neq 0$. Fully simplify each of the following expressions:

(a) $f(x + h)$
(b) $f(x + h) - f(x)$
(c) $\frac{f(x + h) - f(x)}{h}$

**Solution**

(a) $f(x + h) = \frac{2}{3(x + h)}$

(b) $f(x + h) - f(x) = \frac{2}{3(x + h)} - \frac{2}{3x}$

(c) We have the following.

$$\frac{f(x + h) - f(x)}{h} = \frac{\frac{2}{3(x + h)} - \frac{2}{3x}}{h} = \frac{2x - 2(x + h)}{3hx(x + h)} = \frac{-2h}{3hx(x + h)} = \frac{-2}{3x(x + h)}$$

**287.** Find the domain of the function $f(x) = \sqrt{x^2 + x - 6} + \ln(10 - x)$. Write your answer using interval notation.
Solution
We examine the square root and the logarithm separately.

The argument of the square root cannot be negative, hence we must have \( x^2 + x - 6 \geq 0 \). This is equivalent to \((x + 3)(x - 2) \geq 0\). To solve this inequality, we construct a sign chart and test each of the intervals \((-\infty, -3)\), \((-3, 2)\), and \((2, \infty)\). We find that the solution to the inequality is \((-\infty, -3]\cup[2, \infty)\).

The argument of the logarithm cannot be negative or zero, hence we must have \(10 - x > 0\), or \(x < 10\) (or \((-\infty, 10)\) in interval notation).

The domain of \(f\) is the intersection of the solutions to these two inequalities.

\((-\infty, -3] \cup [2, 10)\)
Midterm Exam #2

10 pts 288. Use the graph of \( y = f(x) \) below to answer the questions below. All numbers must be exact.

(a) List the \( x \)-values where \( f \) is not continuous or determine that \( f \) is continuous for all \( x \).
(b) List all vertical asymptotes of \( f \).
(c) List all horizontal asymptotes of \( f \).
(d) Calculate \( \lim_{x \to 8} f(x) \) or determine that the limit does not exist.
(e) At \( x = 7 \), which of the one-sided limits of \( f \) exist? Select the best answer.

(i) the left-hand limit exists, but not the right-hand limit
(ii) the right-hand limit exists, but not the left-hand limit
(iii) both one-sided limits exist
(iv) neither one-sided limit exists

Solution

(a) \( x = 0, 7, 8 \) only
(b) \( x = 0 \) only
(c) \( y = 3 \) only
(d) \( \lim_{x \to 8} f(x) = -1 \)
(e) Choice (iii). Note that \( \lim_{x \to 7^-} f(x) = 2 \) and \( \lim_{x \to 7^+} f(x) = 1 \).

20 pts 289. Consider the piecewise-defined function \( f(x) \) below; \( A \) and \( B \) are unspecified constants and \( g(x) \) is an unspecified function with domain \( [94, \infty) \).

\[
f(x) = \begin{cases} 
Ax^2 + 8 & x < 75 \\
\ln(B) + 6 & x = 75 \\
\frac{x - 75}{\sqrt{x + 6} - 9} & 75 < x < 94 \\
19 & x = 94 \\
g(x) & x > 94
\end{cases}
\]
(a) Find \( \lim_{x \to 75^-} f(x) \). Your answer may be a number or an expression containing \( A \) or \( B \).

(b) Find \( \lim_{x \to 75^+} f(x) \). Your answer may be a number or an expression containing \( A \) or \( B \).

(c) Find the exact values of \( A \) and \( B \) for which \( f \) is continuous at \( x = 75 \).

(d) Suppose \( g(94) = 19 \). What does this imply about \( \lim_{x \to 94} f(x) \)? Select the best answer.

(i) \( \lim_{x \to 94} f(x) \) exists.

(ii) \( \lim_{x \to 94} f(x) \) does not exist.

(iii) It gives no information about \( \lim_{x \to 94} f(x) \).

\[ \text{Solution} \]

(a) \( \lim_{x \to 75^-} f(x) = \lim_{x \to 75^-} (Ax^2 + 8) = A \cdot 75^2 + 8 = 5625A + 8 \)

(b) We have the following:

\[
\lim_{x \to 75^+} f(x) = \lim_{x \to 75^+} \left( \frac{x - 75}{\sqrt{x + 6} - 9} \right) = \lim_{x \to 75^+} \left( \frac{x - 75}{x + 6 - 9} \cdot \frac{\sqrt{x + 6} + 9}{\sqrt{x + 6} + 9} \right) \\
= \lim_{x \to 75^+} \left( \frac{(x - 75)(\sqrt{x + 6} + 9)}{x + 6 - 81} \right) = \lim_{x \to 75^+} \left( \frac{\sqrt{x + 6} + 9}{x + 6 - 81} \right) \\
= \sqrt{81} + 9 = 18
\]

(c) We need \( 5625A + 8 = 18 = f(75) = \ln(B) + 6 \), whence \( A = 10/5625 \) and \( B = e^{12} \).

(d) Choice (iii). Note that \( \lim_{x \to 94^-} f(x) = \lim_{x \to 94^-} \left( \frac{x - 75}{\sqrt{x + 6} - 9} \right) = 19 \). So for \( \lim_{x \to 94^-} f(x) \) to exist, we require only that \( 19 = \lim_{x \to 94^-} f(x) = \lim_{x \to 94^+} g(x) \). However, we are given no information at all about this right-limit of \( g \) since the function value \( g(94) \) is irrelevant to its value.

\[ \text{8 pts 290.} \] The position of a particle (measured in feet) after \( t \) seconds is modeled by the following function.

\[ h(t) = -16t^2 + 96t + 100 \]

(a) Calculate the average velocity of the particle (in feet per second) between \( t = 4 \) and \( t = 5 \).

(b) Find an equation of the secant line between \((4, h(4))\) and \((5, h(5))\).

\[ \text{Solution} \]

(a) \( v = \frac{\Delta h}{\Delta t} = \frac{h(5) - h(4)}{5 - 4} = \frac{-16(25 - 16) + 96(5 - 4)}{1} = -48 \)

(b) The slope of the secant line is \(-48\) and the secant line passes through \((4, h(4)) = (4, 228)\). Hence an equation of the secant line is \( y = 228 - 48(t - 4) \).
291. Suppose $\lim_{x \to 6} |f(x)| = 2$. Which of the following statements must be true about $\lim_{x \to 6} f(x)$?

(i) $\lim_{x \to 6} f(x)$ does not exist.
(ii) $\lim_{x \to 6} f(x) = 2$.
(iii) $\lim_{x \to 6} f(x)$ exists and is equal to either 2 or $-2$, but there is not enough information to determine which of these possibilities must be true.
(iv) There is not enough information about $f(x)$ to determine whether $\lim_{x \to 6} f(x)$ exists.
(v) $\lim_{x \to 6} f(x) = -2$.

Solution

Choice (iv). Consider the following two examples, both of which satisfy the hypothesis $\lim_{x \to 6} |f(x)| = 2$.

- $f(x) = 2$. Then $\lim_{x \to 6} f(x)$ exists and is equal to 2.
- $f(x) = 2$ for $x < 6$ and $f(x) = -2$ for $x \geq 6$. Then $\lim_{x \to 6} f(x)$ does not exist (the left- and right-limits at $x = 6$ are not equal).

Thus it is not possible to determine whether $\lim_{x \to 6} f(x)$ exists.

292. Consider the following function, where $k$ is an unspecified constant.

$$f(x) = \frac{4x^2 - kx}{x^2 + 12x + 32}$$

(a) Find the value of $k$ for which $\lim_{x \to 4} f(x)$ exists.

(b) For the value of $k$ described in part (a), evaluate $\lim_{x \to 4} f(x)$.

Solution

(a) Direct substitution of $x = -4$ into $f(x)$ gives the undefined expression $\frac{64+4k}{0}$. If the number $64 + 4k$ were non-zero, then we would conclude there is a vertical asymptote for $f$ at $x = -4$. However, since $\lim_{x \to -4} f(x)$ exists, we must have $64 + 4k = 0$, whence $k = -16$.

(b) With $k = -16$, we have the following.

$$\lim_{x \to -4} \left( \frac{4x^2 + 16x}{x^2 + 12x + 32} \right) = \lim_{x \to -4} \left( \frac{4x(x + 4)}{(x + 8)(x + 4)} \right) = \lim_{x \to -4} \left( \frac{4x}{x + 8} \right) = -4$$
4 pts 293. Suppose $\lim_{x \to 0} \left( \frac{f(x)}{x} \right) = 8$. Calculate $\lim_{x \to 0} \left( \frac{f(x)}{\sin(6x)} \right)$, if it exists. Select the best answer below.

(i) The limit is either $\infty$ or $-\infty$.
(ii) Not enough information is given to determine whether the limit exists.
(iii) The limit does not exist and is neither $\infty$ nor $-\infty$.
(iv) The limit exists. If you select this answer, also indicate the value of the limit.

Solution
Choice (iv). We have the following:

$$\lim_{x \to 0} \left( \frac{f(x)}{\sin(6x)} \right) = \lim_{x \to 0} \left( \frac{1}{6} \cdot \frac{f(x)}{x} \cdot \frac{6x}{\sin(6x)} \right) = \frac{1}{6} \cdot 8 \cdot 1 = \frac{4}{3}$$

12 pts 294. Consider the following function.

$$f(x) = \frac{x^2 - x - 6}{x^3 - 2x^2 - 3x}$$

(a) Where is $f$ discontinuous?

(b) At the leftmost $x$-value where $f$ is discontinuous, what type of discontinuity does $f$ have?

(i) removable discontinuity
(ii) jump discontinuity
(iii) infinite discontinuity (vertical asymptote)
(iv) other type of discontinuity

(c) At the rightmost $x$-value where $f$ is discontinuous, what type of discontinuity does $f$ have?

(i) removable discontinuity
(ii) jump discontinuity
(iii) infinite discontinuity (vertical asymptote)
(iv) other type of discontinuity

Solution
First we note the following:

$$f(x) = \frac{x^2 - x - 6}{x^3 - 2x^2 - 3x} = \frac{(x + 2)(x - 3)}{x(x + 1)(x - 3)}$$

(a) $f$ is continuous on its domain, hence discontinuous at $x = -1, 0, 3$ only.

(b) Choice (iii). Direct substitution of $x = -1$ into $f(x)$ gives the undefined expression $\frac{-6}{0}$, indicating a vertical asymptote at $x = -1$.

(c) Choice (i). We see that $\lim_{x \to -3} f(x) = \lim_{x \to -3} \left( \frac{x + 2}{x(x + 1)} \right) = \frac{5}{12}$. Since this limit exists, $f$
Answer the questions below for the following function. All numbers you give must be exact.

\[ f(x) = \frac{8 + 6e^x}{9e^x - \pi^6} \]

(a) Evaluate \( \lim_{x \to \infty} f(x) \).

(b) Evaluate \( \lim_{x \to -\infty} f(x) \).

(c) List all vertical asymptotes of \( f \).

Solution

(a) Divide each term by \( e^x \) and recall that \( \lim_{x \to \infty} e^{-x} = 0 \).

\[
\lim_{x \to \infty} \left( \frac{8 + 6e^x}{9e^x - \pi^6} \right) = \lim_{x \to \infty} \left( \frac{8e^{-x} + 6}{9 - \pi^6 e^{-x}} \right) = \frac{0 + 6}{9 - 0} = \frac{2}{3}
\]

(b) Recall that \( \lim_{x \to -\infty} e^x = 0 \).

\[
\lim_{x \to -\infty} \left( \frac{8 + 6e^x}{9e^x - \pi^6} \right) = \frac{8 + 0}{0 - \pi^6} = -\frac{8}{\pi^6}
\]

(c) The denominator vanishes if \( x = \ln(\pi^6/9) \), and the numerator does not vanish at this \( x \)-value. Hence the only vertical asymptote of \( f \) is the line \( x = \ln(\pi^6/9) \).
Midterm Exam #3

9 pts 296. The following limit represents the derivative of a function $f$ at a point $a$.

$$f'(a) = \lim_{h \to 0} \left( \frac{9 \tan \left( \frac{\pi}{6} + h \right) - \frac{9}{\sqrt{3}}}{h} \right)$$

(a) Find a possible pair for $f$ and $a$.
(b) Calculate the value of the limit.

Solution
(a) Recall that the definition of the derivative is:

$$f'(a) = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{h} \right)$$

Let $f(x) = 9 \tan(x)$ and let $a = \frac{\pi}{6}$. Then the given limit is $f'(a)$.

(b) Observe that $f'(x) = 9 \sec^2(x)$, and so the given limit is $9 \sec^2\left( \frac{\pi}{6} \right) = 9 \cdot \frac{4}{3} = 12$.

12 pts 297. Use the graph of $f$ below to determine the sign of each of the indicated values.

(a) $f'(1)$  (b) $f'(2)$  (c) $f'(3.5)$  (d) $f'(7)$

Solution
(a) zero
(b) $f'(2)$ does not exist (the graph of $f$ has a sharp corner at $x = 2$)
(c) negative
(d) $f'(7)$ does not exist ($f$ is not continuous at $x = 7$)

9 pts 298. Let $f(x) = x^9 e^{4x}$.

(a) Find $f'(x)$.
(b) Explain how to find where the tangent line to the graph of $f$ is horizontal.
(c) Find where the graph of $f$ has a horizontal tangent line.
Solution
(a) Use product rule and chain rule.
\[ f'(x) = 9x^8 e^{4x} + x^9 \cdot 4e^{4x} = x^8 e^{4x}(9 + 4x) \]

(b) We must solve the equation \( f'(x) = 0 \) for \( x \).
(c) The solutions to \( f'(x) = 0 \) are \( x = 0 \) and \( x = -\frac{9}{4} \), thus these are the \( x \)-values where \( f \) has a horizontal tangent line.

9 pts 299. Selected values of the functions \( f \) and \( g \) and their derivatives are given in the table below. Use these values to complete the questions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>-4</td>
<td>-1</td>
<td>-9</td>
<td>-3</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( g'(x) )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

(a) Suppose \( h(x) = 5f(x) - 8g(x) \). Find \( h'(1) \).
(b) Suppose \( p(x) = x^2f(x) \). Find \( p'(2) \).
(c) Suppose \( q(x) = f(x^2) \). Find \( q'(2) \).

Solution
(a) We have \( h'(x) = 5f'(x) - 8g'(x) \). Thus
\[ h'(1) = 5f'(1) - 8g'(1) = 5 \cdot (-4) - 8 \cdot 1 = -28 \]

(b) By product rule we have \( p'(x) = 2xf(x) + x^2f'(x) \). Thus
\[ p'(2) = 2 \cdot 2 \cdot f(2) + 4 \cdot f'(2) = 4 \cdot 3 + 4 \cdot (-1) = 8 \]

(c) By chain rule we have \( q'(x) = f'(x^2) \cdot 2x \). Thus
\[ q'(2) = f'(4) \cdot 2 \cdot 2 = (-3) \cdot 4 = -12 \]

12 pts 300. Let \( f(x) \) and \( g(x) \) be functions such that \( f'(-8) = g'(-8) \) and the line tangent to the graph of \( f \) at \( x = -8 \) is \( y = -7x + 6 \). For each part, compute the desired value, if possible.

(a) \( f(-8) \) \hspace{1cm} (b) \( f'(-8) \) \hspace{1cm} (c) \( g(-8) \) \hspace{1cm} (d) \( g'(-8) \)

Solution
(a) The tangent line to \( f \) at a point passes through the graph of \( f \) at the point of tangency. So \( f(-8) \) is equal to the \( y \)-coordinate of the tangent line at \( x = -8 \). Thus
\[ f(-8) = -7 \cdot (-8) + 6 = 62 \]

(b) The slope of the tangent line to \( f \) is the derivative of \( f \) at the point of tangency. Hence \( f'(-8) \) is \(-7\), the slope of the line \( y = -7x + 6 \).
(c) We are not given enough information to determine \( g(8) \). (In particular, the slope of the tangent line to \( g \) at \( x = -8 \) is \(-7\) also, but the \( y \)-intercept need not be 6. In other words, the point of tangency need not be the same for both \( f \) and \( g \).)

(d) We are given that \( f'(-8) = g'(-8) \), whence \( g'(-8) = -7 \).

15 pts  301. Consider the curve defined by the following equation.

\[
Ax^2 - 8xy = B \cos(y) + 3
\]

where \( A \) and \( B \) are unspecified constants.

(a) Find a formula for \( \frac{dy}{dx} \).

(b) Suppose the point \((8, 0)\) is on the curve. Find an equation that \( A \) and \( B \) must satisfy.

(c) Suppose the tangent line to the curve at the point \((8, 0)\) is \( y = 6x - 48 \). Find the values of \( A \) and \( B \).

Solution

(a) Using implicit differentiation, we obtain:

\[
2Ax - 8y - 8x \frac{dy}{dx} = -B \sin(y) \frac{dy}{dx}
\]

Solving for \( \frac{dy}{dx} \) gives:

\[
\frac{dy}{dx} = \frac{2Ax - 8y}{8x - B \sin(y)}
\]

(b) The point \((8, 0)\) must satisfy the equation that defines the curve, whence:

\[
64A = B + 3
\]

(c) We have that \( \frac{dy}{dx} = 6 \) (the slope of the tangent line) when \( x = 8 \) and \( y = 0 \). Hence by part (a) we have:

\[
7 = \frac{16A - 0}{64 - 0} = \frac{A}{4}
\]

Hence \( A = 28 \). From part (b) we then have \( B = 64A - 3 = 1533 \).

12 pts  302. The base of a right triangle is decreasing at a constant rate of 10 cm/sec and in such a way that the triangle always remains a right triangle. At the time when the base is 15 cm and the height is 22 cm, the area of the triangle is increasing by 25 cm\(^2\)/sec. Use this information to answer the questions below. Let \( B \) denote the base of the triangle.

(a) At the described time, what is the sign of \( \frac{dB}{dt} \)?

(b) At the described time, what is the sign of \( \frac{d^2B}{dt^2} \)?

(c) At the described time, at what rate is the height changing?

(d) What are the units of the answer to part (c)?
Solution

(a) We are given that the base is decreasing at the given time, so $\frac{dB}{dt}$ is negative.

(b) We are given that $\frac{dB}{dt}$, the rate at which the base is changing, is constant. Thus $\frac{d^2B}{dt^2}$ is zero.

(c) At any time we have $A = \frac{1}{2}BH$, where $A$, $B$, and $H$ are the area, base, and height of the triangle, respectively. Differentiating with respect to time gives us a total of two equations that hold for any time.

\[
A = \frac{1}{2}BH
\]
\[
\frac{dA}{dt} = \frac{1}{2} \frac{dB}{dt} H + \frac{1}{2} B \frac{dH}{dt}
\]

At the given time, we have: $\frac{dB}{dt} = -10$, $B = 15$, $H = 22$, and $\frac{dA}{dt} = 25$. Substituting this information into the previous two equations gives us two equations that hold only at the described time.

\[
A = 165
\]
\[
25 = -110 + 7.5 \frac{dH}{dt}
\]

Solving for $\frac{dH}{dt}$ gives $\frac{dH}{dt} = 18$.

(d) The units of $\frac{dH}{dt}$ are cm/sec.

12 pts 303. Suppose $f$ is differentiable at $x$ and $g(x) = \frac{16\ln(15x)}{6f(x) - \sqrt{x + 17}}$. Find $g'(x)$.

Solution

We start with quotient rule since the expression for $g(x)$ is a quotient. When we differentiate the numerator we must use chain rule.

\[
g'(x) = \frac{(16 \cdot \frac{1}{15x} \cdot 15) \cdot (6f(x) - \sqrt{x + 17}) - (16\ln(15x) \cdot (6f'(x) - \frac{1}{2\sqrt{x + 17}}))}{(6f(x) - \sqrt{x + 17})^2}
\]