Sections 2.1/2.2: Introduction to Limits

Motivation: average vs. instantaneous velocity

$x(t)$: position of particle

$v(t)$: (instantaneous) velocity of particle

Average velocity over time interval $[t_1, t_2]$

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

Ex. 1

Suppose $x(t) = 16t^2$. Estimate the instantaneous velocity at $t = 2$.

Solution:

We can estimate the velocity at $t=2$ with the average velocity over $[2, 2.1]$.

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(2.1) - x(2)}{2.1 - 2} = \frac{16(2.1)^2 - 16(2)^2}{2.1 - 2} = 65.6$$

We get a better estimate by using a smaller interval.

Interval	Δt	$\bar{v}$
$[2, 2.1]$	0.1	65.6
$[2, 2.01]$	0.01	64.16
$[2, 2.001]$	0.001	64.016
$[1.9, 2]$	0.1	62.4

what is relationship between these two columns of #'s?

"As Δt gets smaller (closer to 0), $\bar{v}$ gets closer to 64."

$[1.99, 2]$	0.01	63.84
$[1.999, 2]$	0.001	63.984

We estimate $v(2) = 64$.

"As Δt gets smaller (closer to 0), $\bar{v}$ gets closer to 64."

This intuition is written symbolically as:

$$\boxed{\lim_{\Delta t \to 0} (\bar{v}) = 64}$$

"limit of $\bar{v}$ as Δt goes to 0 is 64"

Q: How can we calculate $v(2)$ exactly?

A: Use $\bar{v}$ as an estimate, but calculate $\bar{v}$ symbolically in terms of Δt.

$\boxed{\text{Ex.1 cont.}}$

Let $\underbrace{h > 0}$. The average velocity over $\underbrace{[2, 2+h]}_{\text{length} = h}$

(Think of h as a tiny number)

Recall: $x(t) = 16t^2$

$$\bar{v} = \frac{x_{final} - x_{initial}}{t_{final} - t_{initial}} = \frac{x(2+h) - x(2)}{2+h-2}$$

$$= \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(h^2+4h+4) - 64}{h}$$

$$= \frac{16h^2 + 64h \cancel{+64} \cancel{-64}}{h} = \frac{\cancel{h}(16h+64)}{\cancel{h}} = \boxed{16h + 64}$$

So as h gets closer to 0, $\bar{v}$ gets closer to 64. (Why? Plugging in $h=0$ into $16h+64$ gives 64.)

General Definition of Limit

$$\boxed{\lim_{x\to a} f(x) = L}$$

"limit of $f(x)$ as x goes to a is L."

This means the values of $f(x)$ can be made aribitrarily close to L as long as we choose x-values arbitrarily close to a.

Ex. 2

Use a table of values to estimate the limit:

$$\lim_{x\to 3}\left(\frac{x^2-2x-3}{x-3}\right)$$

Solution:

x	$y=f(x)=\dfrac{x^2-2x-3}{x-3}$
2.5	3.5
2.9	3.9
2.99	3.99
3	error
3.01	4.01
3.1	4.1
3.5	4.5

$x \to 3$

$y \to 4$

This suggests the limit is 4.

Ex. 3

Use a graph to estimate the limit.

$$\lim_{x \to 3} \left(\frac{x^2 - 2x - 3}{x - 3} \right)$$

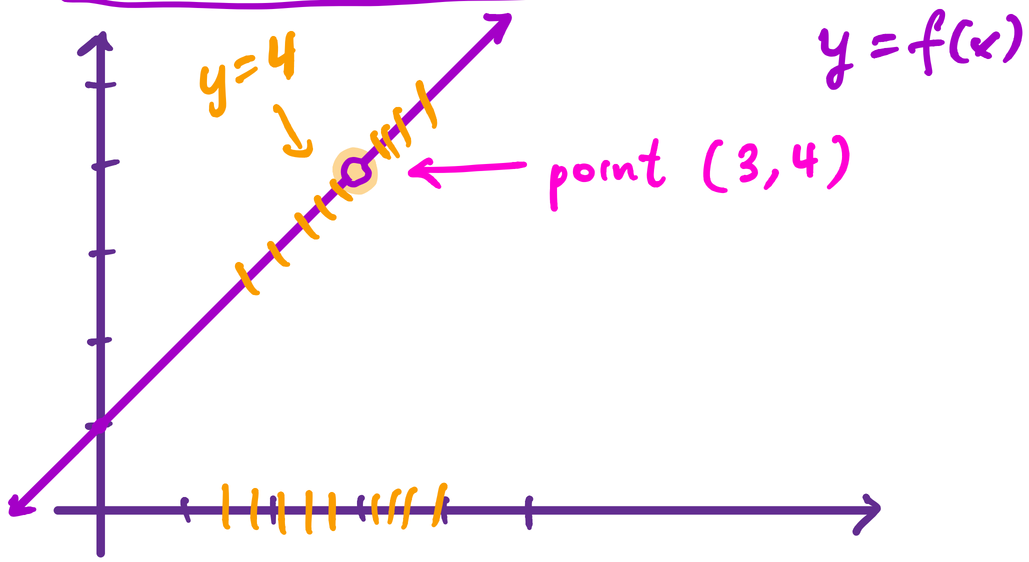
Solution:

$$f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x - 3} & \text{if } x \neq 3 \\ \text{undefined} & \text{if } x = 3 \end{cases}$$

$$= \frac{\cancel{(x-3)}(x+1)}{\cancel{x-3}} = x + 1$$

So $f(x)$ can be written as:

$$\boxed{f(x) = \begin{cases} x + 1 & \text{if } x \neq 3 \\ \text{undefined} & \text{if } x = 3 \end{cases}}$$



Graph suggests $\lim_{x \to 3} f(x) = 4$.

Ex. 4

Calculate the limit using algebra.

$$\lim_{x \to 3} \left(\frac{x^2 - 2x - 3}{x - 3} \right)$$

Solution:

$$\lim_{x\to 3}\left(\frac{x^2-2x-3}{x-3}\right) = \lim_{x\to 3}\left(\frac{(x-3)(x+1)}{x-3}\right) = \lim_{x\to 3}(x+1) = 4$$

algebra (factor) cancel $(x-3)$ (why?) plug in $x=3$

Note: $\frac{x-3}{x-3} \neq 1$ in general (what if $x=3$?)

One-sided Limits

Compare to General Definition of Limit

- Left-sided limits

$$\boxed{\lim_{x\to a^-} f(x) = L}$$

"limit of $f(x)$ as x goes to a from the left is L."

This means the values of $f(x)$ can be made aribitrarily close to L as long as we choose x-values arbitrarily close to a... AND $x < a$.

- Right-sided limits

$$\boxed{\lim_{x\to a^+} f(x) = L}$$

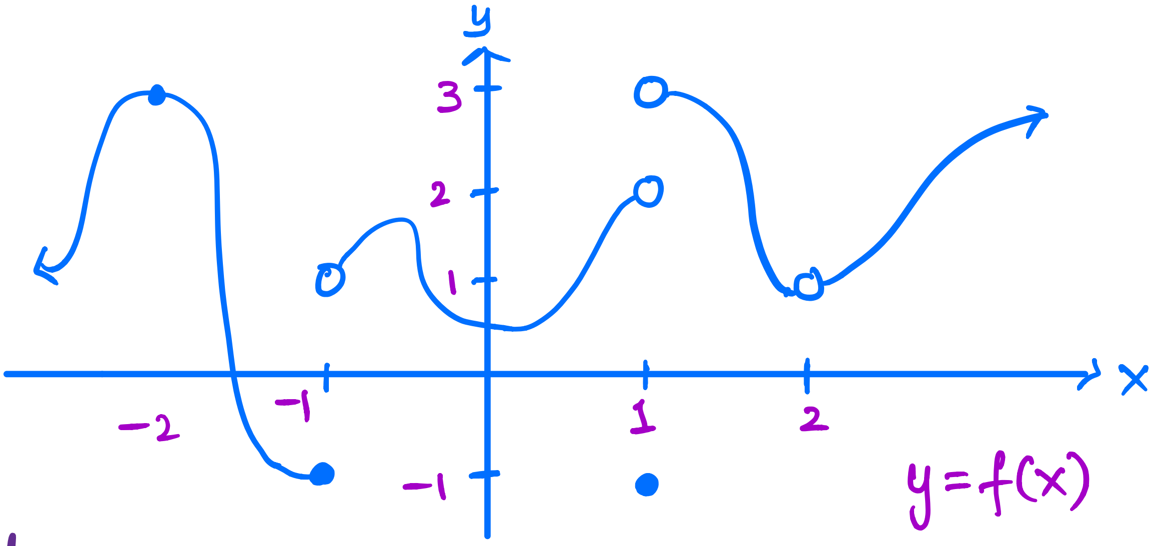
"limit of $f(x)$ as x goes to a from the right is L."

This means the values of $f(x)$ can be made aribitrarily close to L as long as we choose x-values arbitrarily close to a... AND $x > a$.

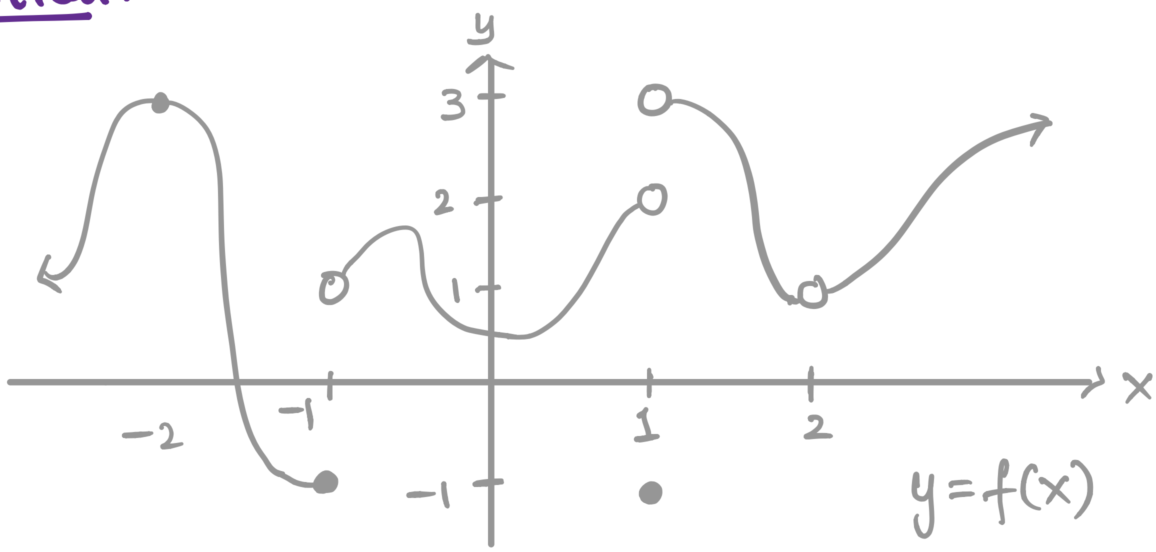
$\boxed{\text{Ex. 5}}$

For $a = -2, -1, 1, 2$: use the graph to calculate

(a) $\lim_{x \to a^-} f(x)$ (b) $\lim_{x \to a^+} f(x)$ (c) $\lim_{x \to a} f(x)$ (d) $f(a)$



Solution:



a	$\lim_{x \to a^-} f(x)$	$\lim_{x \to a^+} f(x)$	$\lim_{x \to a} f(x)$	$f(a)$
-2	3	3	3	3
-1	-1	1	DNE	-1
1	2	3	DNE	-1
2	1	1	1	DNE

* If left and right limits are different, then the two-sided limit does not exist (DNE).

Section 2.3/3.5: Calculating Limits

Direct Substitution Property (DSP)

If $\lim_{x \to c} f(x) = f(c)$, then f has the DSP at $x=c$.

What are some functions with the DSP?

1. polynomials $(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n)$
2. rational
3. algebraic
4. trigonometric
5. exponential $(e^x, 2^x, \ldots)$
6. logarithmic

These functions have the DSP only on their domain

Ex. 1

Calculate $\lim_{x \to 3} \left(\frac{x^2 + x - 12}{x - 3} \right)$.

Solution:

The function $f(x) = \frac{x^2+x-12}{x-3}$ does not have DSP at $x=3$. So we use algebra to transform $f(x)$ into a function with the DSP.

$$\lim_{x \to 3} \left(\frac{x^2 + x - 12}{x - 3} \right) = \lim_{x \to 3} \left(\frac{\cancel{(x-3)}(x+4)}{\cancel{x-3}} \right) = \lim_{x \to 3} (x+4) = 3+4=7$$

limit symbol tells you $x \neq 3$, so cancelation is okay

$x+4$ is a polynomial, so it has the

Note: $\frac{x^2+x-12}{x-3} \neq x+4$ (why? what if $x=3$?)

Ex. 2

Calculate $\lim_{x \to 2} \frac{\sqrt{2} - \sqrt{x}}{4 - x^2}$.

Solution:

Note: D.S. of $x=2$ gives "$\frac{0}{0}$", which is undefined.

① this does not necessarily mean the limit DNE.

② "$\frac{0}{0}$" suggests algebraic cancelation

$$\lim_{x\to 2}\left(\frac{\sqrt{2}-\sqrt{x}}{4-x^2}\cdot\frac{\sqrt{2}+\sqrt{x}}{\sqrt{2}+\sqrt{x}}\right)=\lim_{x\to 2}\left(\frac{2-x}{\underbrace{(4-x^2)(\sqrt{2}+\sqrt{x})}_{\text{keep factored}}}\right)$$

$$=\lim_{x\to 2}\left(\frac{\cancel{2-x}}{\cancel{(2-x)}(2+x)(\sqrt{2}+\sqrt{x})}\right)=\lim_{x\to 2}\left(\frac{1}{(2+x)(\sqrt{2}+\sqrt{x})}\right)$$

Now use DSP!

$$=\frac{1}{(2+2)(\sqrt{2}+\sqrt{2})}=\frac{1}{8\sqrt{2}}$$

$\boxed{\text{Ex. 3}}$

Calculate each limit:

(a) $\lim_{x\to 0}\left(\frac{(x+3)^2-9}{x}\right)$ (b) $\lim_{x\to 4}\left(\frac{\frac{1}{x}-\frac{1}{4}}{x-4}\right)$

Solution:

(a) $$\lim_{x\to 0}\left(\frac{(x+3)^2-9}{x}\right)=\lim_{x\to 0}\left(\frac{x^2+6x+9-9}{x}\right)=\lim_{x\to 0}\left(\frac{x^2+6x}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\cancel{x}(x+6)}{\cancel{x}}\right)=\lim_{x\to 0}(x+6)=0+6=6$$

(b) $\lim\limits_{x\to4}\left(\dfrac{\frac{1}{x}-\frac{1}{4}}{x-4}\cdot\dfrac{4x}{4x}\right)=\lim\limits_{x\to4}\left(\dfrac{4-x}{(x-4)\cdot4x}\right)=\lim\limits_{x\to4}\left(\dfrac{-\cancel{(x-4)}}{\cancel{(x-4)}\cdot4x}\right)$

$=\lim\limits_{x\to4}\left(\dfrac{-1}{4x}\right)=\dfrac{-1}{16}$

$\boxed{\text{Ex. 4}}$

Calculate $\lim\limits_{x\to1}f(x)$ where f is:

$$f(x)=\begin{cases}\dfrac{x^2-x}{x-1} & \text{if } x>1\\ \sqrt{1-x} & \text{if } x\le1\end{cases}$$

Solution:

Note that $x=1$ is a transition point for $f(x)$. So we examine the one-sided limits.

$\boxed{\text{Left limit}}$

$\lim\limits_{\underbrace{x\to1^-}}f(x)=\lim\limits_{x\to1^-}\underbrace{\left(\sqrt{1-x}\right)}=\sqrt{1-1}=0$

use DSP

This means:
1. close to 1
2. $x<1$

$f(x)=\sqrt{1-x}$ if x is close to 1 and $x<1$

$\boxed{\text{Right limit}}$

$\lim\limits_{\underbrace{x\to1^+}}f(x)=\lim\limits_{x\to1^+}\left(\dfrac{x^2-x}{\underbrace{x-1}}\right)=\lim\limits_{x\to1^+}\left(\dfrac{x\cancel{(x-1)}}{\cancel{x-1}}\right)=\lim\limits_{x\to1^+}(x)=1$

This means:
1. close to 1
2. $x>1$

$f(x)=\dfrac{x^2-x}{x-1}$ if x is close to 1 and $x>1$

use DSP

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, $\lim_{x \to 1} f(x)$ DNE.

Ex. 5

Calculate $\lim_{x \to 6} \frac{|x-6|}{x-6}$.

Solution:

Recall definition of $|x|$.

$$|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x \geq 0 \end{cases} \Rightarrow |x-6| = \begin{cases} -(x-6) & \text{if } x<6 \\ x-6 & \text{if } x \geq 6 \end{cases}$$

So $x=6$ is a transition point for $|x-6|$. So we examine one-sided limits.

Left Limit:

$$\lim_{x \to 6^-} \frac{|x-6|}{x-6} = \lim_{x \to 6^-} \left(\frac{-\cancel{(x-6)}}{\cancel{x-6}} \right) = \lim_{x \to 6^-} (-1) = -1$$

This means $x<6$. So $x-6$ is negative.
So $|x-6| = -(x-6)$.

Right Limit:

$$\lim_{x \to 6^+} \frac{|x-6|}{x-6} = \lim_{x \to 6^+} \left(\frac{\cancel{x-6}}{\cancel{x-6}} \right) = \lim_{x \to 6^+} (+1) = +1$$

This means $x>6$. So $x-6$ is positive
So $|x-6| = x-6$

So $\lim_{x \to 6} \frac{|x-6|}{x-6}$ DNE.

Special Limit to Memorize

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} = 1$$

Common mistakes:

① $\frac{\sin(7\theta)}{\theta} \neq \frac{7\sin(\theta)}{\theta} = \ldots$

② $\frac{\cancel{\sin(\theta)}}{\cancel{\theta}} \neq 1$ Need the limit symbol!

Ex.6

Calculate $\lim_{x \to 0} \frac{\sin(2x)}{2x}$.

Solution:

False Solution:

$$\lim_{x \to 0} \frac{\sin(2x)}{2x} = \lim_{x \to 0} \frac{\cancel{2}\sin(x)}{\cancel{2}x} = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

These two limits look similar:

$$\lim_{x \to 0} \frac{\sin(2x)}{2x}, \quad \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$

But in what sense? "$2x$" plays the role of "θ".

① If $\theta = 2x$, then $\frac{\sin(\theta)}{\theta} = \frac{\sin(2x)}{2x}$

② If $\theta = 2x$, then "$\theta \to 0$" means the same thing as "$2x \to 0$" (or "$x \to 0$").

So we can conclude that

$$\lim_{x\to 0} \frac{\sin(2x)}{2x} = 1$$

Similarly ...

$$\lim_{\theta\to 0} \frac{\sin(a\theta)}{a\theta} = 1 \qquad \lim_{\theta\to 0} \frac{a\theta}{\sin(a\theta)} = 1 \qquad (a \neq 0)$$

$\boxed{\text{Ex. 7}}$

Calculate $\displaystyle\lim_{x\to 0} \frac{\tan(8x)}{\sin(3x)}$.

Solution:

$$\lim_{x\to 0}\left(\frac{\tan(8x)}{\sin(3x)}\right) = \lim_{x\to 0}\left(\frac{\sin(8x)}{\cos(8x)\sin(3x)}\right) \longrightarrow \lim_{x\to 0}\cos(8x) = 1$$

$$= \lim_{x\to 0}\left(\frac{\sin(8x)}{8x}\cdot\frac{3x}{\sin(3x)}\cdot\frac{8x}{3x}\cdot\frac{1}{\cos(8x)}\right)$$

$$= \underbrace{\lim_{x\to 0}\left(\frac{\sin(8x)}{8x}\right)}_{=1 \text{ (special)}}\cdot\underbrace{\lim_{x\to 0}\left(\frac{3x}{\sin(3x)}\right)}_{=1 \text{ (special)}}\cdot\underbrace{\lim_{x\to 0}\left(\frac{8x}{3x}\right)}_{=8/3 \text{ (cancel } x)}\cdot\underbrace{\lim_{x\to 0}\left(\frac{1}{\cos(8x)}\right)}_{=1 \text{ (DSP)}}$$

$$= 1 \cdot 1 \cdot \frac{8}{3} \cdot 1 = \frac{8}{3}$$

Section 2.4: Infinite Limits

Consider the following limit:

This means: x is close to 0 AND $x > 0$

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = +\infty \quad \text{(technically, limit DNE)}$$

As $x \to 0^+$, what happens to $\frac{1}{x}$. The values of $\frac{1}{x}$ are positive and they get arbitrarily large

What about this similar limit?

This means: x is close to 0 AND $x < 0$

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = -\infty \quad \text{(technically, limit DNE)}$$

As $x \to 0^-$, what happens to $\frac{1}{x}$. The values of $\frac{1}{x}$ are negative and they get arbitrarily large

What does "arbitrarily large" mean? It does not mean "numbers get bigger and bigger".

Ex:

1, 2, 3, 4, 5, 6, 7, 8, ...

These numbers get arbitrarily large because they eventually surpass any given number and remain larger.

0.9, 0.99, 0.999, 0.9999, . . .

These numbers get bigger and bigger but never surpass 1, so they do not get arbitrarily large.

Master Strategy for Infinite Limits

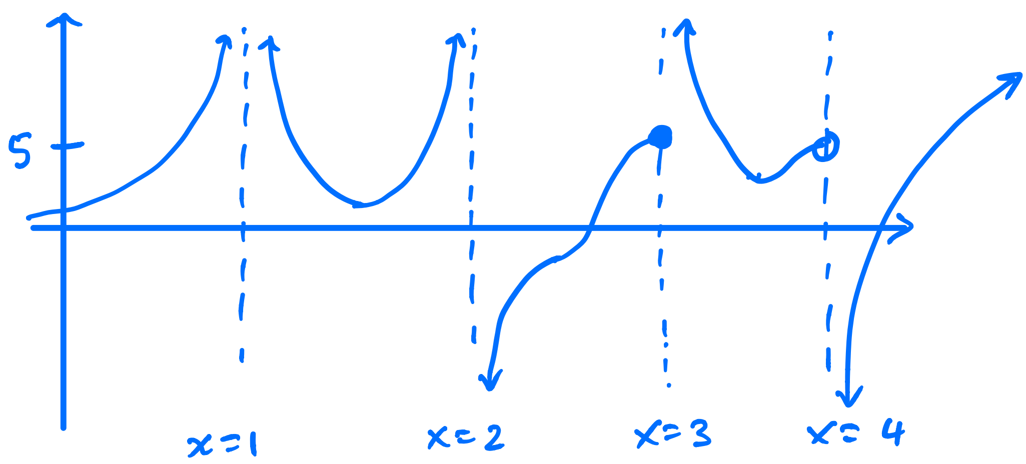
If D.S. gives the expression "$\frac{\text{nonzero \#}}{0}$", then each one-sided limit is infinite. To determine whether limit is $+\infty$ or $-\infty$, we perform a sign analysis of numerator and denominator.

Vertical Asymptote

If either $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ is infinite, then the line $x=a$ is a vertical asymptote of f.

Ex.1

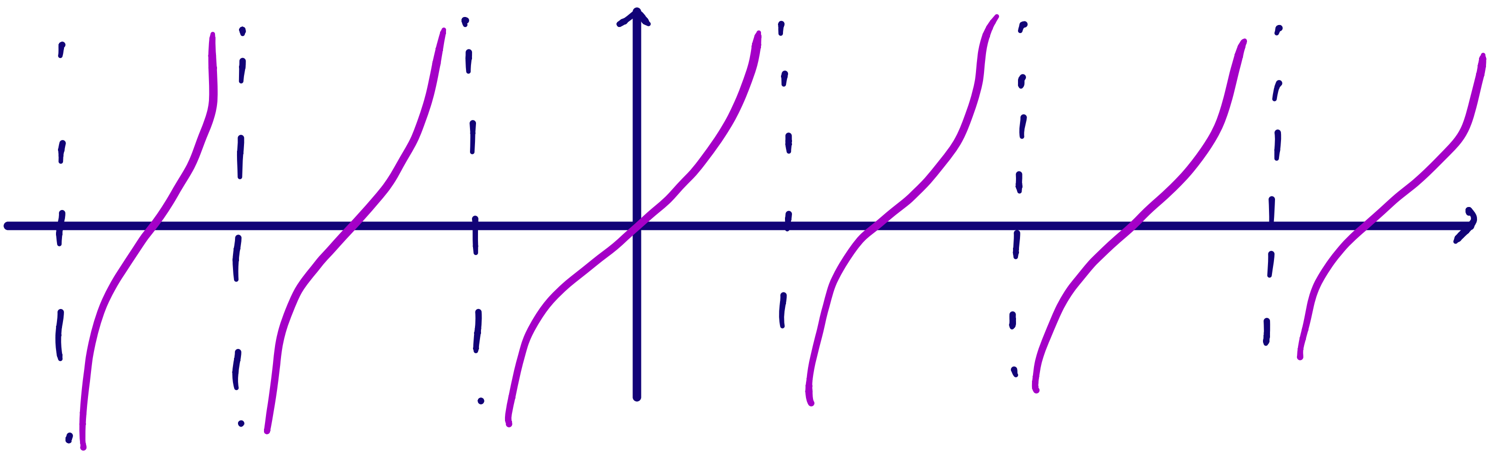
Use the graph to fill in the table.



Solution:

a	$\lim\limits_{x\to a^-} f(x)$	$\lim\limits_{x\to a^+} f(x)$	$\lim\limits_{x\to a} f(x)$	$f(a)$
1	$+\infty$	$+\infty$	$+\infty$	DNE
2	$+\infty$	$-\infty$	DNE	DNE
3	5	$+\infty$	DNE	5
4	5	$-\infty$	DNE	DNE

How many VA's can a function have? Infinitely many? Ex: $f(x) = \tan(x)$



Ex. 2

Find all vertical asymptotes of

$$f(x) = \frac{x}{x^2 - 4}$$

Find the one-sided limits at the leftmost VA.

Solution:

To find candidate VA's, set denominator to 0.

$$x^2 - 4 = 0 \implies x = -2 \quad \text{OR} \quad x = 2$$

D.S. of $x = -2$ (or $x = 2$) into $f(x)$ gives "$\frac{\text{nonzero}}{0}$". So the one-sided limits at $x = -2$ and $x = 2$ are infinite. So $x = -2$ and $x = 2$ are VA's.

$\boxed{x = -2}$

$$\lim_{\underbrace{x \to -2^-}_{\substack{x \text{ is close to } -2 \\ \text{and } x < -2}}} \left(\frac{x}{x^2 - 4}\right) = \frac{-2}{0^+} \infty = -\infty$$

→ D.S. of $x = -2$

→ goes to 0 but $x^2 - 4 > 0$

$$\lim_{x\to -2^+}\left(\frac{x}{x^2-4}\right) = \frac{-2}{0^-}\infty = +\infty$$

D.S. of $x=-2$

goes to 0 but $x^2-4<0$

x is close to -2 and $x>-2$

D.S. gives "$\frac{\text{nonzero}}{0}$", so both limit are infinite.

Ex. 3

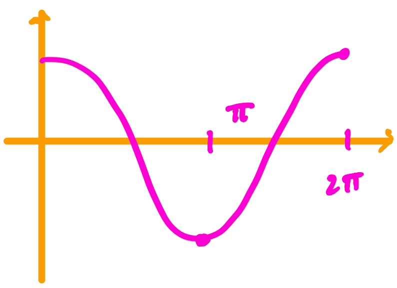
Compute $\displaystyle\lim_{x\to\pi}\frac{x^2}{1+\cos(x)}$.

Solution:

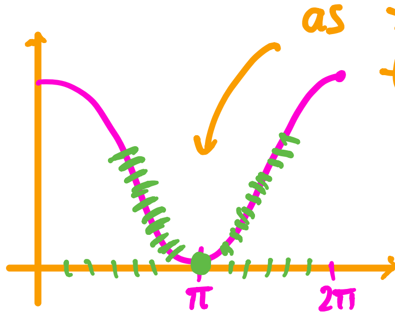
D.S. of $x=\pi$ gives "$\frac{\pi^2}{0}$". So the one-sided limits are infinite.

$$\lim_{x\to\pi^-}\left(\frac{x^2}{1+\cos(x)}\right) = \frac{\pi^2}{0^+}\infty = +\infty$$

$$\lim_{x\to\pi^+}\left(\frac{x^2}{1+\cos(x)}\right) = \frac{\pi^2}{0^+}\infty = +\infty$$



$y=\cos(x)$



as $x\to\pi$, $1+\cos(x)$ goes to 0 but is positive

$y = 1+\cos(x)$

Ex. 4

Find all VA's of $f(x)=\dfrac{x^2-4}{x^2-x-2}$.

Solution:

Observe:

$x^2 - x - 2 \implies x = 2$, $x = -1$

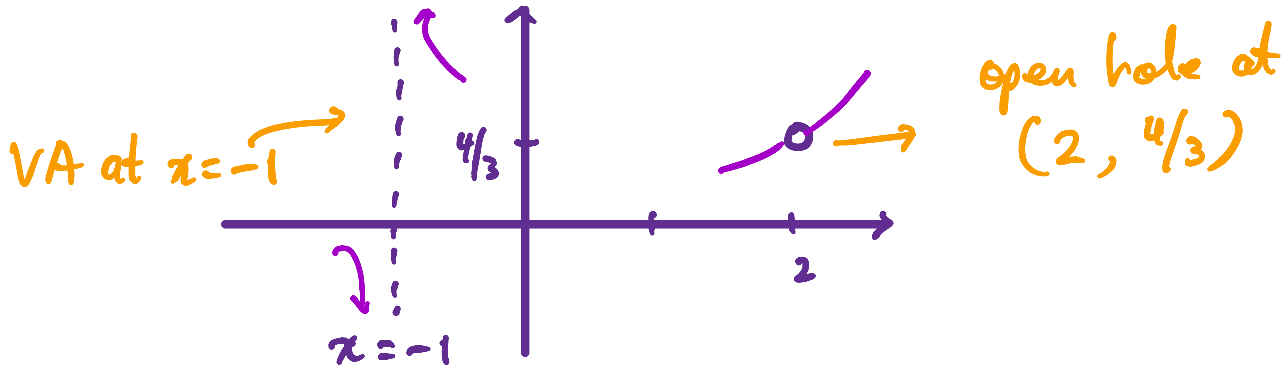
D.S. of $x=-1$ gives "$\frac{-3}{0}$". So $x=-1$ must be a VA.

D.S. of $x=2$ gives "$\frac{0}{0}$". So $x=2$ may or may not be a VA. Need more analysis.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x^2-x-2}\right) = \lim_{x\to 2}\left(\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x+1)}\right)$$

$$= \lim_{x\to 2}\left(\frac{x+2}{x+1}\right) = \frac{2+2}{2+1} = \frac{4}{3}$$

So $x=2$ is not a VA. The only VA is $x=-1$.



Ex. 5

Let $f(x) = \frac{x^3+4x^2+3x}{x^3+2x^2+x}$. Find all VA's and, at each VA, find the one-sided limits

Solution:

Observe:

$0 = x^3+2x^2+x = x(x^2+2x+1) = x(x+1)^2$

$\implies x=0$ or $x=-1$

So our candidate VA's are $x=0$ and $x=-1$.

D.S. of each gives:

$x=0$: $\frac{0}{0}$

$x=-1$: $\frac{0}{0}$

inconclusive, so need more analysis!

First we do some algebra. If $x \neq 0$ and $x \neq -1$, we have:

$$\frac{x^3+4x^2+3x}{x^3+2x^2+x} = \frac{x(x^2+4x+3)}{x(x+1)^2} = \frac{\cancel{x}\,\cancel{(x+1)}(x+3)}{\cancel{x}\,(x+1)^{\cancel{2}\,1}} = \frac{x+3}{x+1}$$

Now we investigate the limits:

$\boxed{x=0}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x+3}{x+1}\right) = \frac{0+3}{0+1} = 3$$

So $x=0$ is not a VA. (Limit is not infinite)

$\boxed{x=-1}$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(\frac{x+3}{x+1}\right)$$

D.S. gives "$\frac{2}{0}$", which indicates infinite one-sided limits!

So $x=-1$ is a VA. Now for the one-sided limits.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\left(\frac{x+3}{x+1}\right) = \frac{2}{0^-}\infty = -\infty$$

test $x=-1.01$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}\left(\frac{x+3}{x+1}\right) = \frac{2}{0^+}\infty = +\infty$$

test $x=-0.99$

Section 2-5: Limits at Infinity

Consider the following limit.

As x gets arbitrarily large and positive...

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

... the values of $\frac{1}{x}$ get arbitrarily close to 0.

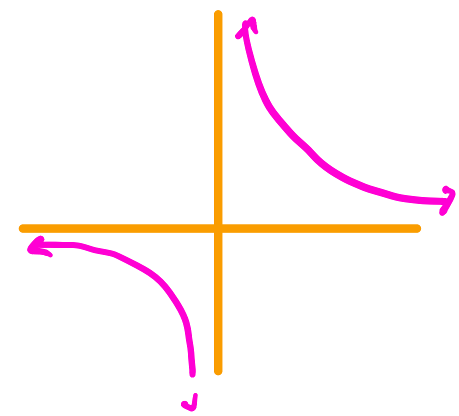
(and $\frac{1}{x}$ is always positive)

Similarly,

$$\lim_{x\to-\infty} \left(\frac{1}{x}\right) = 0$$

x gets arbitrarily large and negative

$\frac{1}{x}$ gets arbitrarily small and negative



In general, we have (for $n>0$)

$$\boxed{\lim_{x\to\infty} \frac{1}{x^n} = \left(\lim_{x\to\infty} \frac{1}{x}\right)^n = 0^n = 0}$$

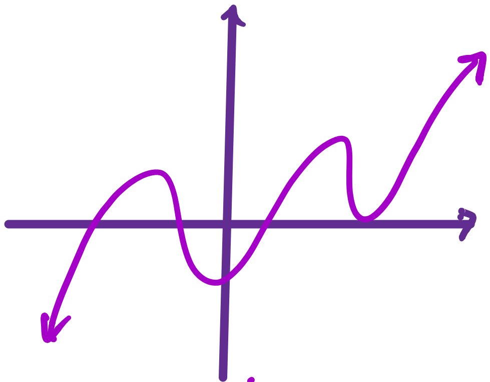
Master Strategy for Limits with $x \to \pm\infty$

If $x\to\infty$ or $x\to-\infty$, then factor out "highest power" of numerator and denominator separately.

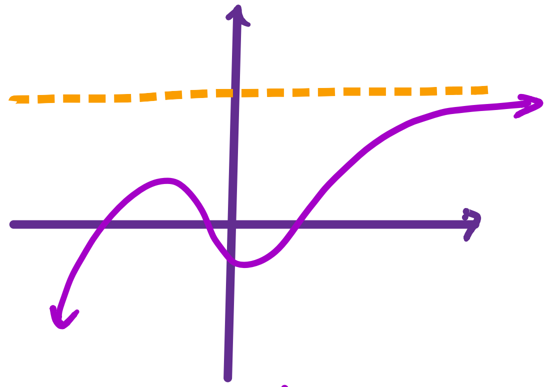
(Or factor out "highest power" of denominator from all terms.)

$\boxed{\text{Horizontal Asymptotes}}$ (L is finite)

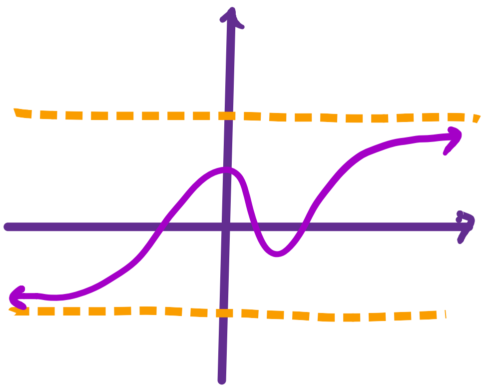
If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then we say that the line $y = L$ is a horizontal asymptote for f.



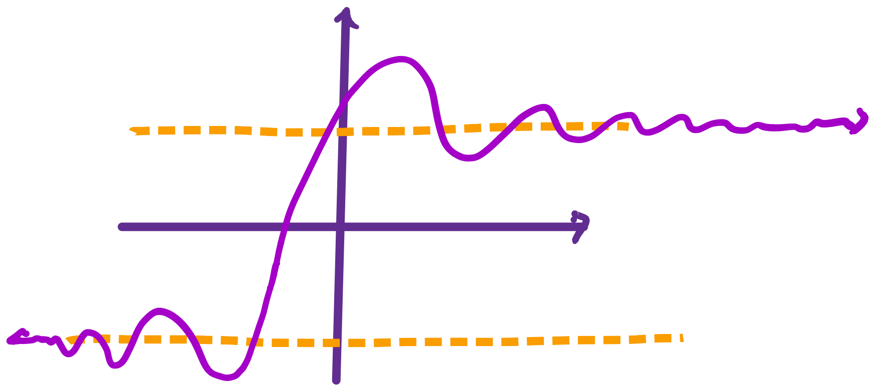
0 HA's



1 HA



2 HA's



2 HA's

$\boxed{\text{Ex. 1}}$

Calculate $\lim\limits_{x\to\infty} \dfrac{3x^2 - 5x + 1}{4x^2 - 7}$.

Solution:

$$\lim_{x\to\infty}\left(\frac{3x^2-5x+1}{4x^2-7}\right) = \lim_{x\to\infty}\left(\underbrace{\frac{x^2}{x^2}} \cdot \underbrace{\frac{3-\frac{5}{x}+\frac{1}{x^2}}{4-\frac{7}{x^2}}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\cancel{x^2}}{\cancel{x^2}}\right) \cdot \lim_{x\to\infty}\left(\frac{3-\frac{5}{x}+\frac{1}{x^2}}{4-7/x^2}\right) = 1 \cdot \frac{3-0+0}{4-0} = \frac{3}{4}$$

Ex. 2

Calculate $\lim\limits_{x\to-\infty}\left(\dfrac{5x^3-2x}{x^2+1}\right)$.

Solution:

$$\lim_{x\to-\infty}\left(\frac{5x^3-2x}{x^2+1}\right)=\lim_{x\to-\infty}\left(\underbrace{\frac{x^3}{x^2}}\cdot\underbrace{\frac{5-2/x^2}{1+1/x^2}}\right)$$

$$=\underbrace{\lim_{x\to-\infty}\left(\frac{x^3}{x^2}\right)}_{=\lim\limits_{x\to-\infty}(x)=-\infty}\cdot\underbrace{\lim_{x\to-\infty}\left(\frac{5-2/x^2}{1+1/x^2}\right)}_{=\frac{5-0}{1+0}=5}=(-\infty)\cdot(5)=-\infty$$

$$\lim_{x\to-\infty}\left(\frac{2}{x^2}\right)=0$$

$$\lim_{x\to-\infty}\left(\frac{1}{x^2}\right)=0$$

Warning: "$\infty\cdot 0$" is undefined (like "$\frac{0}{0}$")

$\infty\cdot 0$

(a) $\lim\limits_{x\to\infty}\left(x\cdot\dfrac{1}{x}\right)=\lim\limits_{x\to\infty}(1)=1$

(b) $\lim\limits_{x\to\infty}\left(x^2\cdot\dfrac{1}{x}\right)=\lim\limits_{x\to\infty}(x)=\infty$

(c) $\lim\limits_{x\to\infty}\left(x\cdot\dfrac{1}{x^2}\right)=\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)=0$

Ex. 3

Calculate $\lim\limits_{x\to-\infty}\left(\dfrac{\sqrt{25x^2+3}}{9x-1}\right)$

Solution:

Note: $\sqrt{x^2-9}\neq x-3$, $\sqrt{x^2+9}\neq x+3$

$$\lim_{x\to-\infty}\left(\frac{\sqrt{25x^2+3}}{9x-1}\right)=\lim_{x\to-\infty}\left(\frac{\sqrt{x^2(25+3/x^2)}}{9x-1}\right)$$

$$=\lim_{x\to-\infty}\left(\frac{\sqrt{x^2}\sqrt{25+3/x^2}}{9x-1}\right)=\lim_{x\to-\infty}\left(\frac{\sqrt{x^2}}{x}\cdot\frac{\sqrt{25+3/x^2}}{9-1/x}\right)$$

$\sqrt{x^2} \neq \pm x$ ← not a function (two values)

$\sqrt{x^2} \neq x$ ← what if $x<0$?

$\sqrt{x^2} = |x|$ ✓

$$=\lim_{x\to-\infty}\left(\frac{|x|}{x}\cdot\frac{\sqrt{25+3/x^2}}{9-1/x}\right)=\lim_{x\to-\infty}\left(\frac{-\cancel{x}}{\cancel{x}}\cdot\frac{\sqrt{25+3/x^2}}{9-1/x}\right)$$

Since $x\to-\infty$, we can assume $x<0$.

So then $|x|=-x$.

$$=\lim_{x\to-\infty}\left(-1\cdot\frac{\sqrt{25+3/x^2}}{9-1/x}\right)=-1\cdot\frac{\sqrt{25+0}}{9-0}=-\frac{5}{9}$$

terms go to 0 as $x\to-\infty$

$\boxed{\text{Ex. 4}}$

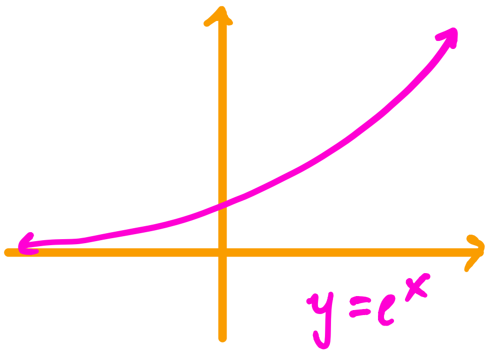
Let $f(x)=\dfrac{3+e^x}{5-4e^x}$. Find all HA's of f.

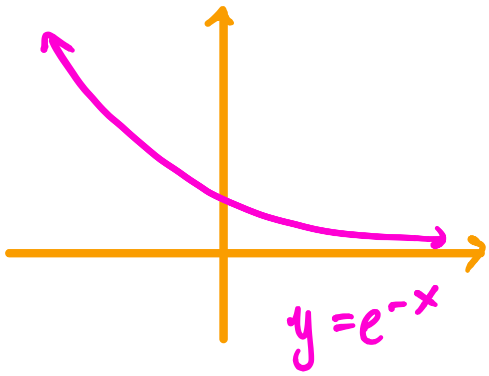
Solution:

To calculate the HA's, we must calculate:

$$\lim_{x\to\infty}f(x) \quad \text{and} \quad \lim_{x\to-\infty}f(x)$$

Recall the following:





$\lim\limits_{x\to\infty}(e^x) = \infty$ $\qquad$ $\lim\limits_{x\to\infty}(e^{-x}) = 0$

$\lim\limits_{x\to-\infty}(e^x) = 0$ $\qquad$ $\lim\limits_{x\to-\infty}(e^{-x}) = \infty$

Now we compute our limits:

$\boxed{x\to-\infty}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{3+e^x}{5-4e^x}\right) = \frac{3+0}{5-0} = \frac{3}{5}$$

$\lim\limits_{x\to-\infty}(e^x) = 0$

$\boxed{x\to\infty}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{3+e^x}{5-4e^x}\right) = ?$$

$\lim\limits_{x\to+\infty}(e^x) = +\infty$, so this gives $\frac{\infty}{-\infty}$, which is like $\frac{0}{0}$ or $\infty\cdot 0$ (need algebra)

This is where we modify our "highest power" strategy:

$$= \lim_{x\to+\infty}\left(\frac{\cancel{e^x}}{\cancel{e^x}}\cdot\frac{3e^{-x}+1}{5e^{-x}-4}\right) = \lim_{x\to\infty}\left(\frac{3e^{-x}+1}{5e^{-x}-4}\right) = \frac{0+1}{0-4} = -\frac{1}{4}$$

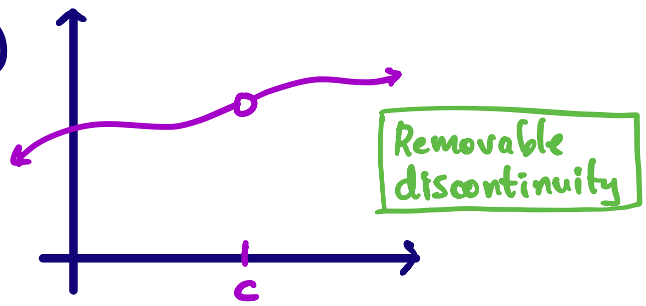
$$\lim_{x\to\infty} (e^{-x}) = 0$$

So the HA's are $y = \frac{3}{5}$ and $y = -\frac{1}{4}$.

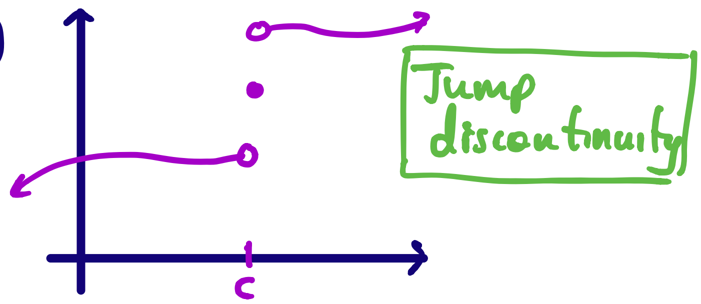
Section 2.6: Continuity

Def: We say f is continuous at $x=c$ if
$$\lim_{x\to c} f(x) = f(c)$$
(i.e., f has the DSP at $x=c$). Otherwise, we say f is discontinuous at $x=c$.

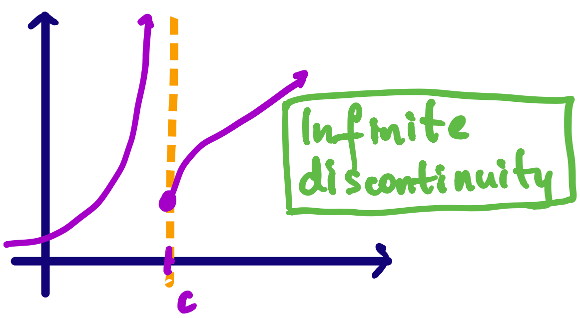
How can f fail to be continuous at $x=c$? Four types:

① 

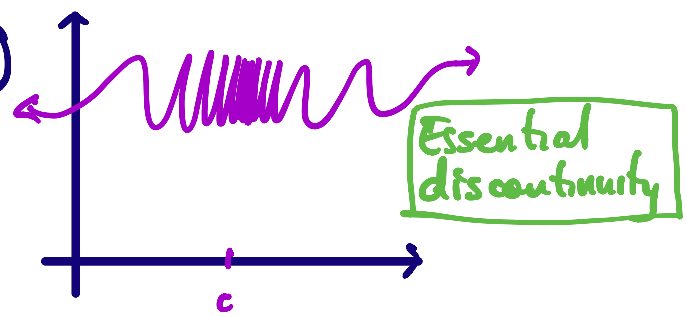
- $\lim_{x\to c} f(x)$ exists
- $f(c)$ undefined OR $f(c) \neq \lim_{x\to c} f(x)$

② 

- one-sided limits both exist but are not equal
- $f(c)$ may or may not be defined

③ 

- vertical asymptote at $x=c$
- at least one one-sided limit is infinite

④ 

- not ①, ②, or ③
- usually infinite oscillation

Ex. 1

Let $f(x) = \dfrac{x^2+x-12}{x-3}$.

(a) Where is f continuous?

(b) At each value of x where f is discontinuous, Can we redefine the value of f so that f is cont.?

Solution:

(a) For what values of x does f have the DSP? For all x in the domain! So f is continuous on $(-\infty, 3) \cup (3, \infty)$.

(b) Right now $f(3)$ is undefined. Can we choose a value for $f(3)$ to make f continuous at $x=3$?

Note: If f were continuous at $x=3$, we would have

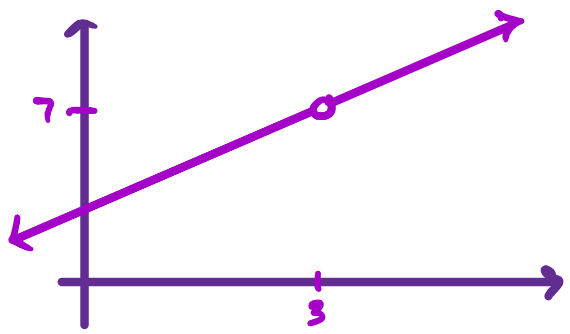
$$\underbrace{\lim_{x\to 3} f(x)}_{} = f(3)$$

This tells us what the new value of $f(3)$ should be!

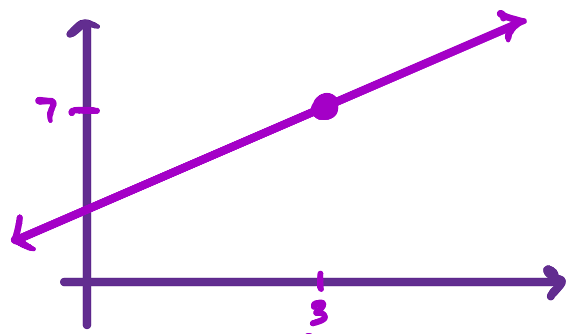
So we compute this limit.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\left(\frac{x^2+x-12}{x-3}\right) = \lim_{x\to 3}\left(\frac{(x+4)\cancel{(x-3)}}{\cancel{x-3}}\right) = \lim_{x\to 3}(x+4) = 7$$

So if we define $f(3)$ to be 7, then f is cont. at $x=3$.



$y=f(x)$
(before redefining $f(3)$)



$y=f(x)$
(after redefining $f(3)$)

Ex. 2

Let $f(x) = \begin{cases} x^2+3, & x<0 \\ x-5, & x \geq 0 \end{cases}$

(a) Where is f continuous?

(b) At each value of x where f is discontinuous, can we redefine the value of f so that f is cont.?

Solution:

(a) Where does f have the DSP? Each "piece" of f has the DSP so the only value of x for which f might not have the DSP is $x=0$ (transition point).
To check whether f is continuous at $x=0$, we check:

$$\lim_{x\to 0} f(x) = f(0)$$

Note: This means the left-limit, right-limit, and function value must all be equal.

Left limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+3) = 0+3 = 3$$

Right limit

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-5) = 0-5 = -5$$

Function value

$$f(0) = (x-5)\Big|_{x=0} = -5$$

These numbers are not all equal! So f is not cont. at $x=0$

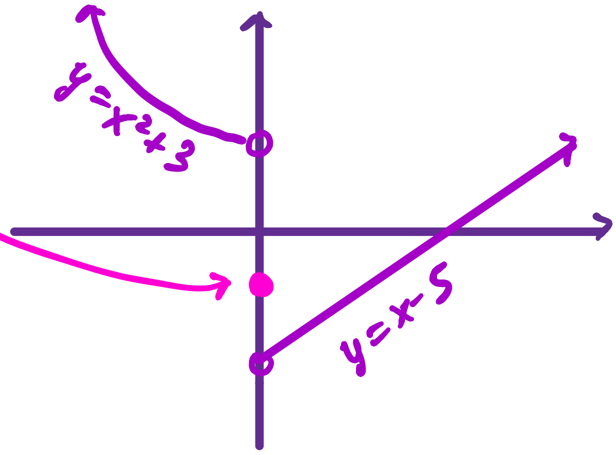
So f is continuous on $(-\infty, 0) \cup (0, \infty)$.

(b) If we can redefine $f(0)$ to make f continuous, we have one candidate value:

$$f(0) = \lim_{x\to 0} f(x) \longleftarrow \text{only possible value that can work}$$

Since $\lim_{x\to 0} f(x)$ does not exist (why?), there is no value we can assign to make f continuous.

Is there a point on the y-axis we can put this point to make f continuous? No!



$y=f(x)$ with $f(0)$ undefined

Ex. 3

Let $f(x) = \begin{cases} \dfrac{x^4-16}{x+2}, & x<-2 \\ a+2b, & x=-2 \\ bx+3, & x>-2 \end{cases}$

where a and b are unspecified constants. Find the values of a and b which make f continuous at $x=-2$, or determine no such values exist.

Solution:

We need the left-limit, right limit, and function value at $x=-2$ to be equal.

Left-limit

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}\left(\frac{x^4-16}{x+2}\right) = \lim_{x\to -2^-}\left(\frac{(x^2-4)(x^2+4)}{x+2}\right)$$

$$= \lim_{x\to -2^-}\left(\frac{(x-2)\cancel{(x+2)}(x^2+4)}{\cancel{x+2}}\right) = \lim_{x\to -2^-}\Big((x-2)(x^2+4)\Big) = -32$$

Right limit

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \underbrace{(bx+3)}_{\text{substitute } x=-2} = -2b+3$$

$\boxed{\text{Function Value}}$

$f(-2) = a+2b$

All three of these numbers must be equal.

$$-32 = -2b+3 = a+2b$$

So we solve for a and b. (Extract two equations)

$$\left.\begin{aligned} -32 &= -2b+3 \\ -32 &= a+2b \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} 2b &= 35 \\ a+2b &= -32 \end{aligned}\right\} \Longrightarrow \begin{aligned} a &= -67 \\ b &= 35/2 \end{aligned}$$

So if $a=-67$ and $b=35/2$, f is continuous at $x=-2$.

$\boxed{\text{Ex. 4}}$

Let $f(x) = \begin{cases} x+a, & x<0 \\ 5, & x=0 \\ \dfrac{\sin(bx)}{x}, & x>0 \end{cases}$

where a and b are unspecified constants.

Find the values of a and b which make f continuous at $x=$, or determine no such values exist.

Solution:

We need the left-limit, right-limit, and function value at $x=0$ to be equal.

$\boxed{\text{Left limit}}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+a) = 0+a = a$$

$\boxed{\text{Right limit}}$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{\sin(bx)}{x}\right) = \lim_{x\to 0^+}\left(\frac{\sin(bx)}{bx}\cdot b\right)$$

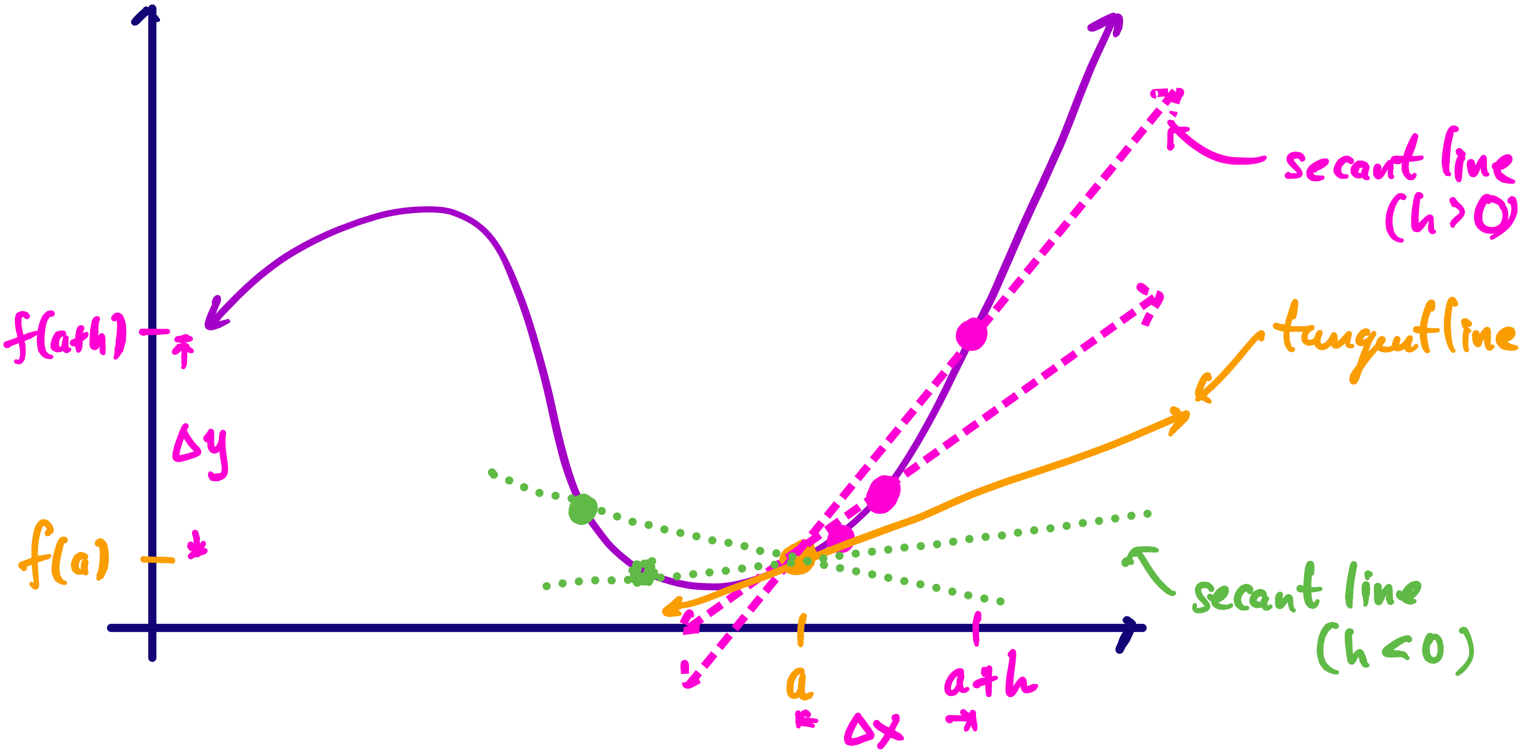
$$= \lim_{x\to 0^+}\left(\frac{\sin(bx)}{bx}\right)\cdot \lim_{x\to 0^+}(b) = 1\cdot b = b$$

Function Value

$f(0) = 5$

All three numbers must be equal, so $a = b = 5$.

Sections 3.1/3.2: Introduction to Derivatives



Slope of secant line (h>0): $m_{sec} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(a+h) - f(a)}{h}$

slope of tangent line: $m_{tan} = \lim\limits_{h \to 0} \left(\dfrac{f(a+h) - f(a)}{h} \right)$

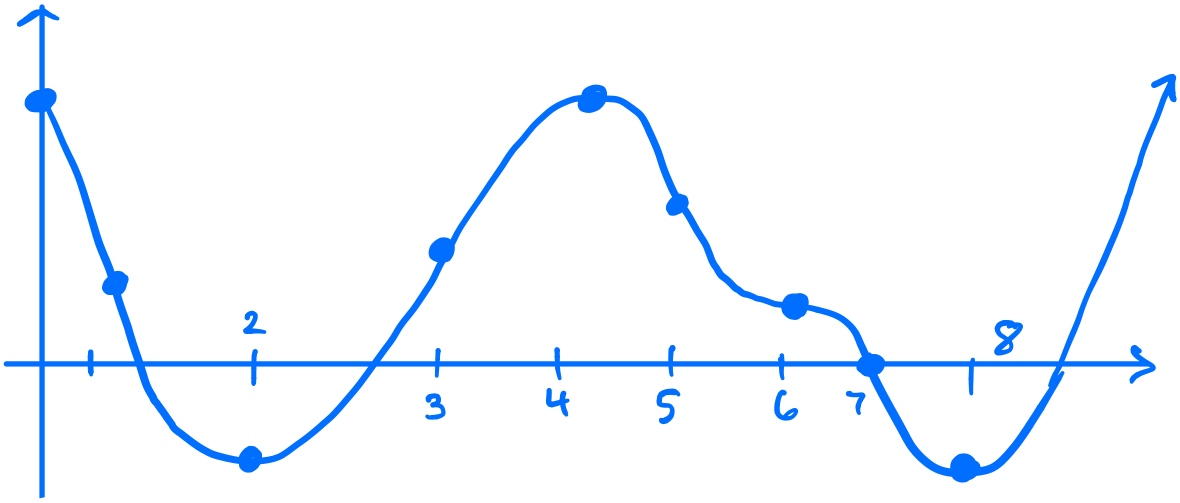
Def: Given the function $f(x)$, the number

$$m = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

is the derivative of f at $x = a$. We denote it as $f'(a)$.

Def: The line tangent to the graph of $y = f(x)$ at $x = a$ is the line that passes through $(a, f(a))$ with slope $f'(a)$.

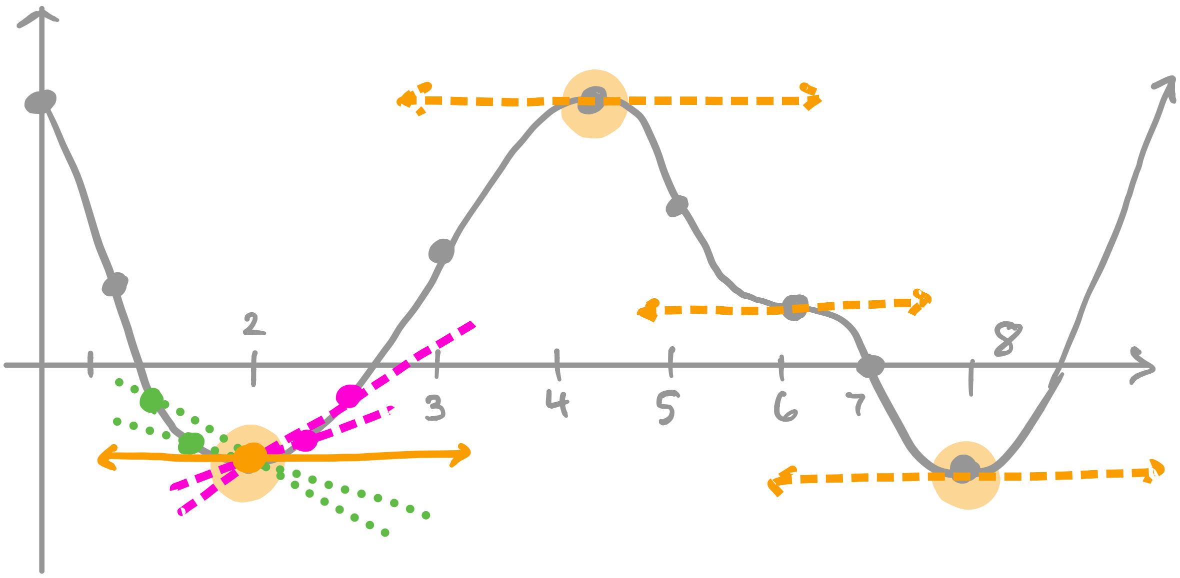
Ex. 1



(a) Where is $f'(x) = 0$? (b) $f'(x) < 0$? (c) $f'(x) > 0$?

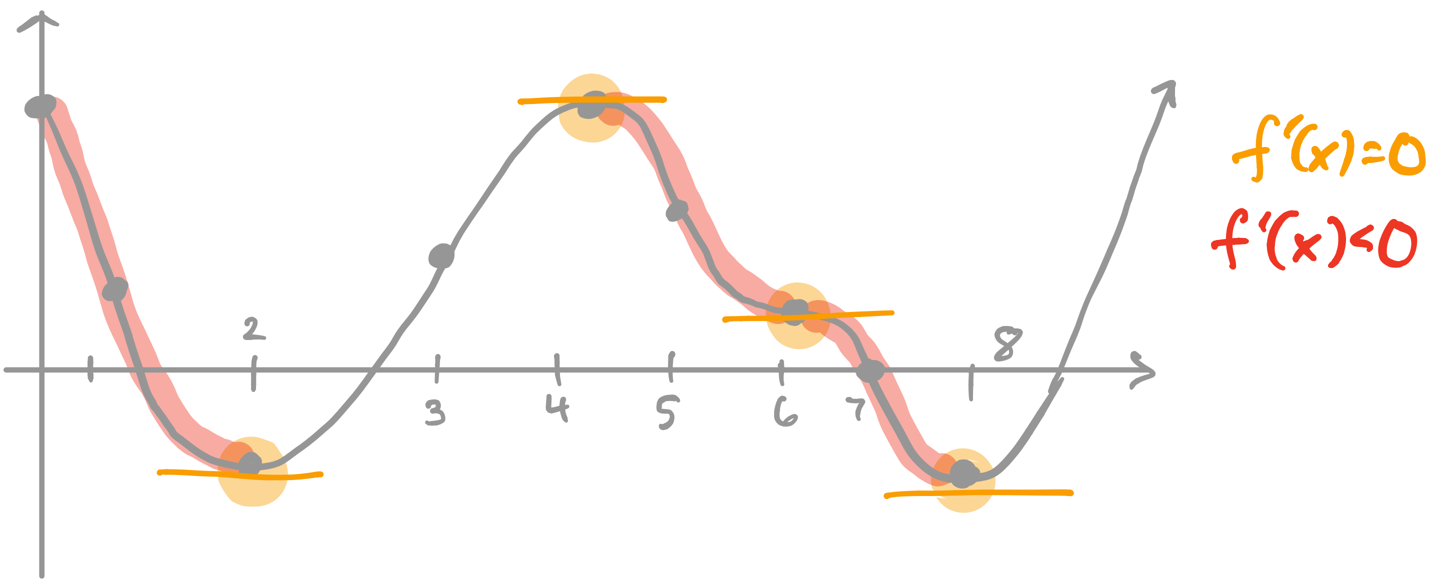
Solution:

(a)



$x = 2, 4, 6, 8$

(b)

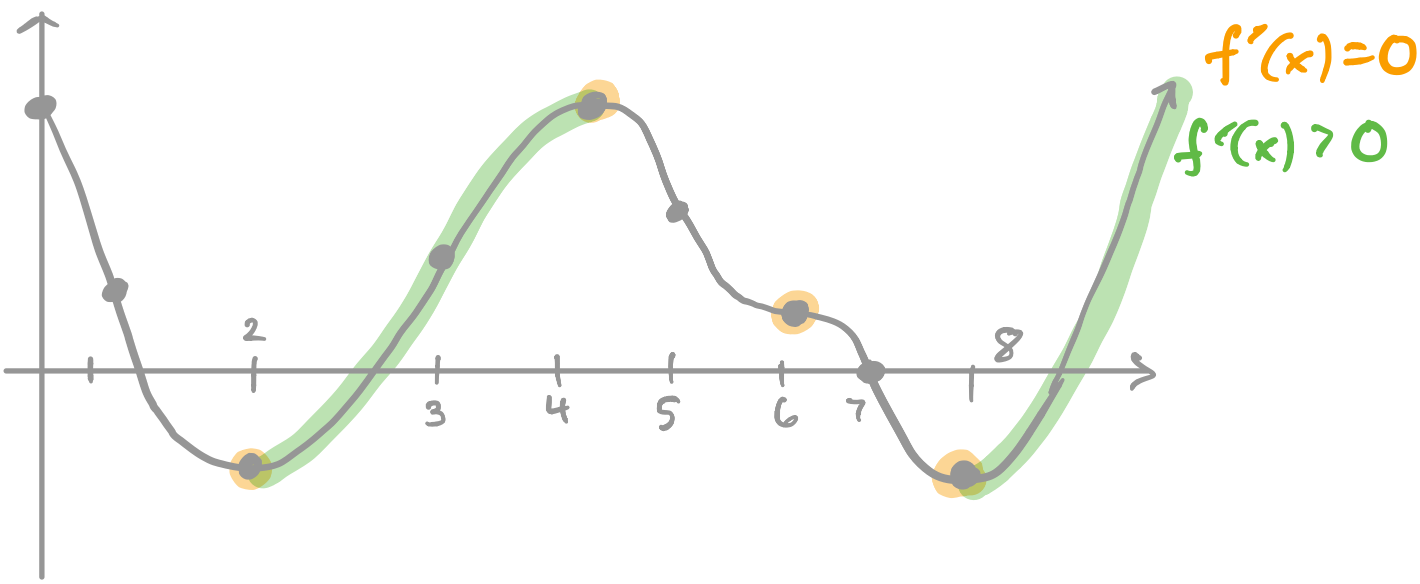


$f'(x) < 0$ on $(0,2) \cup (4,6) \cup (6,8)$

Note: $f'(0)$ dne! (derivative cannot exist at boundary point since we cannot take both one-sided limits)

Note: f is decreasing on: $[0,2]$, $[4,8]$

(c)



$f'(x)>0$ on $(2,4)\cup(8,\infty)$

Note: f is increasing on: $[2,4]$, $[8,\infty)$

$\boxed{\text{Ex. 2}}$

Find tangent line to $f(x)=x^3+2x-1$ at $x=1$.

Solution:

In the earlier language, $a=1$.

Point: $(a, f(a)) = (1,2)$

Slope: $m = f'(1) = \lim\limits_{h\to 0}\left(\dfrac{f(1+h)-f(1)}{h}\right)$

$= \lim\limits_{h\to 0}\left(\dfrac{(1+h)^3+2(1+h)-1 \;-2}{h}\right) = \lim\limits_{h\to 0}\left(\dfrac{\cancel{1}+3h+3h^2+h^3+\cancel{2}+2h-\cancel{1}-\cancel{2}}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{5h+3h^2+h^3}{h}\right) = \lim_{h\to 0}\left(\frac{\cancel{h}(5+3h+h^2)}{\cancel{h}}\right)$$

$$= \lim_{h\to 0}\left(5+3h+h^2\right) = 5+0+0 = 5$$

So our tangent line is:

$$y = 2+5(x-1)$$

$\boxed{\text{Ex. 3}}$

Let $f(x)=\sqrt{x}$. Calculate $f'(x)$ for $x>0$.

<u>Solution</u>:

We use the definition.

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{x}+h-\cancel{x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\right) = \lim_{h\to 0}\left(\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right) = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

$\boxed{\text{Ex. 4}}$

Let $f(x)=|x|$. Calculate:

(a) $f'(-3)$ (b) $f'(0)$

<u>Solution</u>:

(a) By definition,

$$f'(-3) = \lim_{h\to 0}\left(\frac{f(-3+h)-f(-3)}{h}\right) = \lim_{h\to 0}\left(\frac{|-3+h|-3}{h}\right)$$

Note: We assume h is close to 0 (neg. or pos.).
So $(-3+h)$ is close to -3, hence we can assume $(-3+h)$ is neg. So $|-3+h| = -(-3+h)$.

$$= \lim_{h\to 0}\left(\frac{-(-3+h)-3}{h}\right) = \lim_{h\to 0}\left(\frac{\cancel{3}-h-\cancel{3}}{h}\right) = \lim_{h\to 0}\left(\frac{-\cancel{h}}{\cancel{h}}\right) = -1$$

So $f'(-3) = -1$.

(b) By definition,

$$f'(0) = \lim_{h\to 0}\left(\frac{f(0+h)-f(0)}{h}\right) = \lim_{h\to 0}\left(\frac{|h|}{h}\right)$$

Note: See Section 2.3 for details.

- $\lim_{h\to 0^-}\left(\frac{|h|}{h}\right) = \lim_{h\to 0^-}\left(\frac{-h}{h}\right) = -1$
- $\lim_{h\to 0^+}\left(\frac{|h|}{h}\right) = \lim_{h\to 0^+}\left(\frac{h}{h}\right) = +1$

So $f'(0)$ dne. (Even though f is continuous at $x=0$!)

$\boxed{\text{Ex. 5}}$

Find tangent line to $f(x) = \frac{1}{x}$ at $x=3$.

Solution:

Point: $(3, f(3)) = (3, 1/3)$

Slope: $f'(3) = \lim_{h\to 0}\left(\frac{f(3+h)-f(3)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{3+h}-\frac{1}{3}}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{3-(3+h)}{h\cdot 3\cdot(3+h)}\right) = \lim_{h\to 0}\left(\frac{\cancel{3}-\cancel{3}-h}{3h\,(3+h)}\right)$$

$$= \lim_{h\to 0}\left(\frac{-\cancel{h}}{3\cancel{h}\,(3+h)}\right) = \lim_{h\to 0}\left(\frac{-1}{3\,(3+h)}\right) = \frac{-1}{3\,(3)} = \frac{-1}{9}$$

The tangent line:

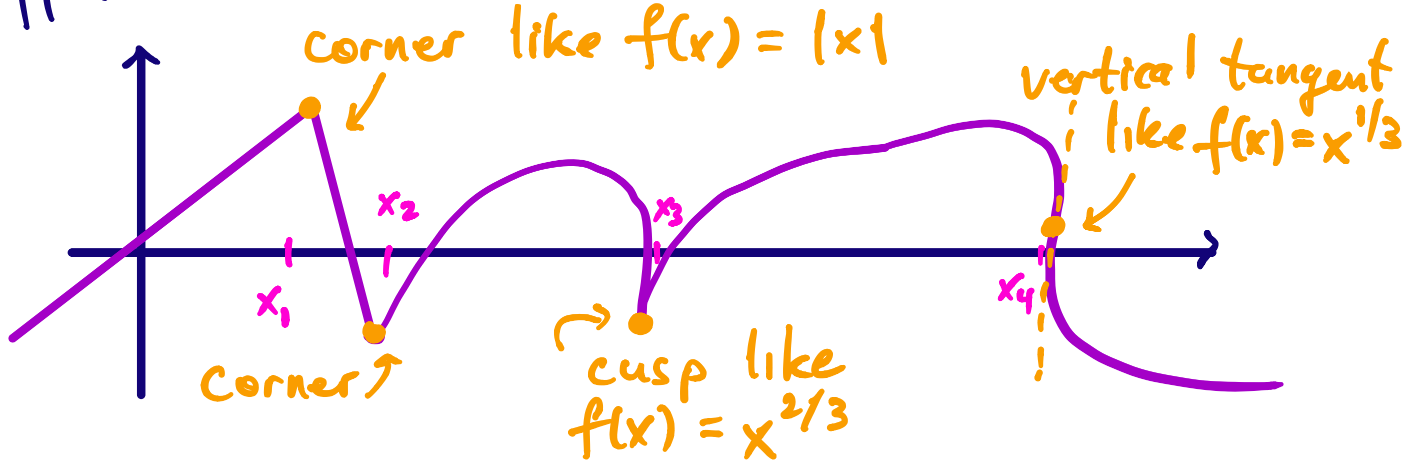
$$y = \frac{1}{3} - \frac{1}{9}(x-3)$$

$\boxed{\text{How can a function fail to be differentiable?}}$

Q: What is the relationship between continuity and differentiability?

A: Continuity is <u>necessary</u> for differentiability. So if f is <u>not</u> continuous at $x=a$, then f is <u>not</u> differentiable at $x=a$.

So if f is continuous, how can f fail to be differentiable?



Sections 3.3, 3.4, 3.5, 3.9: Derivative Rules

$f(x)$	$f'(x)$
c	0
x^n	nx^{n-1}
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec(x)^2$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc(x)^2$

Power Rule

Advanced Rules

$F(x)$	$F'(x)$	
$f+g$	$f'+g'$	Sum
cf	cf'	
fg	$f'g+fg'$	Product
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	Quotient

Ex. 1

Verify $\frac{d}{dx}\left(\tan(x)\right) = \sec(x)^2$ using the other rules.

Solution:

We use Quotient Rule.

$$\frac{d}{dx}\left(\frac{\overbrace{\sin(x)}^{f}}{\underbrace{\cos(x)}_{g}}\right) = \frac{\overbrace{\cos(x)}^{f'} \cdot \overbrace{\cos(x)}^{g} - \overbrace{\sin(x)}^{f}\overbrace{(-\sin(x))}^{g'}}{\underbrace{\cos(x)^2}_{g^2}}$$

$$= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2} = \sec(x)^2$$

Ex. 2

Calculate $\frac{d}{dx}\left(\frac{7x^2}{x^3\sqrt{x}}\right)$.

Solution:

Rewrite function in simpler form first.

$$\frac{7x^2}{x^3\sqrt{x}} = 7x^2x^{-3}x^{-1/2} = 7x^{-3/2}$$

Now we use power rule.

$$\frac{d}{dx}\left(7x^{-3/2}\right) = 7\cdot\frac{d}{dx}\left(x^{-3/2}\right) = 7\cdot\left(-\frac{3}{2}\cdot x^{-5/2}\right) = -\frac{21}{2}x^{-5/2}$$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

$\boxed{\text{Ex. 3}}$

Calculate $h'(x)$ if $h(x) = \dfrac{x\sqrt{x}\tan(x)}{e^x - e^3}$.

Solution:

First simplify. Then use Quotient Rule.

$$h(x) = \frac{\underbrace{x^{3/2}\tan(x)}_{f}}{e^x - e^3}$$

$$h'(x) = \frac{\overbrace{\left(\frac{3}{2}x^{1/2}\tan(x) + x^{3/2}\sec(x)^2\right)}^{f'}\cdot\overbrace{\left(e^x - e^3\right)}^{g} - \overbrace{x^{3/2}\tan(x)}^{f}\cdot\overbrace{e^x}^{g'}}{\underbrace{\left(e^x - e^3\right)^2}_{g^2}}$$

$\boxed{\text{Scratch Work}}$

- $f(x) = \underbrace{x^{3/2}}_{F}\underbrace{\tan(x)}_{G}$ (Product Rule)

$$f'(x) = \underbrace{\frac{3}{2}x^{1/2}}_{F'}\cdot\underbrace{\tan(x)}_{G} + \underbrace{x^{3/2}}_{F}\cdot\underbrace{\sec(x)^2}_{G'}$$

- $g(x) = e^x - e^3$

$$g'(x) = \frac{d}{dx}\left(e^x - e^3\right) = \frac{d}{dx}\left(e^x\right) - \underbrace{\frac{d}{dx}\left(e^3\right)} = e^x - 0 = e^x$$

e^3 is a constant!

$\boxed{\text{Ex. 4}}$

Find the tangent line to $f(x)$ at $x=1$.

$$f(x) = x^3 - \frac{3}{x^2}$$

Solution:

Point: $(1, f(1)) = (1, -2)$

Slope: $f(x) = x^3 - 3x^{-2}$ d/dx

$$f'(x) = 3x^2 - 3\left(-2x^{-3}\right)$$

$f'(x) = 3x^2 + 6x^{-3}$ ← recipe for slopes

$f'(1) = 3 + 6 = 9$ ← slope of tangent @ $x=1$

Equation: $y = -2 + 9(x-1)$

$\boxed{\text{Ex. 5}}$

Find all horizontal tangent lines of f:

$$f(x) = 3x^5 e^x$$

Solution:

Our initial goal is to solve $f'(x) = 0$.

$$f(x) = \underbrace{3x^5}_{F}\underbrace{e^x}_{G}$$

$$f'(x) = 3\cdot\big(\underbrace{5x^4}_{F'}\cdot\underbrace{e^x}_{G} + \underbrace{x^5}_{F}\cdot\underbrace{e^x}_{G'}\big)$$

$$f'(x) = 3x^4e^x(5+x)$$

Now solve $f'(x) = 0$.

$$3x^4e^x(5+x) = 0$$

$3x^4 = 0$ or $\cancel{e^x = 0}$ or $5+x=0$

$x = 0$ or no solution or $x = -5$

The equations of the horizontal tangents are:

$\boxed{x=0}$ $\qquad$ $\boxed{x=-5}$

$f(0) = 0$ $\qquad$ $f(-5) = 3(-5)^5e^{-5} = -9375e^{-5}$

$y = 0$ $\qquad$ $y = -9375e^{-5}$

Ex. 6

Find all horizontal tangent lines to

$$f(x) = 4x^3\ln(x)$$

Solution:

First find $f'(x)$.

$$f(x) = \underbrace{4x^3}_{F}\underbrace{\ln(x)}_{G}$$

$$f'(x) = \underbrace{12x^2}_{F'}\cdot\underbrace{\ln(x)}_{G} + \underbrace{4x^3}_{F}\cdot\underbrace{\frac{1}{x}}_{G'}$$

$$f'(x) = 12x^2\ln(x) + 4x^2$$

$f'(x) = 4x^2(3\ln(x) + 1)$

Now solve $f'(x) = 0$. (Why? Horizontal lines have slope 0.)

$$4x^2(3\ln(x) + 1) = 0$$

$4x^2 = 0$ or $3\ln(x) + 1 = 0$

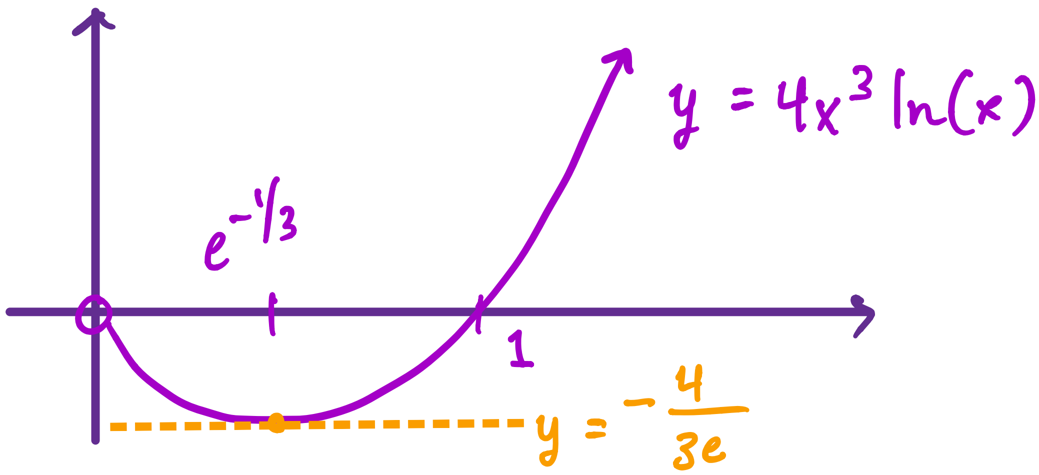
$\cancel{x = 0}$ or $x = e^{-1/3}$

domain of f is: $(0, \infty)$

A horizontal tangent line occurs at $x = e^{-1/3}$ only. Its equation is:

$$y = f(e^{-1/3}) = 4(e^{-1/3})^3 \ln(e^{-1/3}) = -\frac{4}{3e}$$

$$f(x) = 4x^3\ln(x)$$



Section 3.6: Derivatives as Rates of Change

$\frac{dy}{dx}$?

- rate of change of y with respect to x
- $\frac{\Delta y}{\Delta x}$ is approximation of $\frac{dy}{dx}$
- if x changes by Δx, then y changes by $\Delta y = \left(\frac{\Delta y}{\Delta x}\right)\Delta x \approx f'(a)\,\Delta x$

One-dimensional Motion

$x(t)$: position

$v_{av} = \dfrac{x(a+\Delta t) - x(a)}{\Delta t}$: average velocity over $[a, a+\Delta t]$

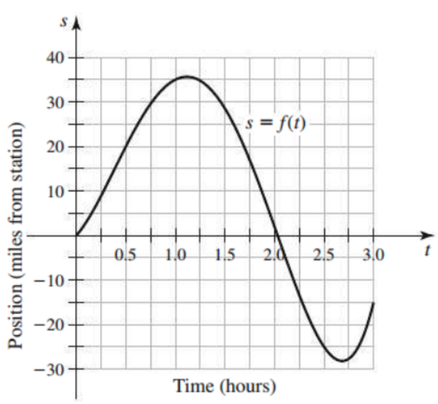
$v(t) = \dfrac{dx}{dt}$: (instantaneous) velocity

$|v(t)| = \left|\dfrac{dx}{dt}\right|$: speed

$a(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$: acceleration

Ex. 1

Graph shows position of patrol car relative to station ($s=0$). Car initially heads north at 9:00am. Time t is measured in hours since 9:00am.



(a) What is the average velocity in the first 45 minutes?

(b) What is the average velocity on $[0.25, 0.75]$? Is this a good estimate of the velocity at 9:30am?

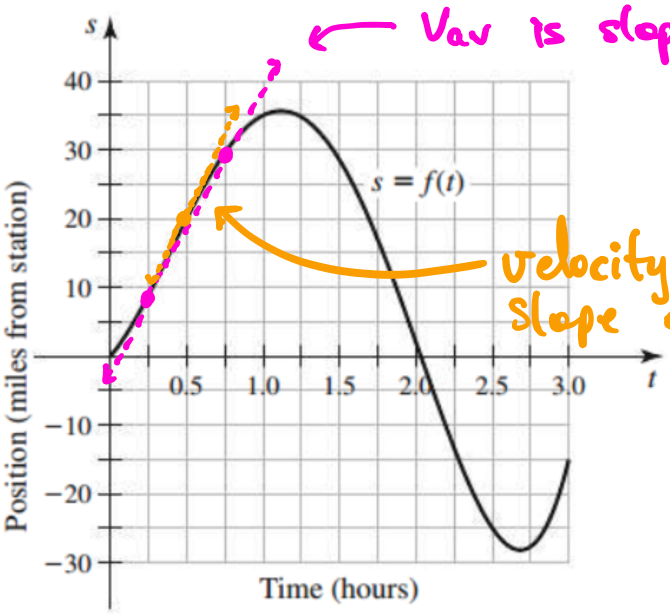
(c) What is the velocity at 11am? In which direction is the car moving?

(d) Describe the motion of the car from 9am to noon.

Solution:

(a) $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(0.75) - s(0)}{0.75 - 0} = \dfrac{30 - 0}{0.75} = 40$ mph

(b) $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(0.75) - s(0.25)}{0.75 - 0.25} = \dfrac{30 - 8}{0.5} = 44$ mph

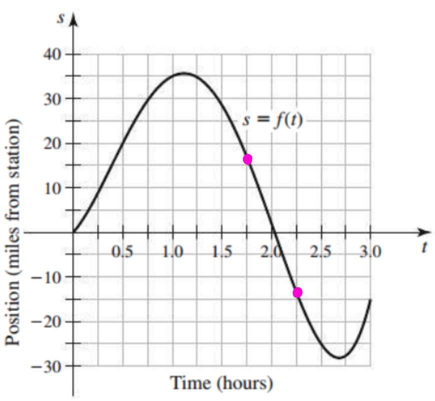


V_{av} is slope of secant line

velocity at 9:30am is slope of tangent line

Yes. 44 mph is good estimate of velocity at 9:30am.

(c) The velocity at 11am is the slope of the tangent line at $t = 2$. We will use the average velocity on $[1.75, 2.25]$ to estimate $v(2)$.

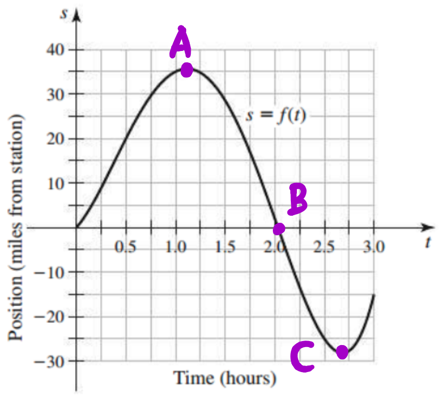


$$v(2) \approx V_{av} = \frac{s(2.25) - s(1.75)}{2.25 - 1.75}$$

$$= \frac{(-13) - (17)}{0.5} = -60 \text{ mph}$$

Since $-60 < 0$, the car is headed back to the station (south).

(d) The car initially moves away from station (north). At about 10:05am (A) the car reaches it max distance from the station (35 miles) and then reverses direction (south). At about 11:05am (B) the car passes the station. At about 11:40am (C) the car reaches its max distance from the station (28 miles). Then it reverses direction again.



Ex. 2

A stone is thrown vertically from a 48-ft cliff with initial velocity of 32 ft/s. The height of the stone is:

$$h(t) = -16t^2 + 32t + 48$$

(a) What is the velocity after t seconds?
(b) When does stone reach max height?
(c) What is the max height?
(d) When does it hit the ground?
(e) What is the impact velocity?
(f) When is the speed increasing?

Solution:

(a) $v(t) = h'(t) = -32t + 32$ $\leftarrow$ ft/s

(b) $v(t) = 0 \Rightarrow t = 1$ $\leftarrow$ sec.

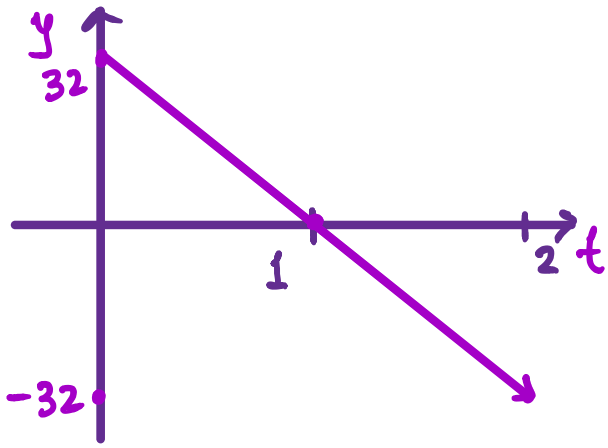
(c) $h(1) = -16 + 32 + 48 = 64$ $\leftarrow$ ft.

(d) $h(t) = 0$

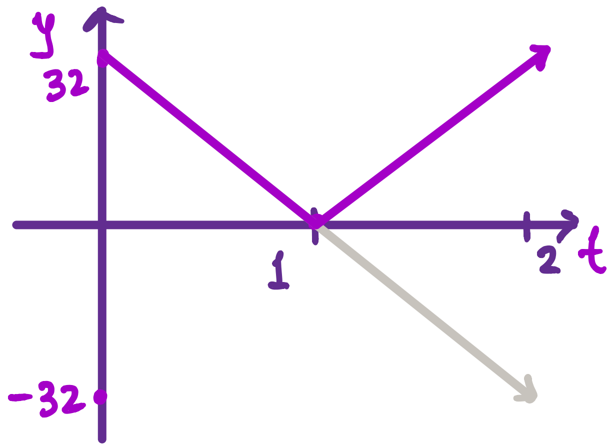
$$-16t^2 + 32t + 48 = 0$$
$$-16(t^2 - 2t - 3) = 0$$
$$-16(t-3)(t+1) = 0$$
$$t = 3 \quad \text{or} \quad \cancel{t = -1}$$

(e) $v(3) = -32 \cdot 3 + 32 = -64$ ft/s

(f)



$y = v(t)$
$(y = -32t + 32)$
velocity



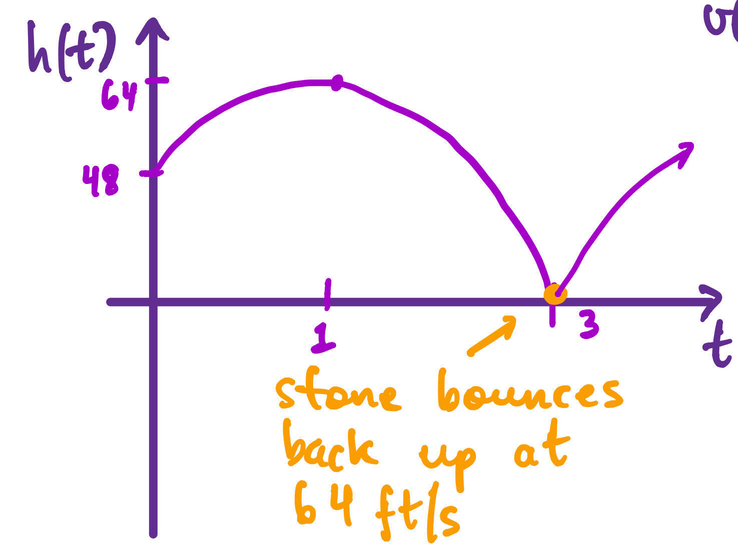
$y = |v(t)|$
$(y = |-32t + 32|)$
Speed

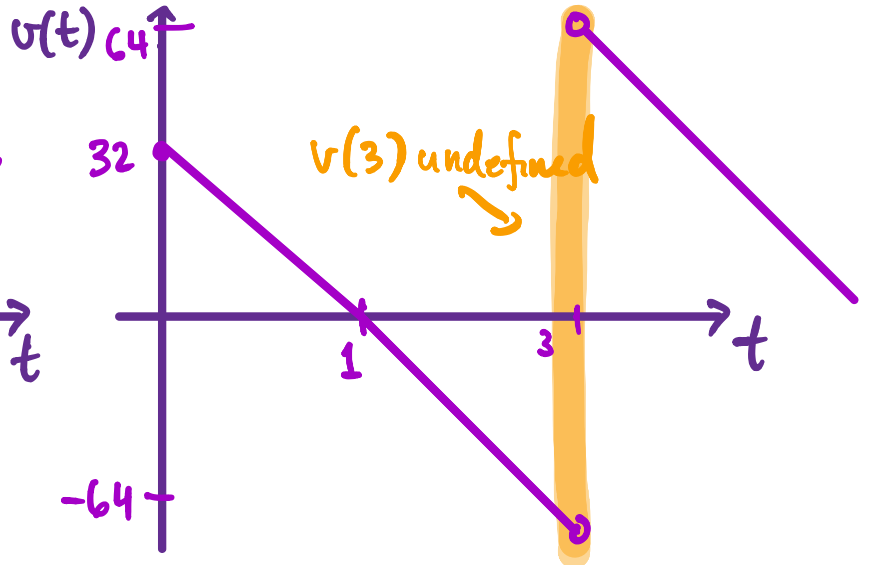
The speed is increasing on $(1, 3)$.
(The velocity is increasing on no interval.)

Ex. 3

Same as Ex. 2, but when stone reaches the ground it hits a spring elastically, bouncing back up with the same speed. Graph the height $h(t)$ and velocity $v(t)$.

<u>Solution:</u>





Ex.4

A coffee vendor has collected data on the price of coffee in her store over the last year. The price of coffee t weeks since when the data collection began was

$$p(t) = 0.02t^2 - 0.1t + 6$$

dollars per pound.

For each part, you must give correct units as part of your answer.

(a) How much did the price of one pound of coffee increase in the first ten weeks after the data collection began?

(b) What was the average rate at which the price of one pound of coffee changed over the same ten-week period mentioned in part (a)?

The vendor also found that, in a given week, the local consumers bought approximately

$$D(p) = \frac{2500}{p^2+1}$$

pounds of coffee when the price was p dollars per pound. That is, D is the *weekly demand* of the consumers.

(c) Calculate $D'(7)$ and explain its precise meaning in the given context.

(d) At what rate was the weekly demand for coffee changing with respect to time exactly ten weeks after data collection began?

Solution:

(a) $\Delta p = p(10) - p(0) = 7 - 6 = 1$ dollar

(b) $\dfrac{\Delta p}{\Delta t} = \dfrac{1}{10}$ $\dfrac{\text{dollars}}{\text{week}}$

(c) $D'(p) = -\dfrac{5000p}{(p^2+1)^2}$

$$D'(7) = \frac{-5000 \cdot 7}{(49+1)^2} = -14 \ \frac{\text{pounds}}{\text{dollar}}$$

When the price is \$7 per pound, the demand is decreasing by 14 pounds/dollar. In other words, if this rate were constant, each \$1 increase in the price would lead to a 14-pound decrease in demand.

(d)

$$\underset{\text{demand}}{D} \xrightarrow{\text{depends on}} \underset{\text{price}}{P} \xrightarrow{\text{depends on}} \underset{\text{time}}{t}$$

We use chain rule:

$$\left.\frac{dD}{dt}\right|_{t=10} = \left.\frac{d}{dt}\Big(D(p(t))\Big)\right|_{t=10}$$

$$= \left. D'(p(t))\, p'(t) \right|_{t=10}$$

$$= D'(p(10))\, p'(10)$$

$$= D'(7) \cdot p'(10)$$

$$= (-14)(0.3) = -4.2 \ \frac{\text{pounds}}{\text{week}}$$

$$\boxed{\begin{array}{l} \text{Recall:} \\ p(t) = 0.02t^2 - 0.1t + 6 \\ p(10) = 7 \\ p'(t) = 0.04t - 0.1 \\ p'(10) = 0.3 \end{array}}$$

Section 3.7: Chain Rule

How do we differentiate....

$f(x) = \sin(x)$ $\qquad f'(x) = \cos(x)$

$f(x) = 2\sin(x)$ $\qquad f'(x) = 2\cos(x)$

$f(x) = 2x\sin(x)$ $\qquad f'(x) = 2\sin(x) + 2x\cos(x)$

$f(x) = \sin(2x)$ $\qquad f'(x) = ???$

<u>Thm</u>: (Chain Rule)

If f and g are differentiable, then

$$\frac{d}{dx}\left[f(g(x))\right] = \underbrace{f'}_{\text{derivative of outside}}(\underbrace{g(x)}_{\text{evaluated at inside}})\underbrace{g'(x)}_{\text{derivative of inside}}$$

Ex. 1

Calculate $\frac{d}{dx}\left(\sin(x^2)\right)$ and $\frac{d}{dx}\left(\sin(x)^2\right)$.

<u>Solution</u>:

$h(x) = \sin(x^2)$

Outside: $f(x) = \sin(x)$

Inside: $g(x) = x^2$

$h'(x) = \cos(x^2)\cdot 2x$

$h(x) = \sin(x)^2$

Outside: $f(x) = x^2$

Inside: $g(x) = \sin(x)$

$h'(x) = 2\sin(x)^1\cdot\cos(x)$

Ex. 2

Calculate $\frac{d}{dx}\left(e^{\tan(x)}\right)$ and $\frac{d}{dx}\left(\ln(x^3+x)\right)$.

<u>Solution</u>:

$h(x) = e^{\tan(x)}$

Outside : $f(x) = e^{x}$

Inside : $g(x) = \tan(x)$

$h'(x) = e^{\tan(x)} \cdot \sec(x)^2$

$h(x) = \ln(x^3+x)$

Outside: $f(x) = \ln(x)$

Inside: $g(x) = x^3+x$

$h'(x) = \frac{1}{x^3+x} \cdot (3x^2+1)$

$\boxed{\text{Ex. 3}}$

Calculate $\frac{d}{dx}\left(\sin(e^x)\cos(3x)\right)$.

Solution:

Use product rule, then chain rule on each answer.

$$h(x) = \underbrace{\sin(e^x)}_{f}\underbrace{\cos(3x)}_{g}$$

$$h'(x) = \underbrace{\cos(e^x)\cdot e^x}_{f'}\underbrace{\cos(3x)}_{g} + \underbrace{\sin(e^x)}_{f}\cdot\underbrace{(-\sin(3x))\cdot 3}_{g'}$$

$\boxed{\text{Ex. 4}}$

Calculate $\frac{d}{dx}\left(\sqrt{\frac{x^3}{1-x}}\right)$.

Solution:

Use chain rule first.

Outside: $x^{1/2}$

Inside: $\frac{x^3}{1-x}$

$$h'(x) = \frac{1}{2}\left(\frac{x^3}{1-x}\right)^{-1/2} \cdot \underbrace{\frac{3x^2\cdot(1-x) - x^3\cdot(-1)}{(1-x)^2}}_{\text{derivative of inside}}$$

Note: The main stencil you use is very

sensitive to how you write h (x).

$$h(x) = \left(\frac{x^3}{1-x}\right)^{1/2} = \frac{x^{3/2}}{(1-x)^{1/2}} = x^{3/2}(1-x)^{-1/2}$$

Chain Rule Quotient Rule Product Rule

Ex. 5

Calculate $\frac{d}{dx}\left[\ln\left(\tan\left(e^{3x-5}\right)\right)\right]$.

Solution:

We use chain rule multiple times.

$$h(x) = \ln\left(\tan\left(e^{3x-5}\right)\right)$$

$$h'(x) = \frac{1}{\tan\left(e^{3x-5}\right)} \cdot \sec^2\left(e^{3x-5}\right) \cdot e^{3x-5} \cdot 3$$

Ex.6

Let $f(x) = x\sqrt{1-3x}$. Find all horizontal tangent lines.

Solution:

First find $f'(x)$.

$$f(x) = x \cdot (1-3x)^{1/2}$$

$$f'(x) = 1 \cdot (1-3x)^{1/2} + x \cdot \frac{1}{2}(1-3x)^{-1/2} \cdot (-3)$$

Now solve $f'(x) = 0$.

$$(1-3x)^{1/2}\left((1-3x)^{1/2} - \frac{3}{2}x(1-3x)^{-1/2}\right) = 0 \cdot (1-3x)^{1/2}$$

$$(1-3x)^{1} - \frac{3}{2}x = 0$$

$$1 - \frac{9}{2}x = 0$$

$$x = \frac{2}{9}$$ ← Check: is $x = \frac{2}{9}$ in domain of f? Yes.

So the only horizontal tangent is:

$$y = f\left(\tfrac{2}{9}\right) = \left(x\sqrt{1-3x}\right)\Big|_{x=2/9} = \frac{2}{9\sqrt{3}}$$

$\boxed{\text{Ex. 7}}$

Values of f, f', g, and g' are in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	-1	-4	4	2
1	-1	-3	2	-4
2	-4	3	1	-1

(a) Let $F(x) = \dfrac{f(x)}{g(x)}$.

Calculate $F'(0)$.

(b) Let $G(x) = f(x\,g(x))$.

Calculate $G'(1)$.

Solution:

(a) First find $F'(x)$. Use Quotient Rule.

$$F'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

Now put $x = 0$.

$$F'(0) = \frac{(-4)(4) - (-1)(2)}{(4)^2} = -\frac{7}{8}$$

(b) First find $G'(x)$. Use Chain Rule.

$$G(x) = f(x\,g(x))$$

Outside: $f(x)$ Inside: $x\,g(x)$

$$G'(x) = \underbrace{f'(xg(x))}_{\substack{\text{derivative of} \\ \text{outside evaluated} \\ \text{at inside}}} \cdot \underbrace{(1 \cdot g(x) + x \cdot g'(x))}_{\substack{\text{derivative of inside} \\ \text{(product rule)}}}$$

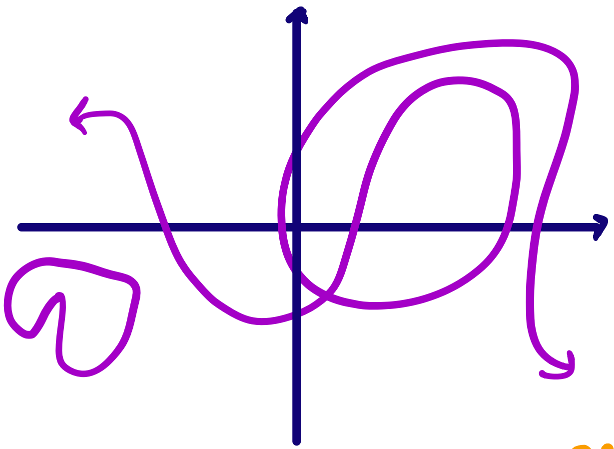
Now put $x = 1$.

$$G'(1) = f'(g(1)) \cdot (g(1) + g'(1))$$

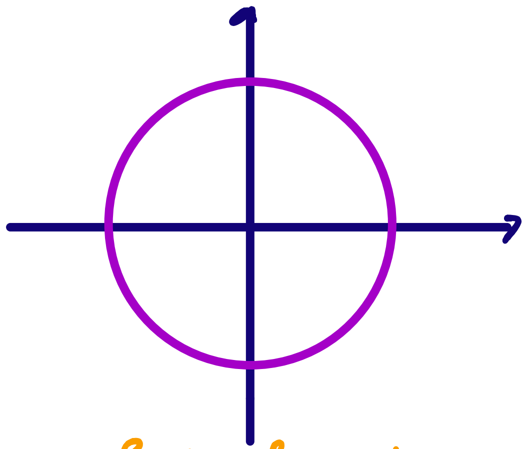
$$= f'(2) \cdot (2 + (-4))$$

$$= (3)(-2) = -6$$

Section 3.8 : Implicit Differentiation

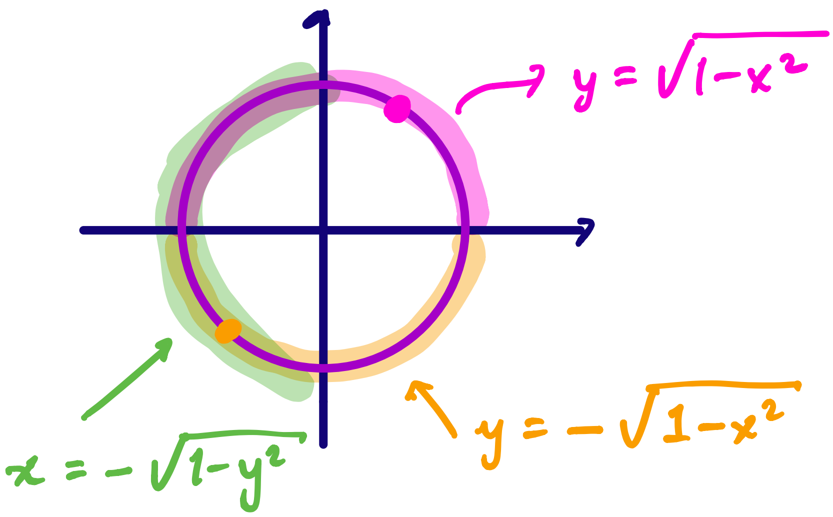


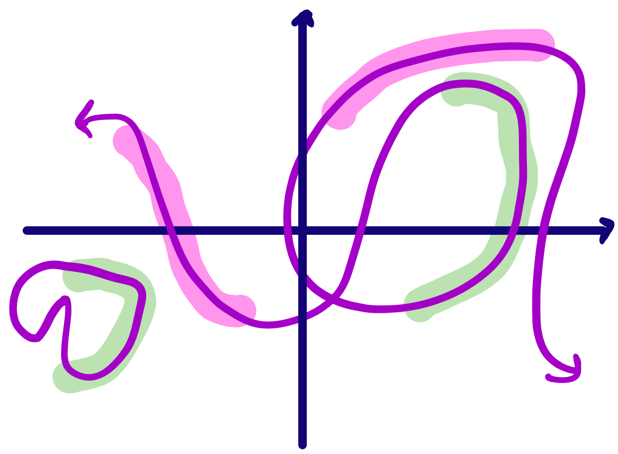
$x^3 + 2xy + y^4 = 3$??



$x^2 + y^2 = 1$

Sometimes x and y are related by an equation but we cannot solve for one as a function of the other. These equations define local, implicit functions.





Even though we have no hope of finding an explicit formulas for these functions, we can still do calculus.

Ex. 1

Suppose $x^2 + y^2 = 1$. Find tangent line to $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Solution:

Note: We make no attempt to solve for x or y!

But $y = f(x)$ locally!

$$x^2 + f(x)^2 = 1 \qquad (1)$$

The slope of the tangent line is $\frac{dy}{dx}$, or $f'(x)$. We can find $f'(x)$ from (1) by simply differentiating all terms and solving for $f'(x)$.

$$2x + \underbrace{2f(x)f'(x)}_{} = 0 \qquad (2)$$

Chain Rule!

Scratch Work

- $g(x) = [f(x)]^2$

Outside: x^2 Inside: $f(x)$

$g'(x) = 2f(x) \cdot f'(x)$

We will typically write (2) as:

$$2x + 2y\frac{dy}{dx} = 0 \qquad (2)^*$$

Now solve algebraically for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{x}{y}$$

For implicit functions, okay for dy/dx to depend on x and y.

Now for the tangent line.

Point: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Slope: $-\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$

Equation: $y = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

Ex. 2

Suppose x and y are implicitly related by:

$$x^2 + 3y^2 + xy = 10$$

Find $\frac{dy}{dx}$ for a given point on this curve.

Solution:

Use implicit differentiation wrt. x.

$$x^2 + 3y^2 + \underbrace{x}_{f}\underbrace{y}_{g} = 10$$

Product rule

$$2x + 6y\frac{dy}{dx} + \underbrace{1}_{f'} \cdot \underbrace{y}_{g} + \underbrace{x}_{f} \cdot \underbrace{\frac{dy}{dx}}_{g'} = 0$$

Now algebraically solve for dy/dx.

$$6y\frac{dy}{dx} + x\frac{dy}{dx} = -2x - y$$

$$(6y + x)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{6y + x}$$

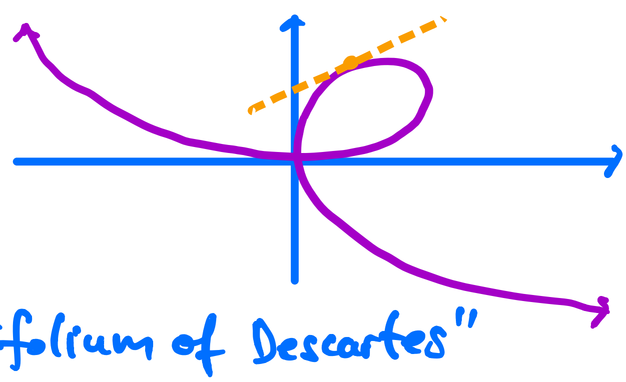
algebra, no calculus

Ex. 3

Find an equation of the line tangent to the graph of

$$x^3 + y^3 = 3xy$$

at $(2/3, 4/3)$.



"folium of Descartes"

Solution:

Use implicit differentiation.

$$x^3 + \underbrace{y^3}_{\text{Chain}} = \underbrace{3xy}_{\text{product}}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

Substitute point and solve for dy/dx.

$$3\left(\frac{2}{3}\right)^2 + 3\left(\frac{4}{3}\right)^2\frac{dy}{dx} = 3\left(\frac{4}{3}\right) + 3\left(\frac{2}{3}\right)\frac{dy}{dx}$$

$$\frac{4}{3} + \frac{16}{3}\frac{dy}{dx} = 4 + 2\frac{dy}{dx}$$

$$4 + 16\frac{dy}{dx} = 12 + 6\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4}{5}$$

So the equation of the tangent line is:

$$y = \frac{4}{3} + \frac{4}{5}\left(x - \frac{2}{3}\right)$$

$\boxed{\text{Ex. 4}}$

Let $f(x) = x^x$. Find $f'(x)$.

Solution:

- Power Rule? No! Exponent not constant
- Exponential Rule? No! Base not constant

No rule has x in both base and exponent.

$\boxed{\text{Method 1}}$ (Rewrite function)

$$f(x) = x^x = e^{\ln(x^x)} = \underbrace{e^{x\ln(x)}}_{\text{Chain Rule!}}$$

Outside: e^x

Inside: $x\ln(x)$

$$f'(x) = e^{x\ln(x)} \cdot \left(1 \cdot \ln(x) + x \cdot \frac{1}{x}\right) = x^x(\ln(x)+1)$$

$\boxed{\text{Method 2}}$ (Logarithmic differentiation)

original $\qquad y = x^x$

log of both sides $\qquad \ln(y) = \ln(x^x)$

log rules $\qquad \ln(y) = x\ln(x)$

implicit diff. $\qquad \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln(x) + x \cdot \frac{1}{x}$

solve for $\frac{dy}{dx}$. $\qquad \frac{dy}{dx} = y(\ln(x)+1)$

replace y $\qquad f'(x) = x^x(\ln(x)+1)$

$\boxed{\text{Ex. 5}}$

Suppose $\sin(x+y) = x + \cos(y)$. Find $\frac{dy}{dx}$.

Solution:

Use implicit differentiation.

$$\cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right) = 1 - \sin(y)\frac{dy}{dx}$$

Now algebraically solve for dy/dx.

$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = 1 - \sin(y)\frac{dy}{dx}$$

$$\left(\cos(x+y) + \sin(y)\right)\frac{dy}{dx} = 1 - \cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{1 - \cos(x+y)}{\cos(x+y) + \sin(y)}}$$

Ex. 6

Suppose $\ln(1+xy) = x^2 + y$. Find $\frac{dy}{dx}$.

Solution:

Use implicit differentiation:

$$\frac{1}{1+xy} \cdot \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) = 2x + \frac{dy}{dx}$$

Now algebraically solve for $\frac{dy}{dx}$.

$$y + x\frac{dy}{dx} = \left(2x + \frac{dy}{dx}\right)(1+xy)$$

$$y + x\frac{dy}{dx} = 2x(1+xy) + (1+xy)\frac{dy}{dx}$$

$$x\frac{dy}{dx} - (1+xy)\frac{dy}{dx} = 2x(1+xy) - y$$

$$(x - 1 - xy)\frac{dy}{dx} = 2x + 2x^2y - y$$

$$\boxed{\frac{dy}{dx} = \frac{2x + 2x^2y - y}{x - 1 - xy}}$$

Section 3.11 : Related Rates

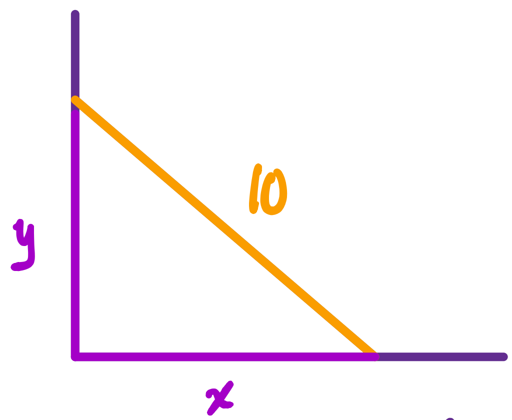
$\frac{dy}{dt}$: rate of change of y with respect to time

Note: All variables are assumed functions of time

Ex. 1

A ladder of length of $l = 10$ ft is leaning against a wall. Suppose the bottom of the ladder slides away from the wall at 2 ft/sec. How fast is the top of the ladder sliding down the wall when the top is 8 ft from the ground?

Solution:



Given Information

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = ?$ when $y = 8$

for all t: $x^2 + y^2 = 100 \quad (1)$

To introduce $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we use implicit diff. with respect to time.

$$x(t)^2 + y(t)^2 = 100$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad (2)$$

Equations (1) and (2) hold for all time t. Now we substitute information specific to described time

Hold for all t

(1) $x^2+y^2=100$

(2) $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

Substitute $\frac{dx}{dt}=2$ and $y=8$

Hold for specific time

$(1)^*$ $x^2+64=100$

$(2)^*$ $4x+16\frac{dy}{dt}=0$

Solve for $\frac{dy}{dt}$

From $(1)^*$, we get $x=6$. Then from $(2)^*$, we get

$$24+16\frac{dy}{dt}=0 \Longrightarrow \frac{dy}{dt}=-\frac{3}{2}\ (\text{ft./sec.})$$

The top of the ladder is sliding down at 1.5 ft/sec.

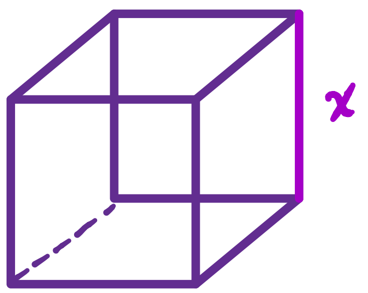
General Strategy for Related Rates

1. Draw diagram and label variables. (Distinguish between constants and non-constants.)
2. Gather all given information, including equations that relate the variables <u>for all time</u>.
3. Implicity differentiate the relations wrt. t.
4. Substitute all given values.
5. Solve for desired value (correct units!)

Ex. 2

The total surface area of a cube is changing at a rate of 12 in^2/sec. when the length of one of its sides is 10 in. At what rate is the volume of the cube changing at that time?

Solution:



Given Information

$\frac{dS}{dt} = 12$ $\quad \frac{dV}{dt} = ?$ $\quad$ when $x = 10$

$V = x^3, \quad S = 6x^2$

Now we implicitly differentiate:

(1) $V = x^3$ $\qquad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$ (3)

(2) $S = 6x^2$ $\qquad \frac{dS}{dt} = 12x \frac{dx}{dt}$ (4)

Now substitute given values ($x = 10$ and $\frac{dS}{dt} = 12$):

(1)* $V = 1000$ $\qquad \frac{dV}{dt} = 300 \frac{dx}{dt}$ (3)*

(2)* $S = 600$ $\qquad$ Solve for this $\qquad 12 = 120 \frac{dx}{dt}$ (4)*

From (4)*, we get $\frac{dx}{dt} = \frac{1}{10}$. So then from (3)*:

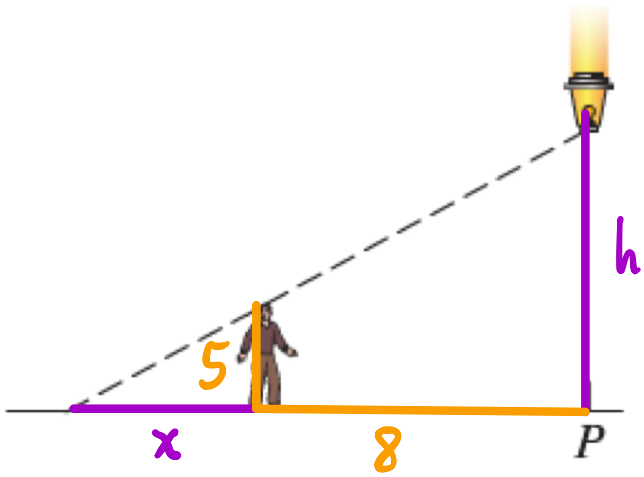
$$\frac{dV}{dt} = 300 \cdot \left(\frac{1}{10}\right) = 30 \text{ in}^3/\text{sec}.$$

Ex. 3

A 5ft-tall person stands still 8 feet from point P, which is directly below a lantern. At the moment when the lantern is 15 feet above the ground, the lantern is falling at a rate of 4 ft./sec.

At what rate is the length of the person's shadow changing at that moment?

Solution:

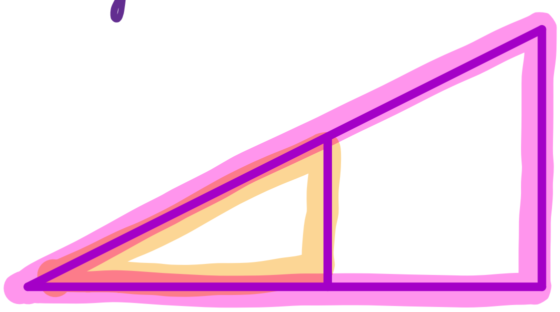


Given Information

$$\frac{dh}{dt} = -4$$

$$\frac{dx}{dt} = ? \text{ when } h = 15$$

To find an equation for x and h, we use similar triangles.



$$\frac{\text{Large length}}{\text{Large height}} = \frac{\text{small length}}{\text{small height}}$$

$$\frac{x+8}{h} = \frac{x}{5}$$

Rearranging the equation gives:

$$h = 5 + \frac{40}{x} \qquad (1)$$

Now differentiate:

$$\frac{dh}{dt} = -\frac{40}{x^2} \cdot \frac{dx}{dt} \qquad (2)$$

Now substitute $dh/dt = -4$ and $h = 15$.

$(1)^*$ $15 = 5 + \frac{40}{x}$ $\qquad$ $-4 = -\frac{40}{x^2} \cdot \frac{dx}{dt}$ $(2)^*$

From $(1)^*$ we get $x = 4$. Then from $(2)^*$ we get.

$$-4 = \frac{-40}{16} \cdot \frac{dx}{dt} \implies \frac{dx}{dt} = \frac{8}{5} \text{ ft./sec.}$$

Section 4.1: Extreme Values

Ex. 1

Find the abs. extrema of $f(x) = x^3 - 6x^2 + 8$ on $[1,6]$.

Solution:

(Note: f is continuous and $[1,6]$ is closed & bounded.)

First find critical numbers of f:

$$f'(x) = 3x^2 - 12x = 3x(x-4)$$

- $f'(x)$ dne: none
- $f'(x) = 0$: $\cancel{x=0}$, $x = 4$

  not in $[1,6]$

Now construct list of candidate values.

	x	$y = x^3 - 6x^2 + 8 = x^2(x-6) + 8$
critical #	4	$16(-2) + 8 = -24$
endpoints	1	$1(-5) + 8 = 3$
	6	$36(0) + 8 = 8$

The abs. min. is -24 and the abs. max. is 8.

Ex. 2

Find the abs. extrema of $f(x) = (x^2 - 16)^{2/3} + 20$ on $[-5, 5]$.

Solution:

First find the critical numbers:

$$f'(x) = \frac{2}{3}(x^2 - 16)^{-1/3} \cdot 2x = \frac{4x}{3(x^2-16)^{1/3}}$$

- $f'(x)$ dne: $x^2 - 16 = 0 \Longrightarrow x = -4$ or $x = 4$
- $f'(x) = 0$: $x = 0$

Now we make a table of values:

x	$y = (x^2-16)^{2/3}+20$
critical # -4	20
4	20
0	$(-16)^{2/3}+20 = 16^{2/3}+20$
endpoints -5	$9^{2/3}+20$
5	$9^{2/3}+20$

The abs. min. is 20 and the abs. max is $16^{2/3}+20$.

$\boxed{\text{Ex. 3}}$

Find abs. extrema of $f(x) = x - \dfrac{4x}{x+1}$ on $[0,3]$.

Solution:

Find the critical #'s:

$$f'(x) = 1 - \frac{4}{(x+1)^2}$$

• $f'(x)$ dne: none

• $f'(x) = 0$: $1 - \dfrac{4}{(x+1)^2} = 0 \Rightarrow (x+1)^2 = 4$

$$\Rightarrow \cancel{x = -3} \text{ or } x = 1$$

not in $[0,3]$

Now make a table of values:

x	$y = x - \dfrac{4x}{x+1}$
critical # 1	$1 - \frac{4}{2} = -1$

endpoints	0	0
	3	$3-\frac{12}{4} = 0$

The abs. min. is -1 and the abs. max is 0.

Ex. 4

Find the absolute extrema of $f(x) = x - \ln(x)$ on $\left[\frac{1}{e^3}, e\right]$. (Hint: $2 < e < 3$.)

Solution:

Find critical numbers:

$$f'(x) = 1 - \frac{1}{x}$$

- $f'(x)$ dne : none
- $f'(x) = 0$: $x = 1$

Now make a table of values:

	x	$y = x - \ln(x)$
Critical #	1	$1 - 0 = 1$
endpoints	$1/e^3$	$\frac{1}{e^3} - (-3) = \frac{1}{e^3} + 3$
	e	$e - 1$

Recall: $2 < e < 3$

The abs. min. is 1 and the abs. max is $\frac{1}{e^3} + 3$.

Ex. 5

Find the abs extrema of $f(x) = e^{-x}\sin(x)$ on $[0, 2\pi]$.

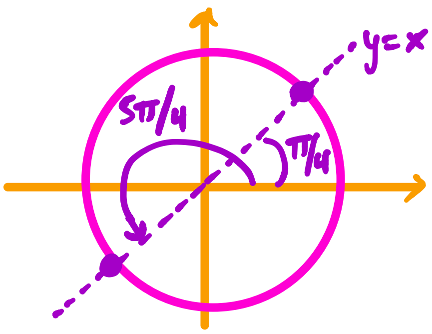
Solution:

Find critical numbers:

$$f'(x) = e^{-x}\cdot(-1)\sin(x) + e^{-x}\cdot\cos(x)$$

$$f'(x) = e^{-x}\left(\cos(x) - \sin(x)\right)$$

- $f'(x)$ dne: none
- $f'(x) = 0$: $\underbrace{e^{-x}}_{\text{always positive, so cancels out}}\left(\cos(x) - \sin(x)\right) = 0$



$$\cos(x) - \sin(x) = 0$$

$$\cos(x) = \sin(x)$$

$$x = \frac{\pi}{4} \quad \text{or} \quad x = \frac{5\pi}{4}$$

Now make a table of values:

	x	$y = e^{-x}\sin(x)$
critical	$\pi/4$	$e^{-\pi/4}\cdot\frac{1}{\sqrt{2}}$
	$5\pi/4$	$e^{-5\pi/4}\cdot\left(-\frac{1}{\sqrt{2}}\right)$
endpoints	0	0
	2π	0

The abs. min. is $\frac{-e^{-5\pi/4}}{\sqrt{2}}$ and the abs. max is $\frac{e^{-\pi/4}}{\sqrt{2}}$.

$\boxed{\text{Ex.6}}$

Find abs. extrema of $f(x) = x^4 e^{-2x}$ on $[-1, 10]$.

Solution:

Find the critical numbers:

$$f'(x) = e^{-2x} \cdot 2x^3 \cdot (2-x)$$

- $f'(x)$ dne: none

- $f'(x) = 0$: $x=0$, $x=2$

Make table of values.

	x	$y = x^4 e^{-2x}$
critical	0	0
	2	$16 \cdot e^{-4} = \frac{16}{e^4} = \left(\frac{2}{e}\right)^4 < 1$
endpoints	-1	$e^2 > 1$
	10	$10000 e^{-20} = \frac{10000}{e^{20}} \approx \frac{10000}{2^{20}}$ very tiny

The abs. min. is 0 and the abs. max is e^2.

Ex. 7

Find abs. extrema of $f(x) = x - 2\sin(x)$ on $[0, 2\pi]$.

Solution:

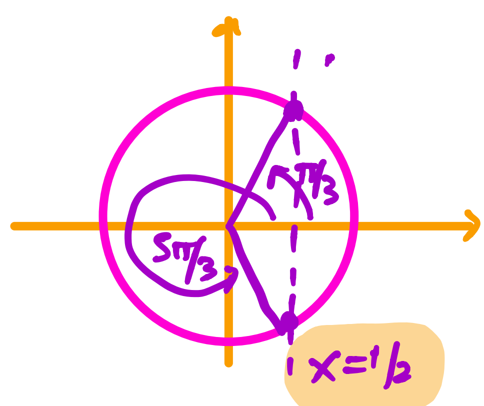
Find the critical numbers.

$$f'(x) = 1 - 2\cos(x)$$

- $f'(x)$ dne: none

- $f'(x) = 0$: $\cos(x) = \frac{1}{2}$

$$x = \frac{\pi}{3}, \quad x = \frac{5\pi}{3}$$



Now make a table of values:

	x	$y = x - 2\sin(x)$	
critical	$\pi/3$	$\frac{\pi}{3} - \sqrt{3}$	$\leftarrow$ only negative #
	$5\pi/3$	$\frac{5\pi}{3} + \sqrt{3}$	
endpoints	0	0	
	2π	2π	

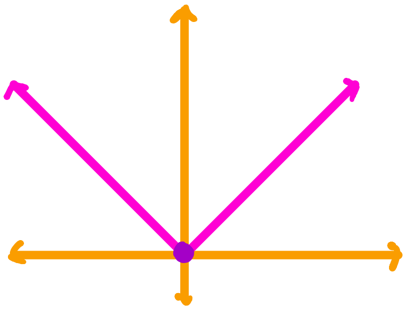
The abs. min. is $\frac{\pi}{3} - \sqrt{3}$ and the abs. max is $\frac{5\pi}{3} + \sqrt{3}$.

Section 4.1 Supplement: Catalog of Non-differentiable Functions

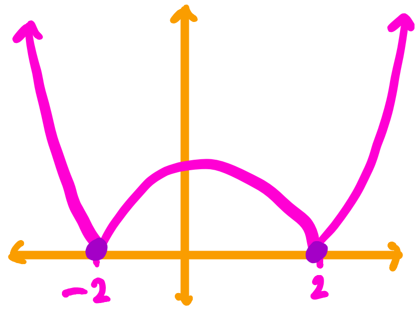
In Section 3.1, we learned how to recognize non-differentiable functions graphically. The notes below describle how to recognize analytically functions that are continuous but not differentiable. There are three main categories:

① Absolute Value: $f(x) = |x|$

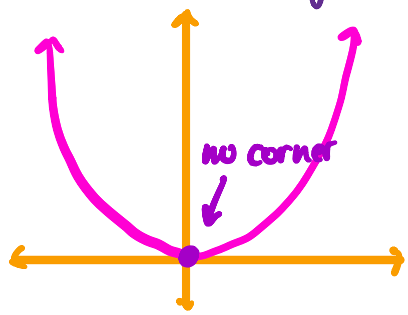
The function f is not differentiable at $x=0$. The graph of f has a sharp corner at $x=0$. So the function $h(x) = |g(x)|$ is (possibly) not differentiable where $g(x)=0$.



$y = |x|$



$y = |x^2-4|$

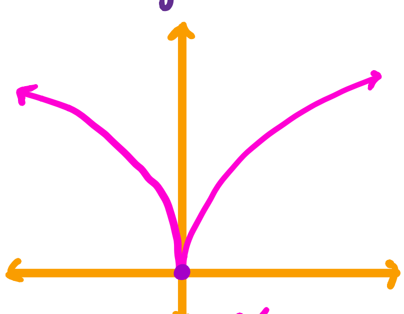


$y = |x^3|$

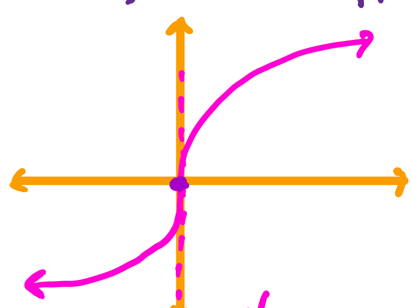
Ex: The function $h(x) = |x^2-4|$ is not differentiable at $x=-2$ and $x=2$. ($x^2-4=0 \iff x=\pm 2$)

② Power Functions: $f(x) = x^n$ $(0<n<1)$

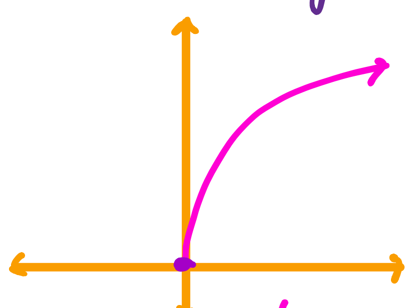
The function f is not differentiable at $x=0$. The graph has a cusp or vertical tangent at $x=0$. So the function $h(x) = g(x)^n$ is (possibly) not differentiable where $g(x)=0$



$y = x^{2/3}$

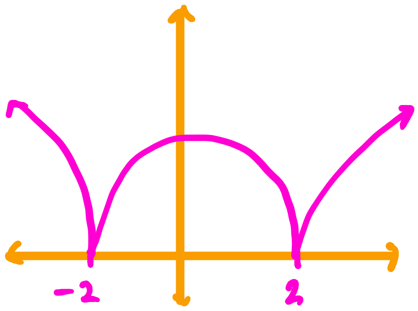


$y = x^{1/3}$

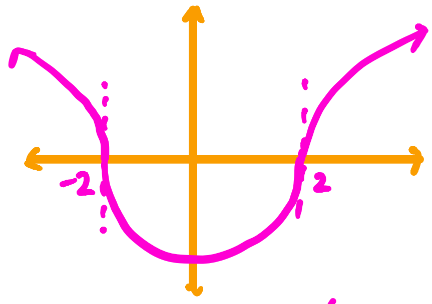


$y = x^{1/2}$

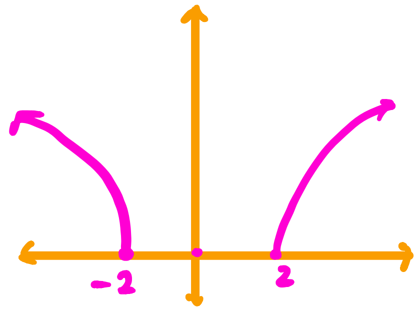
$\boxed{\text{Ex:}}$ The function $h(x) = (x^2-4)^{2/3}$ is not differentiable at $x=-2$ and $x=2$. $(x^2-4=0 \iff x=\pm 2)$. Similarly, both $F(x) = (x^2-4)^{1/3}$ and $G(x) = (x^2-4)^{1/2}$ are not differentiable at $x=-2$ and $x=2$.



$y=(x^2-4)^{2/3}$



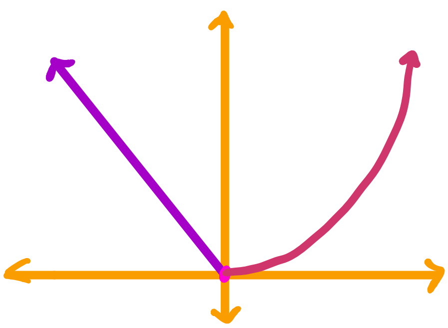
$y=(x^2-4)^{1/3}$



$y=(x^2-4)^{1/2}$

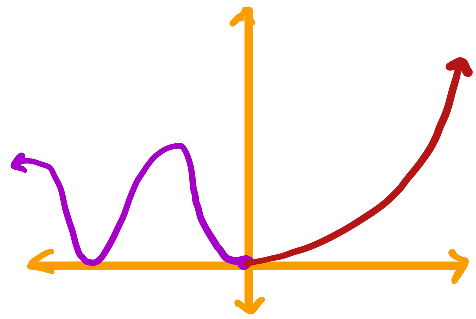
③ Piecewise-defined Functions

Very often (but not always), a piecewise-defined function is not differentiable at the transition points.



$$y=\begin{cases} -x & \text{if } x<0 \\ x^2 & \text{if } x\geq 0 \end{cases}$$

not differentiable at $x=0$



$$y=\begin{cases} 1-\cos(x) & \text{if } x<0 \\ x^2 & \text{if } x\geq 0 \end{cases}$$

differentiable at $x=0$

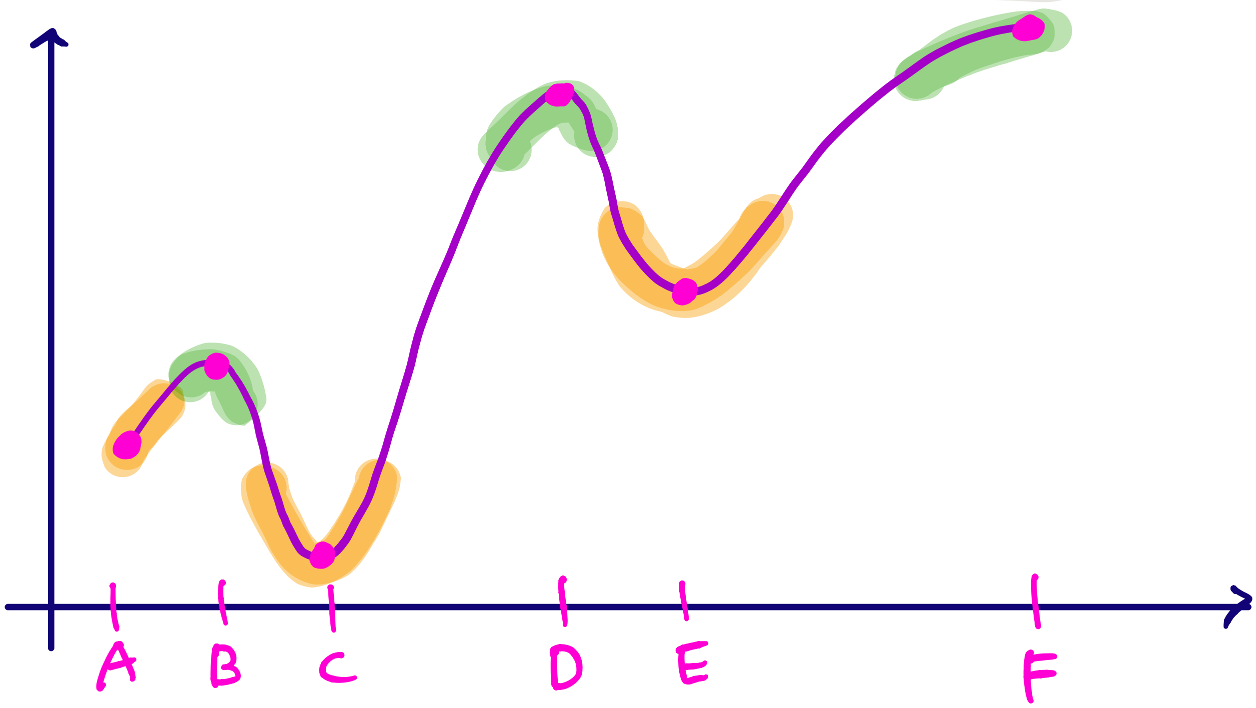
Section 4.1 Supplement: Conceptual Background

Basic Definitions:

Absolute minimum value of f on $[a,b]$	Relative minimum value of f at $x=c$
If $f(c)$ is the abs. min. value of f on $[a,b]$, then $f(c)$ is the least possible value of f for all x in $[a,b]$.	If $f(c)$ is a relative min. value of f, then $f(c)$ is the least possible value of f for x near c.

- similar definitions for absolute maximum and relative maximum (replace "least" with "greatest")
- "global" = "absolute" and "local" = "relative"
- "extremum" means minimum or maximum

Locating local extreme values graphically



Relative Minimum Values: ~~f(A)~~, f(C), f(E)

your textbook does not allow relative extrema at boundary points

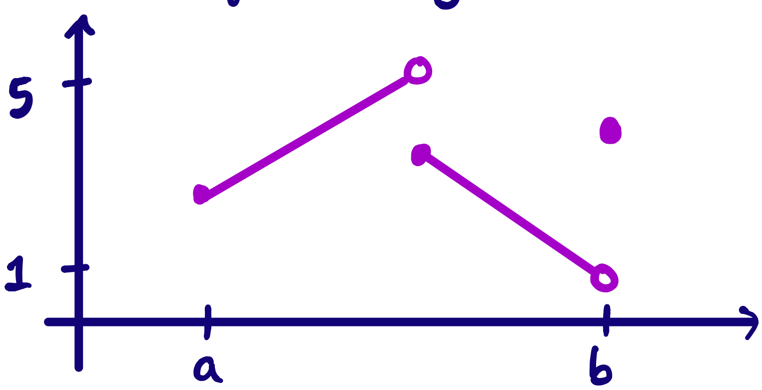
also absolute minimum

Relative Maximum Values: f(B), f(D), ~~f(F)~~

your textbook does allow absolute extrema at boundary points

Thm: (Extreme Value Theorem, EVT)
Suppose f is continuous 1 on the closed 2, bounded 3 interval $[a,b]$. Then the absolute min. and max. of f on $[a,b]$ exist.

What can go wrong if f is not continuous?



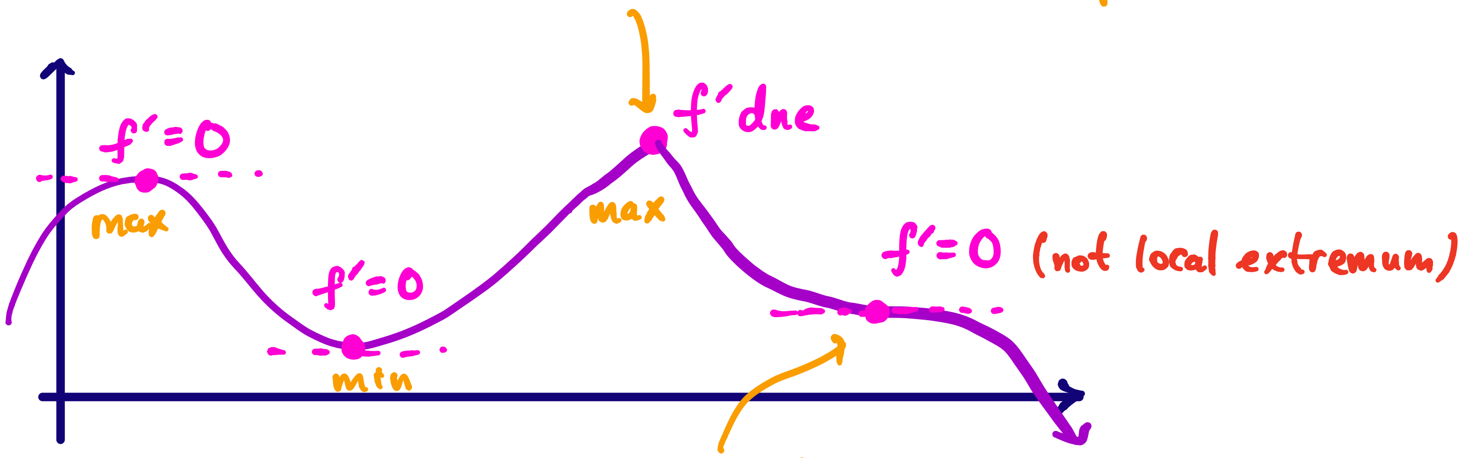
domain: $[a,b]$
no absolute min.
no absolute max.

Def: A number c in the interior of the domain of f is a critical number if $f'(c)$ DNE or $f'(c)=0$.

Thm: (Fermat)
If $f(c)$ is a local extremum, then c is a critical number.

* Why do we need to check where $f'(x)$ dne?

possible for local extremum to occur at corners or cusps



not all critical numbers give rise to local extrema

Algorithm for finding absolute extrema of f on $[a,b]$

① Is f continuous on $[a,b]$?

- If no, stop. (EVT concludes nothing.)
- If yes, continue to ②.

② Find critical numbers of f in (a,b).

- find where $f'(x)$ dne
- solve the equation $f'(x) = 0$

③ Make a table of candidate extreme values of $f(x)$:

- values of f at critical numbers
- values of f at endpoints ($x=a$ and $x=b$

④ Least candidate value is absolute min.
Greatest candidate value is absolute max.

* Discard other values
* Okay for abs. extremum to occur at more than one x-value.

Sections 4.3/4.4 : Graphing functions

Ex. 1

Graph $f(x) = x^3 - 12x^2$ on $[-1, 9]$.

Solution:

$$f(x) = x^3 - 12x^2 = x^2(x-12)$$
$$f'(x) = 3x^2 - 24x = 3x(x-8)$$
$$f''(x) = 6x - 24 = 6(x-4)$$

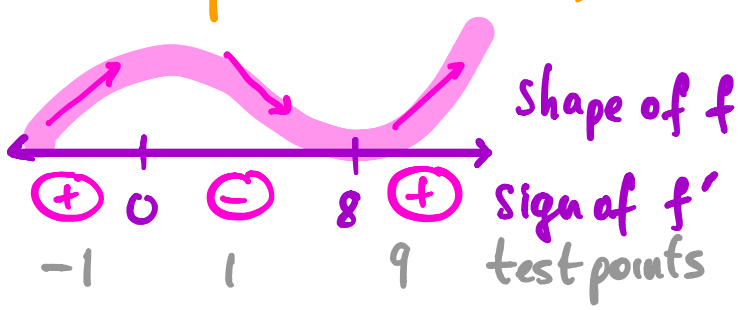
① Information from $f(x)$:

- vertical asymptotes: none
- horizontal asymptotes: none

② Information from $f'(x)$:

- construct sign chart for $f'(x)$:

cut points: $x = 0$, $x = 8$



$f'(x) = 3x(x-8)$

$f'(-1) = ⊖ \quad ⊖ \quad = ⊕$

$f'(1) = ⊕ \quad ⊖ \quad = ⊖$

$f'(9) = ⊕ \quad ⊕ \quad = ⊕$

f is decreasing on: $[0, 8]$

f is increasing on: $(-\infty, 0]$, $[8, \infty)$

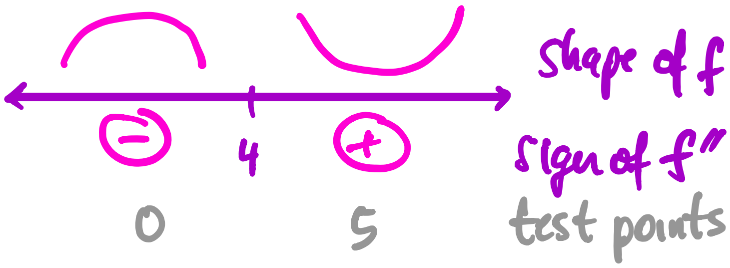
f has a local min @ $x = 8$

f has a local max @ $x = 0$

③ Information from $f''(x)$:

- construct a sign chart for $f''(x)$:

cut points: $x = 4$



$f''(x) = 6(x-4)$

$f''(0) = \ominus$

$f''(5) = \oplus$

f is concave down on: $(-\infty, 4]$

f is concave up on: $[4, \infty)$

f has an inflection pt @ $x = 4$

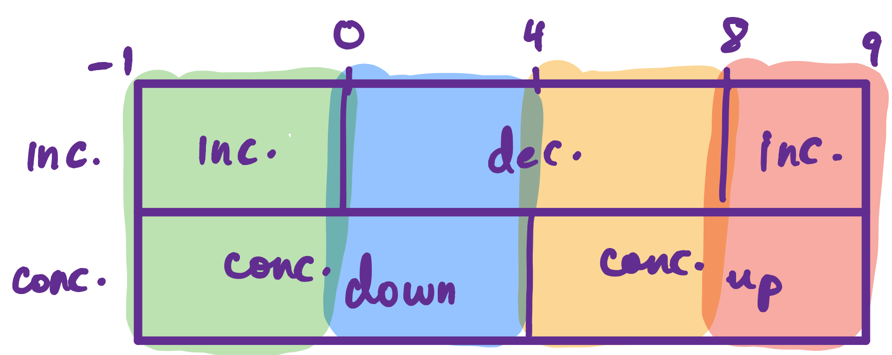
4 Summary and graph

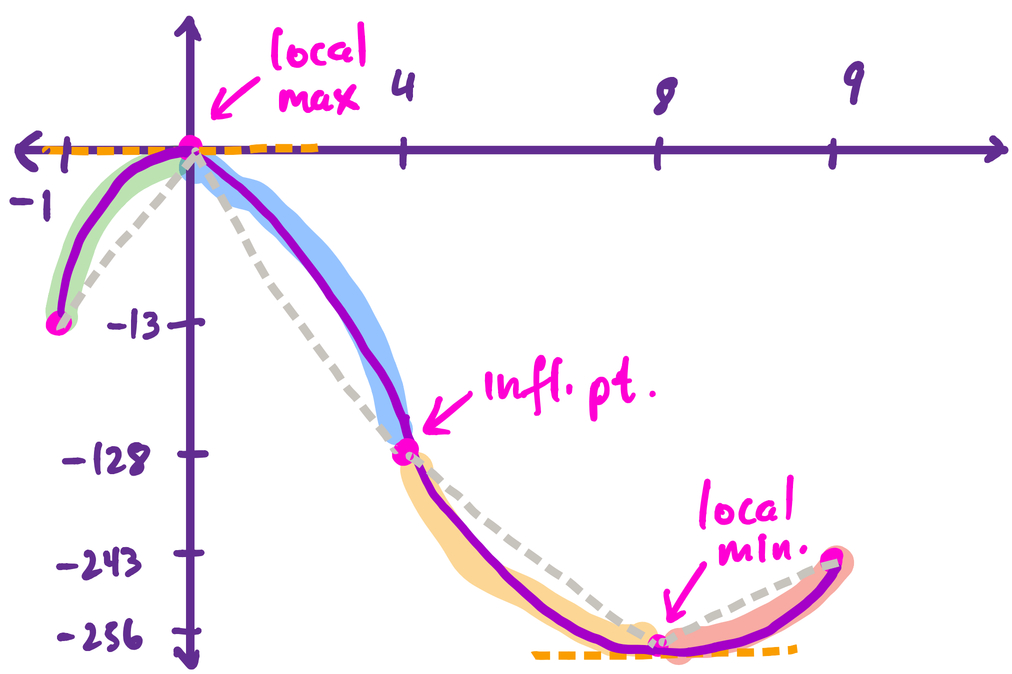
$f(x) = x^2(x-12)$ on $[-1, 9]$

Important Points

x	y	type
-1	-13	endpt.
0	0	local max
4	-128	infl. pt.
8	-256	local min
9	-243	endpt.

Shape of Graph





Ex. 2

Graph $f(x) = \dfrac{x}{x^2-4}$.

Solution:

$$f'(x) = \frac{-(x^2+4)}{(x^2-4)^2} \qquad f''(x) = \frac{12x(x^2+12)}{(x^2-4)^3}$$

① Information from $f(x)$:

- vertical asymptotes: $x=-2$, $x=2$
- horizontal asymptotes: $y=0$

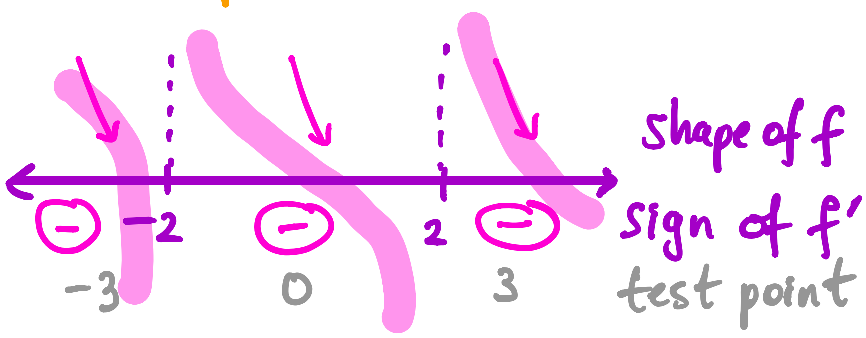
$\frac{-\infty}{\infty}$ $$\lim_{x\to-\infty}\left(\frac{x}{x^2-4}\right) \overset{H}{=} \lim_{x\to-\infty}\left(\frac{1}{2x}\right) = \frac{1}{-\infty} = 0$$

$\frac{\infty}{\infty}$ $$\lim_{x\to\infty}\left(\frac{x}{x^2-4}\right) \overset{H}{=} \lim_{x\to\infty}\left(\frac{1}{2x}\right) = \frac{1}{\infty} = 0$$

② Information from $f'(x)$:

- construct a sign chart for $f'(x)$

cut points: $x=-2$, $x=2$



$$f'(x) = \frac{-(x^2+4)}{(x^2-4)^2}$$

$$f'(-3) = \frac{-}{+} = -$$

$$f'(0) = \frac{-}{+} = -$$

$$f'(3) = \frac{-}{+} = -$$

f is decreasing on: $(-\infty,-2)$, $(-2,2)$, $(2,\infty)$

f is increasing on: $\emptyset$

f has a local min @ nowhere

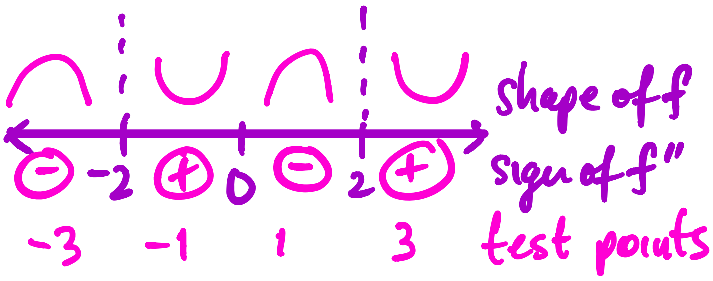
f has a local max @ nowhere

Bonus: $\lim_{x\to-2^+} f(x) = \infty$ (why?)

③ Information from $f''(x)$:

- construct a sign chart for $f''(x)$:

cut points: $x=-2$, $x=0$, $x=2$



$$f''(x) = \frac{12x(x^2+12)}{(x^2-4)^3}$$

$f''(-3) = \frac{(-)(+)}{(+)} = (-)$ $\quad f''(1) = \frac{(+)(+)}{(-)} = (-)$

$f''(-1) = \frac{(-)(+)}{(-)} = (+)$ $\quad f''(3) = \frac{(+)(+)}{(+)} = (+)$

f is concave down on: $(-\infty,-2)$, $[0,2)$

f is concave up on: $(-2,0]$, $(2,\infty)$

f has an infl. pt @ $x=0$

④ Summary and graph:

Important Features:

$(0,0)$: inflection point

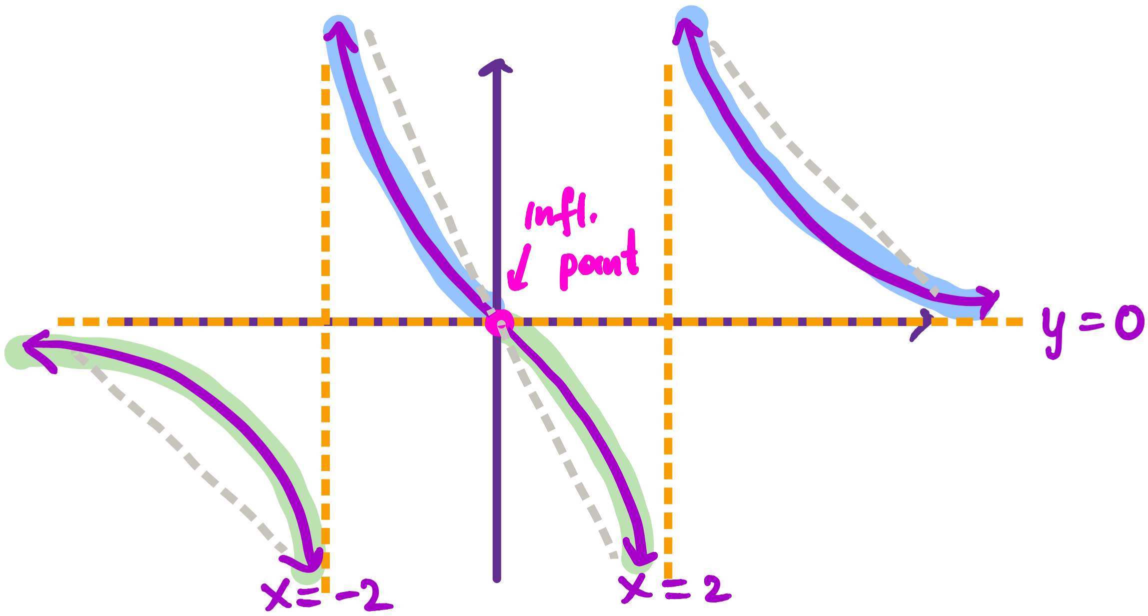
$y=0$: HA

$x=-2$: VA

$x=2$: VA

Summary

	-2	0	2	
inc.	dec	dec		dec
conc.	down	up	down	up



Ex. 3

Graph $y = x^2 e^{-x}$.

Solution:

$f'(x) = x(2-x)e^{-x}$ $\qquad$ $f''(x) = (x^2 - 4x + 2)e^{-x}$

① Information from $f(x)$:

- vertical asymptotes: none
- horizontal asymptotes: $y = 0$

(a) $\lim\limits_{x \to -\infty} (\underbrace{x^2}_{\infty} \cdot \underbrace{e^{-x}}_{\infty}) = \infty$

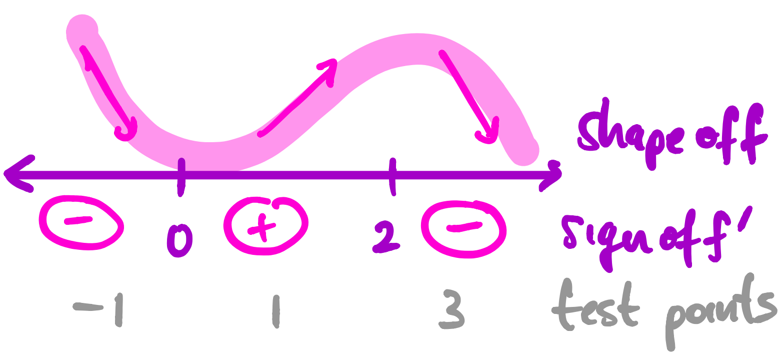
(b) $\lim\limits_{x \to \infty} (\underbrace{x^2}_{\infty} \cdot \underbrace{e^{-x}}_{0}) = \lim\limits_{x \to \infty} \left(\dfrac{x^2}{e^x}\right) \overset{H}{=} \lim\limits_{x \to \infty} \left(\dfrac{2x}{e^x}\right) \overset{H}{=} \lim\limits_{x \to \infty} \left(\dfrac{2}{e^x}\right) = 0$

$\dfrac{\infty}{\infty}$ $\qquad$ $\dfrac{\infty}{\infty}$ $\qquad$ $\dfrac{2}{\infty}$

② Information from $f'(x)$:

- construct a sign chart for $f'(x)$:

cut points: $x = 0$, $x = 2$



$f'(x) = x(2-x)e^{-x}$

$f'(-1) = - \; + \; + \; = -$

$f'(1) = + \; + \; + \; = +$

$f'(3) = + \; - \; + \; = -$

f is decreasing on: $(-\infty, 0]$, $[2, \infty)$

f is increasing on: $[0, 2]$

f has a local min @ $x = 0$

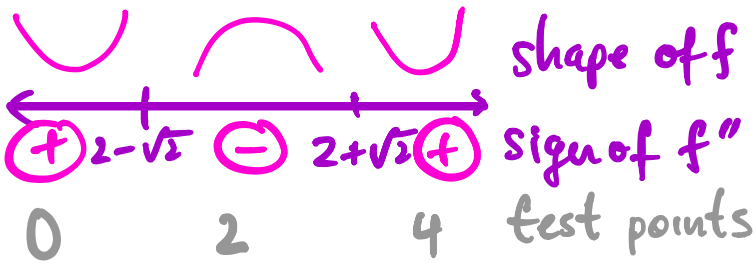
f has a local max @ $x = 2$

③ Information from $f''(x)$:

- construct a sign chart for $f''(x)$:

cut points: $x = 2-\sqrt{2}$, $x = 2+\sqrt{2}$

$$f''(x) = (x^2-4x+2)e^{-x}$$



$f''(0) = (+)(+) = (+)$

$f''(2) = (-)(+) = (-)$

$f''(4) = (+)(+) = (+)$

f is concave down on: $[2-\sqrt{2},\ 2+\sqrt{2}]$

f is concave up on: $(-\infty,\ 2-\sqrt{2}]$, $[2+\sqrt{2},\ \infty)$

f has inflection points @: $x = 2-\sqrt{2}$, $x = 2+\sqrt{2}$

④ Summary and graph:

Important Features:

$x = 0$: local min

$x = 2-\sqrt{2}$: infl. pt.

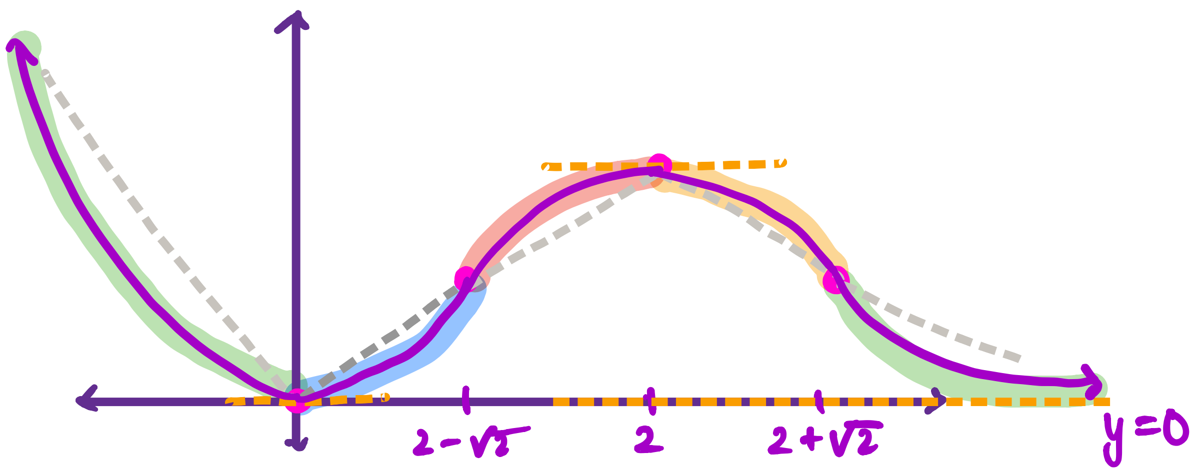
$x = 2$: local max

$x = 2+\sqrt{2}$: infl. pt.

$y = 0$: HA

Summary:

	0	$2-\sqrt{2}$	2	$2+\sqrt{2}$
inc.	dec	inc.		dec.
conc.		up	down	up



Ex. 4

Graph $f(x) = \dfrac{x^3}{x-1}$.

Solution:

$$f'(x) = \frac{x^2(2x-3)}{(x-1)^2} \qquad f''(x) = \frac{2x(x^2-3x+3)}{(x-1)^3}$$

① Information from $f(x)$:

- vertical asymptotes: $x=1$
- horizontal asymptotes: none

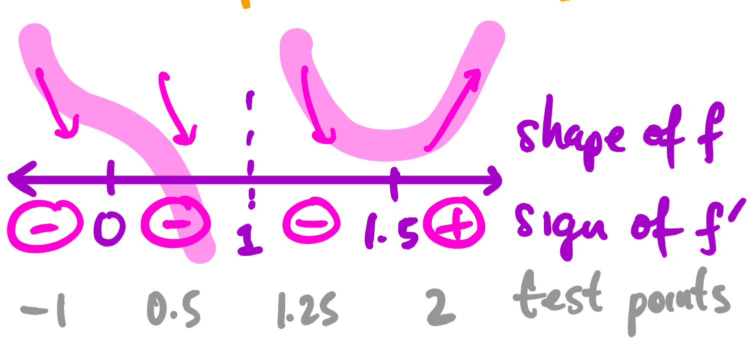
$$\lim_{x\to\pm\infty}\left(\frac{x^3}{x-1}\right) \overset{H}{=} \lim_{x\to\pm\infty}\left(\frac{3x^2}{1}\right) = \infty$$

$\frac{\infty}{\infty}$

② Information from $f'(x)$:

- construct a sign chart

cut points: $x=0$, $x=1.5$, $x=1$



$$f'(x) = \frac{x^2(2x-3)}{(x-1)^2}$$

$f'(-1) = \frac{\oplus\ominus}{\oplus} = \ominus$ $\quad f'(1.25) = \frac{\oplus\ominus}{\oplus} = \ominus$

$f'(0.5) = \frac{\oplus\ominus}{\oplus} = \ominus$ $\quad f'(2) = \frac{\oplus\oplus}{\oplus} = \oplus$

f is decreasing on: $(-\infty, 1)$, $(1, 3/2]$

f is increasing on: $[3/2, \infty)$

f has local min @: $x = 3/2$

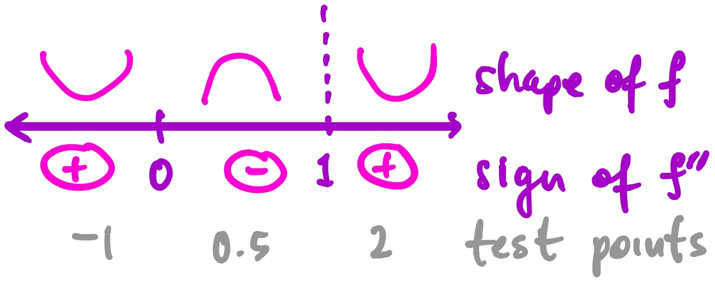
f has local max @: none

③ Information from $f''(x)$:

- construct a sign chart for $f''(x)$:

cut points: $x = 0$, $x = 1$

$x^2 - 3x + 3 = 0 \Rightarrow x = \dfrac{3 \pm \sqrt{3 - 4(3)(1)}}{2}$ (no solution)



$f''(x) = \dfrac{2x(x^2 - 3x + 3)}{(x-1)^3}$

$f''(-1) = \dfrac{\ominus \oplus}{\ominus} = \oplus$ $\quad f''(2) = \dfrac{\oplus \oplus}{\oplus} = \oplus$

$f''(0.5) = \dfrac{\oplus \oplus}{\ominus} = \ominus$

f is concave down on: $[0, 1)$

f is concave up on: $(-\infty, 0]$, $(1, \infty)$

f has inflection points @ $x = 0$

④ Summary and Graph:

Important Features

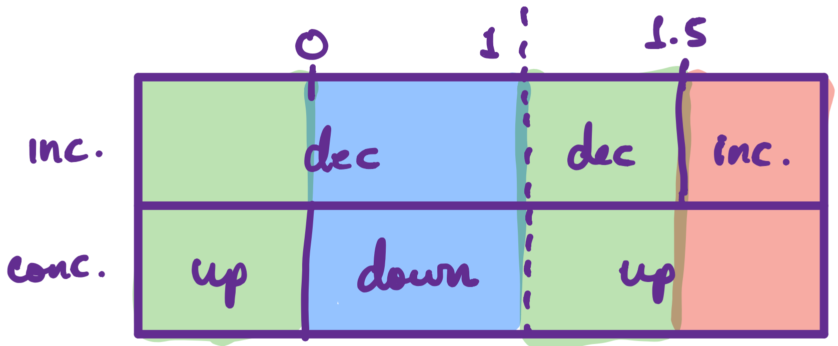
$(0,0)$: inflection pt.

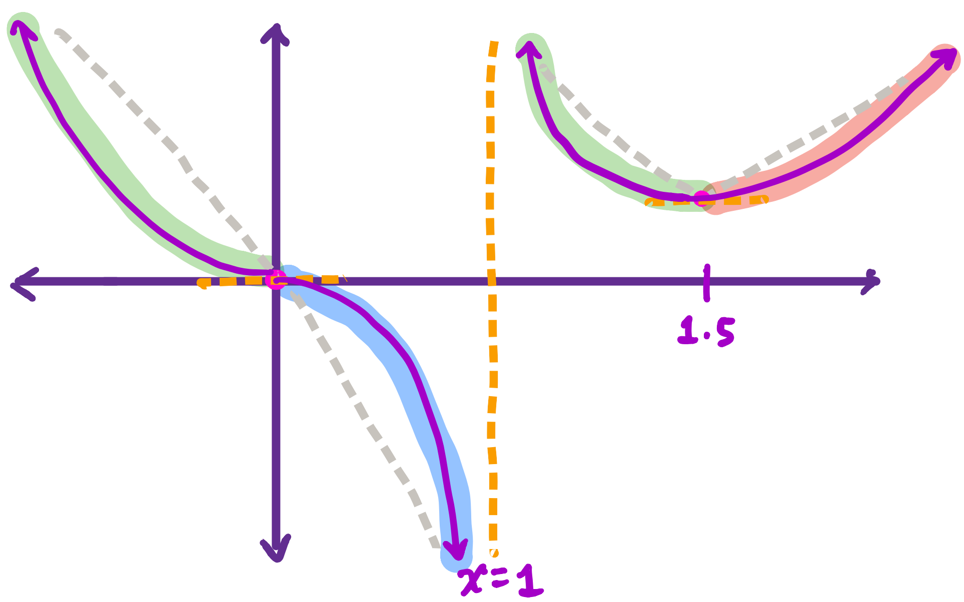
$x = 1$: VA

$x = 1.5$: local min

$\lim_{x \to \pm\infty} f(x) = \infty$

Summary:





Ex. 5

Graph $f(x) = x(x-2)^3$ on $[-1, 3]$.

Solution:

$f'(x) = 2(2x-1)(x-2)^2$ $\qquad$ $f''(x) = 12(x-1)(x-2)$

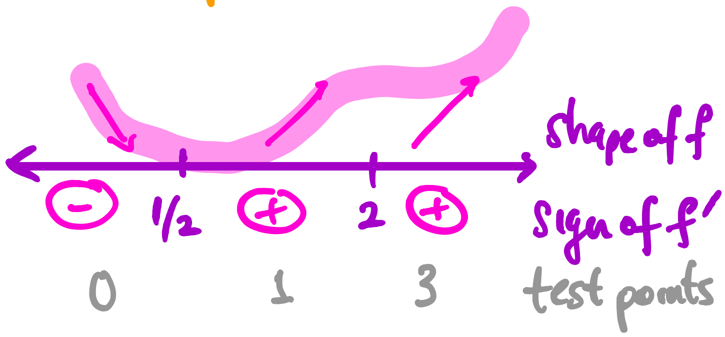
① Information from $f(x)$:

- vertical asymptotes: none
- horizontal asymptotes: none

② Information from $f'(x)$:

- construct a sign chart for $f'(x)$

cut points: $x = \frac{1}{2}$, $x = 2$



$f'(x) = 2(2x-1)(x-2)^2$

$f'(0) = (+)(-)(+) = (-)$

$f'(1) = (+)(+)(+) = (+)$

$f'(3) = (+)(+)(+) = (+)$

f is decreasing on: $(-\infty, 1/2]$

f is increasing on: $[1/2, \infty)$

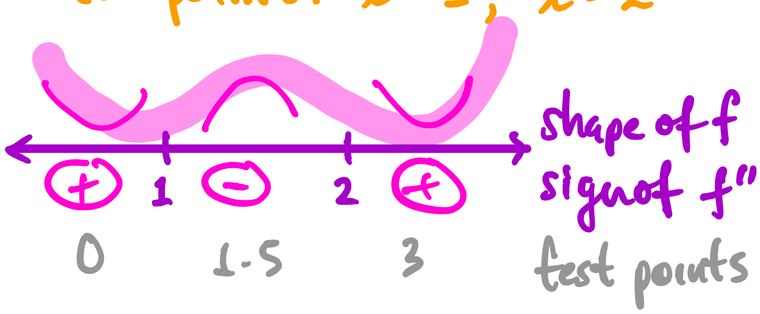
f has a local min @ $x = 1/2$

f has a local max @ nowhere

③ Information from $f''(x)$:

- construct a sign chart for $f''(x)$:

cut points: $x = 1$, $x = 2$



$f''(x) = 12(x-1)(x-2)$

$f''(0) = (+)(-)(-) = (+)$

$f''(1.5) = (+)(+)(-) = (-)$

$f''(3) = (+)(+)(+) = (+)$

f is concave down on: $[1, 2]$

f is concave up on: $(-\infty, 0]$, $[2, \infty)$

f has inflection points @ $x = 1$, $x = 2$

④ Summary and graph

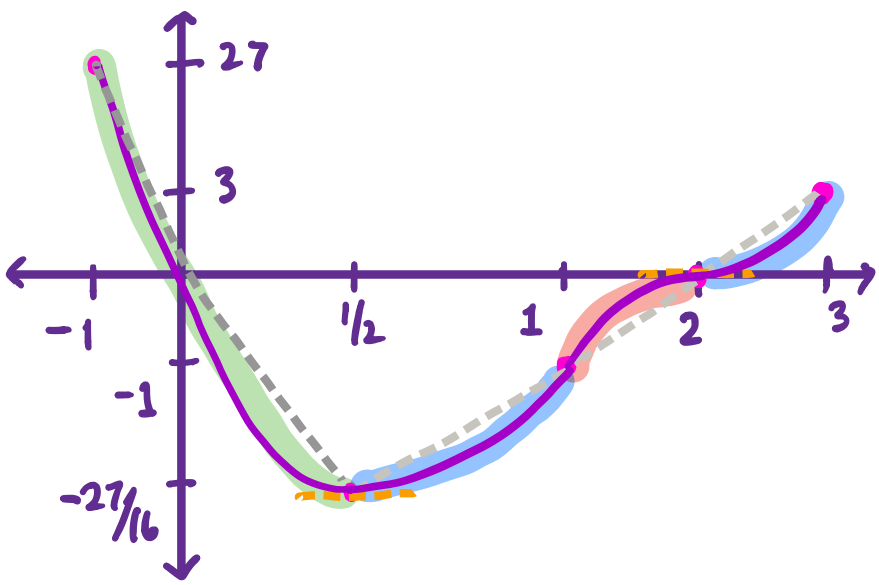
$$f(x) = x(x-2)^3 \text{ on } [-1, 3]$$

Important points:

x	y	type
-1	27	endpt
$1/2$	$-27/16$	local min
1	-1	infl. pt.
2	0	infl. pt.
3	3	endpt.

Summary:

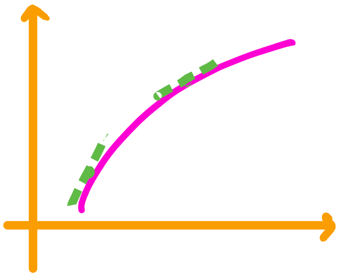
	-1 to $1/2$	$1/2$ to 1	1 to 2	2 to 3
inc	dec.		inc.	
conc.		up	down	up



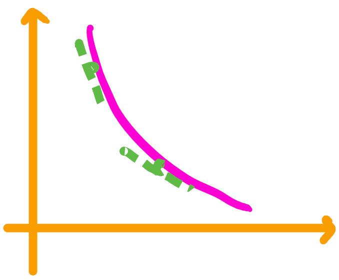
Section 4.3 Supplement: Conceptual Background

First Derivative $f'(x)$

What f' says about the graph of f:

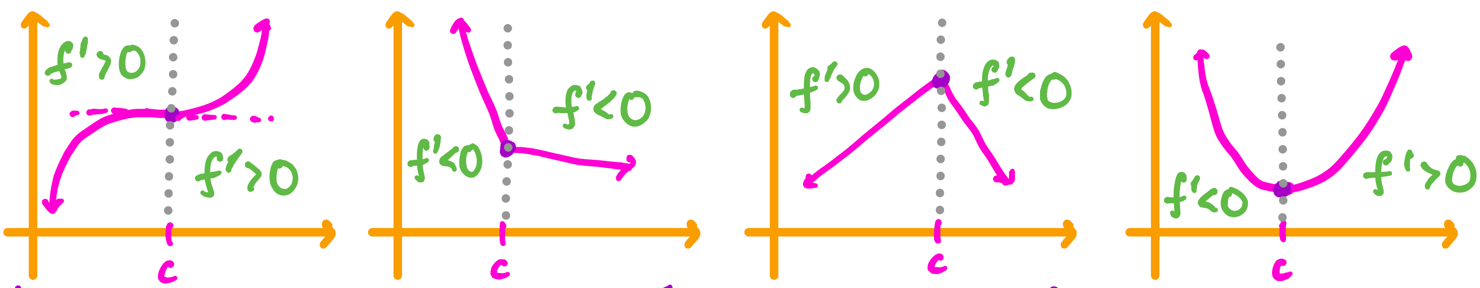


- $f'(x) > 0$ (tangents have positive slope)
- f is increasing (as x increases, y increases)



- $f'(x) < 0$ (tangents have negative slope)
- f is decreasing (as x increases, y decreases)

What about local extreme values? Suppose $x = c$ is a critical point of f. (So $f'(c) = 0$ or $f'(c)$ dne.)



- f' does <u>not</u> change sign at $x = c$
- f has no local extremum at $x = c$

- f' changes from ⊕ to ⊖ at $x = c$
- local max

- f' changes from ⊖ to ⊕ at $x = c$
- local min

Summary of information from $f'(x)$:

Sign of $f'(x)$ on (a, b)	Shape of $f(x)$ on (a, b)
⊖	decreasing
⊕	increasing

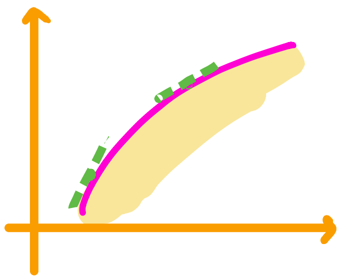
Sign change of f' at $x=c$	Classification of $f(c)$
$\ominus$ to $\oplus$	local minimum
$\oplus$ to $\ominus$	local maximum
no change	not a local extremum

* Assume $x=c$ is a critical point ($f'(c)=0$ or $f'(c)$ dne)

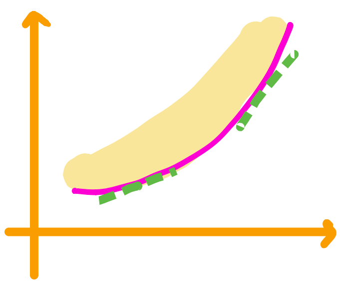
Second Derivative $f''(x)$

What f'' says about the graph of f:

Note: $f'(x)>0$ in both graphs but how is $f'(x)$ changing?

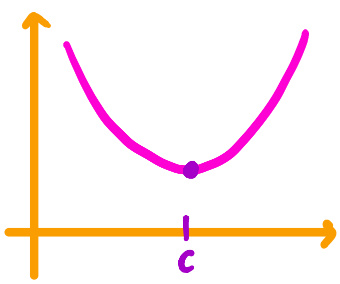


- $f'(x)$ is decreasing (slope gets less positive)
- $f''(x)<0$ (concave down)
- graph of f is below tangent lines

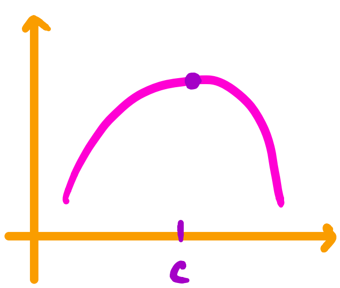


- $f'(x)$ is increasing (slope gets more positive)
- $f''(x)>0$ (concave up)
- graph of f is above tangent lines

What about local extreme values? Suppose $x=c$ is a critical point of f and f'' is continuous at $x=c$.



- $f'(c)=0$
- $f''(c)>0$
- $f(c)$ is a local minimum



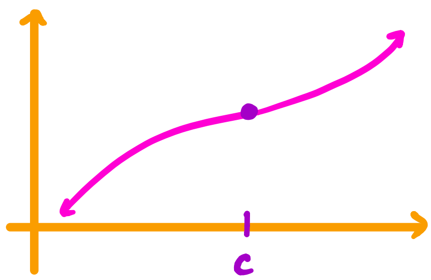
- $f'(c)=0$
- $f''(c)<0$
- $f(c)$ is a local maximum

Summary of information from $f''(x)$:

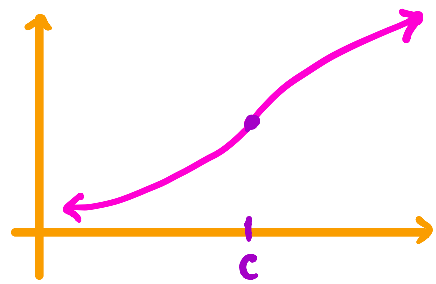
Sign of $f''(x)$ on (a,b)	Shape of $f(x)$ on (a,b)
$-$	concave down
$+$	concave up

$f'(c)=0$, sign of $f''(c)$	Classification of $f(c)$
$-$	local maximum
$+$	local minimum
zero	unknown

* Inflection points occur where $f(x)$ is continuous and $f''(x)$ changes sign



inflection from concave down to concave up



inflection from concave up to concave down

Graphing $y=f(x)$

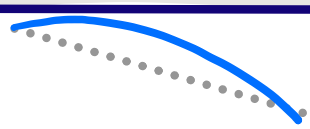
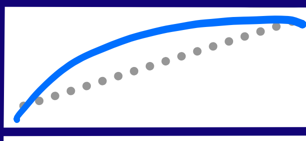
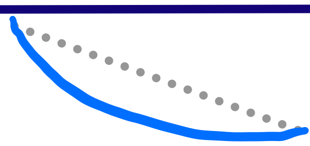
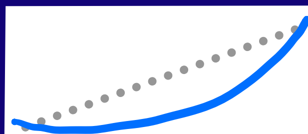
1. Info from $f(x)$:
   - points on graph
   - vertical asymptotes
   - horizontal asymptotes
2. Info from $f'(x)$:
   - find where $f'(x)=0$ or where $f'(x)$ dne
   - construct sign chart for $f'(x)$
   - infer intervals of increase/decrease
   - determine local extrema

③ Info from $f''(x)$:

- find where $f''(x) = 0$ or where $f''(x)$ dne
- construct sign chart for $f''(x)$
- infer intervals of concavity
- determine inflection points
- (optional: verify local extrema)

④ Graph $y = f(x)$:

- list important points (local extrema, inflection pts, etc.)
- summarize all info from f, f', and f''
- use chart below to sketch graph:

concavity \ increase	decreasing	increasing
concave down		
concave up		

Section 4.5: Optimization

Ex. 1

The difference of two numbers is 10. Find their minimum possible product.

Solution:

Goal: Rephase as "find absolute extremum of f on I."

Let x and y be the two numbers. Then the function we want to minimize is:

$$p(x,y) = xy$$ objective function

(The function f has two variables!) However, x and y are not independent of each other. Instead,

$$x - y = 10$$ constraint equation

We use the constraint to write the objective in terms of one variable.

$$x = y + 10$$ ← substitute for x in objective

So our objective is now:

$$p(y+10, y) = (y+10)y = y^2 + 10y$$

Goal: Find the absolute minimum value of:

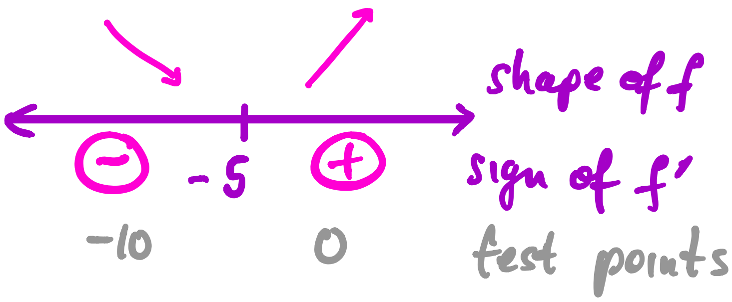
$$f(y) = y^2 + 10y$$

on the interval $(-\infty, \infty)$.

Find the critical points:

$$f'(y) = 2y + 10 = 0 \implies y = -5$$

Since the interval $(-\infty, \infty)$ is unbounded, f is not guaranteed to have both a min and a max value. We construct a sign chart for $f'(y)$.



$f'(y) = 2(y+5)$

$f'(-10) = -$

$f'(0) = +$

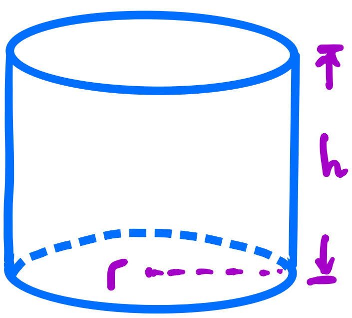
So $y=-5$ gives a local minimum. Since there is only one critical point, $y=-5$ gives an absolute minimum. So the minimum product is

$$f(-5) = (y^2 + 10y)\Big|_{y=-5} = -25$$

Bonus: What is the maximum possible product? There is no local max, hence no absolute max. Additionally, $\lim\limits_{y \to \pm\infty} f(y) = \infty$, so the product can be arbitrarily large.

Ex. 2

A cylindrical tank has volume $2000\pi \text{ m}^3$. Find the dimensions of the tank with the smallest possible surface area.



Hint:

$A = 2\pi r^2 + 2\pi r h$

$V = \pi r^2 h$

Solution:

We want to minimize the function:

$$A(r,h) = 2\pi r^2 + 2\pi r h \qquad \text{objective function}$$

subject to the constraint:

$$2000\pi = \pi r^2 h \qquad \text{constraint equation}$$

Solving h in terms of r gives:

$$h = \frac{2000}{r^2}$$

So now our objective is:

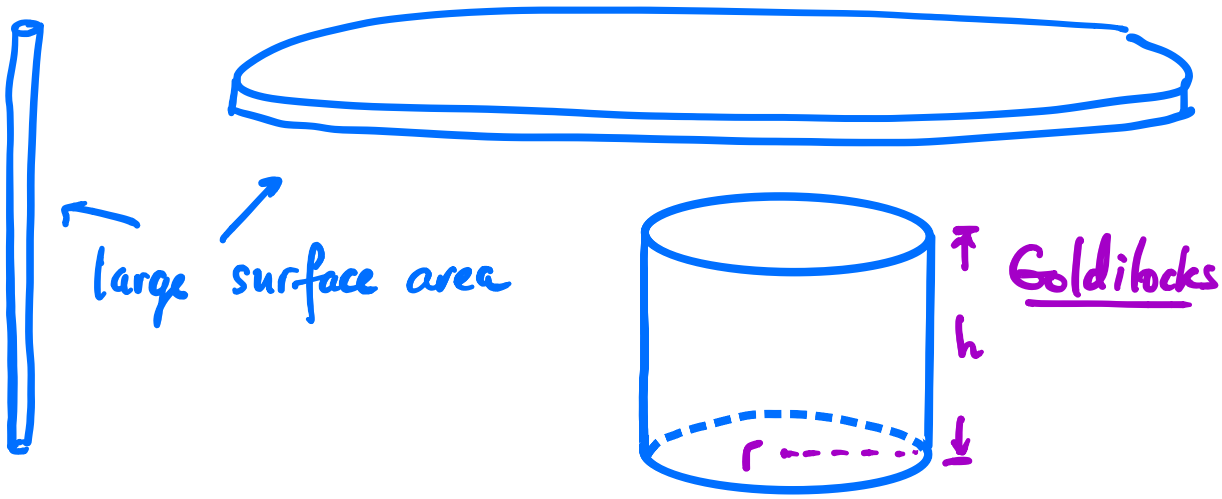
$$A\left(r, \frac{2000}{r^2}\right) = 2\pi r^2 + 2\pi r\left(\frac{2000}{r^2}\right) = 2\pi\left(r^2 + \frac{2000}{r}\right)$$

Goal: Find value of r that gives abs. min. of

$$f(r) = 2\pi\left(r^2 + \frac{2000}{r}\right)$$

on the interval $(0, \infty)$.

r close to 0 large h $\qquad$ r very large, small height



Find the critical points:

$$f'(r) = 2\pi\left(2r - \frac{2000}{r^2}\right) = 0 \implies r = 10$$

Observe that $f''(r) = 2\pi\left(2 + \frac{4000}{r^3}\right)$, which is positive for all $r > 0$. So $f(r)$ is concave up on $(0, \infty)$, so $r = 10$ gives a local minimum. Since there is only one critical point, $r = 10$ must give an absolute minimum. So the dimensions of the tank are $r = 10$ and $h = 20$.

$\boxed{\text{Ex 3}}$

A rectangular box has total surface area 450 in^2, and the length is three times its width. Find the dimensions of such a box with the largest possible volume.

<u>Solution:</u>

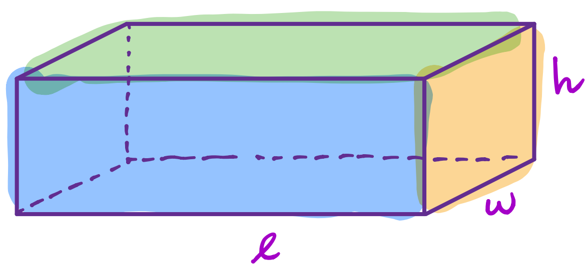
Let ℓ, w, and h be the length, width, and height of the box, respectively. We want to maximize:

$$V(\ell, w, h) = \ell w h$$ objective function

subject to the two constraints:

(1) $\ell = 3w$
("length is three times width")
(2) $450 = 2(\ell w + \ell h + wh)$
("total surface area is 450")

constraint equations



Use the constraints to write ℓ and h in terms of w:

(2)
$$450 = 2(\ell w + \ell h + wh)$$
$$225 = (3w)w + 3wh + wh$$
$$225 = 3w^2 + 4wh$$
$$h = \frac{225 - 3w^2}{4w}$$

So our objective in terms of w is:

$$V\left(3w, w, \frac{225-3w^2}{4w}\right) = 3w \cdot w \cdot \frac{225-3w^2}{4w} = \frac{3}{4}(225w - 3w^3)$$

What is the interval of interest?

$$\ell \geq 0 \implies 3w \geq 0 \implies w \geq 0$$
$$w \geq 0 \implies w \geq 0$$
$$h \geq 0 \implies \frac{225-3w^2}{4w} \geq 0 \implies \begin{cases} w \neq 0 \\ w \leq \sqrt{225/3} = \sqrt{75} \end{cases}$$

So our interval is $(0, \sqrt{75}]$.

<u>Goal</u>: Find the value of w that gives abs. max of
$$f(w) = \frac{3}{4}(225w - 3w^3)$$
on the interval $[0, \sqrt{75}]$.

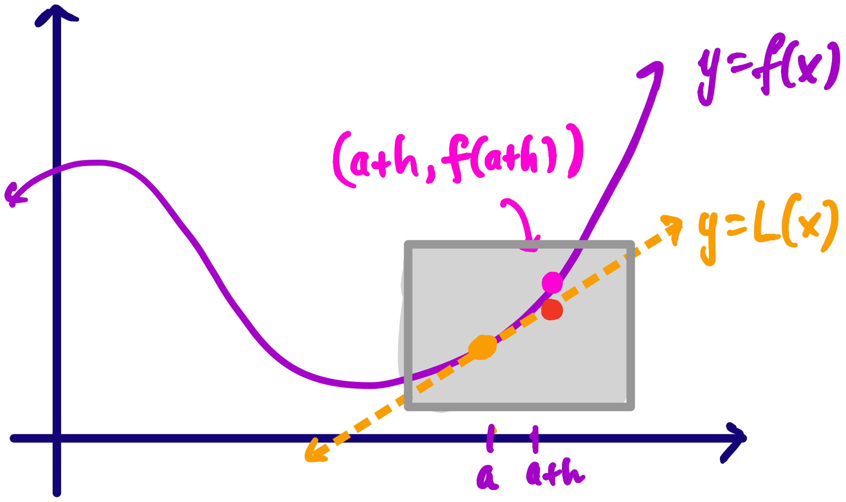
Find the critical points:

$$f'(w) = \frac{3}{4}(225 - 9w^2) = 0 \implies w = 5 \text{ or } \cancel{w = -5}$$

Observe that $f''(w) = \frac{3}{4}(-18w)$, which is negative for all $w>0$. So f is concave down for $w>0$. Hence $w=5$ must give a local max. There is only one critical point, so $w=5$ gives abs. max.

The dimensions of the box are $w=5$, $l=15$, $h=7.5$.

Section 4.6: Linear Approximation



If f is differentiable at $x=a$, then the tangent line to f at $x=a$ approximates f near the point of tangency.

Ex. 1

Use linear approximation to estimate $\tan\left(\frac{\pi}{4}+0.01\right)$.

Solution:

Put $f(x)=\tan(x)$. We use the tangent line at $x=\pi/4$.

Point: $\left(\frac{\pi}{4}, 1\right)$ Slope: $\sec(x)^2\Big|_{x=\pi/4} = 2$

Equation: $y = 1+2\left(x-\pi/4\right)$

Note: This means that if x is near $\frac{\pi}{4}$, then

$$\tan(x) \approx 1+2\left(x-\pi/4\right)$$

So we have:

$$\tan\left(\frac{\pi}{4}+0.01\right) \approx 1+2\left(\frac{\pi}{4}+0.01-\frac{\pi}{4}\right) = 1.02$$

Ex. 2

Use linear approximation to estimate $18^{1/4}$.

Solution:

Put $f(x) = x^{1/4}$. Find the tangent line at $x=16$.

Point: $(16, 2)$ Slope: $\frac{1}{4}x^{-3/4}\Big|_{x=16} = \frac{1}{32}$

Equation: $y = 2 + \frac{1}{32}(x-16)$

Note: This means that if x is near 16, then

$$x^{1/4} \approx 2 + \frac{1}{32}(x-16)$$

So we have:

$$18^{1/4} \approx 2 + \frac{1}{32}(18-16) = 2 + \frac{1}{16} = 2.0625$$

$\boxed{\text{Ex. 3}}$

Concentration of a drug in bloodstream t hours after injection is modeled by

$$C(t) = \frac{100t}{t^2+1}$$

Use linear approximation to estimate the change in the concentration in the period from 2 to 2.1 hours after injection. Did the concentration increase or decrease?

Solution:

The exact change in concentration is:

$$\Delta C = C(2.1) - C(2)$$

Find tangent line to $C(t)$ at $t=2$.

Point: $(2, C(2))$

Slope: $C'(t) = \dfrac{(t^2+1)\cdot 100 - 100t\cdot 2t}{(t^2+1)^2} = \dfrac{100(-t^2+1)}{(t^2+1)^2}$

$$C'(2) = \frac{100(-4+1)}{(4+1)^2} = -12$$

Equation: $y = C(2) - 12(t-2)$

Note: This means that if t is near 2, then

$$C(t) \approx C(2) - 12(t-2)$$

So now we have:

$$\Delta C = C(2.1) - C(2) \approx -12(2.1-2) = -1.2$$

Since $\Delta C < 0$, the concentration decreased.

Terminology in Business & Economics

x: # of units sold/produced

$p(x)$: price per unit if x units sold
(demand function)

$R(x)$: total revenue from selling first x units
(revenue = (# of units sold) · (price per unit) = $xp(x)$)

$C(x)$: total cost of first x units
$C(0)$: sunk cost

Marginal Quantities

$MQ(x)$: additional amount of "Q" achieved if 1 more unit is produced/sold, assuming x units are currently produced/sold.

$MQ(x) = Q(x+1) - Q(x)$

There is a standard approximation for $MQ(x)$:
The tangent line to $Q(x)$ at $x=a$ is:

$$y = Q(a) + Q'(a)(x-a)$$

Note: If x is near a, then

$$Q(x) \approx Q(a) + Q'(a)(x-a)$$

Since $a+1$ is near a,

$$Q(a+1) \approx Q(a) + Q'(a)(a+1-a)$$

Rearranging gives:

$$\underbrace{Q(a+1) - Q(a)}_{= MQ(a)} \approx Q'(a)$$

Summary:

- Marginal Cost $\begin{cases} C(x+1) - C(x) & \text{exact} \\ C'(x) & \text{approximate} \end{cases}$

- Marginal Revenue $\begin{cases} R(x+1) - R(x) & \text{exact} \\ R'(x) & \text{approximate} \end{cases}$

Ex. 4

The total revenue from selling x widgets is

$$R(x) = 40 - \frac{200}{x+5}$$

(a) Write an expression for the exact revenue from the 6th unit.

(b) Using marginal analysis, estimate the revenue from the 6th unit.

Solution:

(a) $R(6) - R(5)$

(b) We use the approximation

$$R(6) - R(5) \approx R'(5)$$

We have:

$$R'(5) = \frac{d}{dx}\left(40 - \frac{200}{x+5}\right)\Bigg|_{x=5} = \left(\frac{200}{(x+5)^2}\right)\Bigg|_{x=5} = 2$$

Ex. 5

The position of a particle on the x-axis is:

$$x(t) = 100 + 8t^{3/4} - 5t$$

Use linear approximation to estimate the particle's change in position between $t=81$ and $t=83$.

Solution:

The exact change in position is:

$$\Delta x = x(83) - x(81)$$

We use the tangent line at $t = 81$.

Point: $(81, x(81))$

Slope: $x'(81) = \left(6t^{-1/4} - 5\right)\Big|_{t=81} = 2 - 5 = -3$

Equation: $\boxed{y = x(81) - 3(t - 81)}$

So now we have:

$$\underbrace{x(83)}_{\text{use tangent line to estimat}} - x(81) \approx \left[x(81) - 3(83-81)\right] - x(81) = -6$$

Section 4.7: L'Hôspital's Rule

Indeterminate Forms

Undefined expressions which do not give information on their own to calculate the limit.

<u>Quotients</u>

$\frac{0}{0}$, $\frac{\infty}{\infty}$

L'Hospital's Rule (LR) applies directly only to quotients

<u>Products</u>

$0 \cdot \infty$

<u>Differences</u>

$\infty - \infty$, $-\infty + \infty$

<u>Exponents</u>

1^{∞}, 0^{0}, ∞^{0}

We will ultimately use LR for these forms, but we must express the limit as a quotient first.

<u>Thm</u>: (L'Hôspital's Rule)

Suppose $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ (or both infinite). Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

as long as the limit on the right sides exists or is infinite.

Ex. 1

Calculate each limit:

(a) $\lim_{x\to 3} \dfrac{x^2-3x}{x^3-3x^2-x+3}$ (b) $\lim_{x\to 1} \dfrac{e^x-e}{\ln(x)}$

Solution:

D.S. of the respective value of x gives $\frac{0}{0}$ for both parts. So we may use LR.

(a) $\lim_{x\to 3}\left(\dfrac{x^2-3x}{x^3-3x^2-x+3}\right) \overset{H}{=} \lim_{x\to 3}\left(\dfrac{2x-3}{3x^2-6x-1}\right) = \dfrac{3}{8}$

(b) $\lim_{x\to 1}\left(\dfrac{e^x-e}{\ln(x)}\right) \overset{H}{=} \lim_{x\to 1}\left(\dfrac{e^x}{1/x}\right) = \dfrac{e^1}{1} = e$

$\boxed{\text{Ex. 2}}$

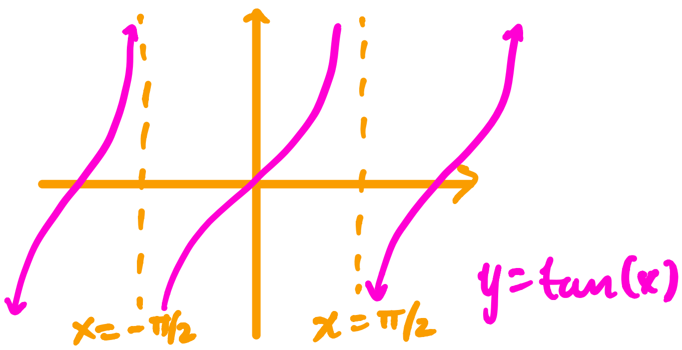
Calculate each limit.

(a) $\lim_{x\to \pi/2^+}\left(\dfrac{\cos(x)}{1-\sin(x)}\right)$ (b) $\lim_{x\to 0} \dfrac{\sin(3x)-3x+\frac{9x^3}{2}}{x^5}$

Solution:

D.S. of the respective value of x gives $\frac{0}{0}$ for both parts. So we may use LR.

(a) $\lim_{x\to \frac{\pi}{2}^+}\left(\dfrac{\cos(x)}{1-\sin(x)}\right) \overset{H}{=} \lim_{x\to \frac{\pi}{2}^+}\left(\dfrac{-\sin(x)}{-\cos(x)}\right) = \lim_{x\to \frac{\pi}{2}^+} \tan(x)$



From the graph, we see that

$$\lim_{x\to \frac{\pi}{2}^+} \tan(x) = -\infty$$

(6) $$\lim_{x\to 0}\left(\frac{\sin(3x)-3x+\frac{9x^3}{2}}{x^5}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{3\cos(3x)-3+\frac{27x^2}{2}}{5x^4}\right)$$

$$\overset{H}{=} \lim_{x\to 0}\left(\frac{-9\sin(3x)+27x}{20x^3}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{-27\cos(3x)+27}{60x^2}\right)$$

$$\overset{H}{=} \lim_{x\to 0}\left(\frac{81\sin(3x)}{120x}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{243\cos(3x)}{120}\right) = \frac{243}{120}$$

Ex. 3

Calculate $\lim_{x\to\infty}\left(\frac{\sqrt{x^2+1}}{x}\right)$.

Solution:

We have the indeterminate form "$\frac{\infty}{\infty}$". So we use LR.

$$\lim_{x\to\infty}\left(\frac{(x^2+1)^{1/2}}{x}\right) \overset{H}{=} \lim_{x\to\infty}\left(\frac{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x}{1}\right) = \lim_{x\to\infty}\left(\frac{x}{(x^2+1)^{1/2}}\right)$$

$$\overset{H}{=} \lim_{x\to\infty}\left(\frac{1}{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x}\right) = \lim_{x\to\infty}\left(\frac{(x^2+1)^{1/2}}{x}\right)$$

So LR gives us an endless loop. So what do we do?
Use techniques from Section 2.5.

$$\lim_{x\to\infty}\left(\frac{(x^2+1)^{1/2}}{x}\right) = \lim_{x\to\infty}\left(\frac{|x|}{x}\cdot\left(1+\frac{1}{x^2}\right)^{1/2}\right) = \lim_{x\to\infty}\left(\left(1+\frac{1}{x^2}\right)^{1/2}\right)$$

$$= (1+0)^{1/2} = 1$$

↳ Since $x\to\infty$, $|x| = x$
Hence $|x|/x = 1$

Ex. 4

Calculate each limit.

(a) $\lim_{x\to\infty} \left(x^2 e^{-x}\right)$ (b) $\lim_{x\to 0^+} \left(x \ln(x)\right)$

Solution:

Both limits give "$0\cdot\infty$". So we rewrite each limit as a quotient before using LR.

(a) $$\underset{(0\cdot\infty)}{\lim_{x\to\infty} \left(x^2 e^{-x}\right)} = \underset{(\infty/\infty)}{\lim_{x\to\infty} \left(\frac{x^2}{e^x}\right)} \overset{H}{=} \underset{(\infty/\infty)}{\lim_{x\to\infty} \left(\frac{2x}{e^x}\right)} \overset{H}{=} \lim_{x\to\infty} \left(\frac{2}{e^x}\right)$$

$$= \frac{2}{\infty} = 0$$

(b) $$\underset{(0\cdot(-\infty))}{\lim_{x\to 0^+} \left(x \ln(x)\right)} = \underset{\left(\frac{-\infty}{+\infty}\right)}{\lim_{x\to 0^+} \left(\frac{\ln(x)}{x^{-1}}\right)} \overset{H}{=} \lim_{x\to 0^+} \left(\frac{1/x}{-x^{-2}}\right)$$

$$= \lim_{x\to 0^+} (-x) = 0$$

$\boxed{\text{Ex. 5}}$

Calculate $\lim_{x\to 0} \left(\cos(x)\right)^{3/x^2}$.

Solution:

D.S. of $x=0$ gives the indeterminate form "1^∞". So we must write the limit as a quotient.

$$L = \lim_{x\to 0} \left(\cos(x)\right)^{3/x^2}$$

We will calculate $\ln(L)$ instead.

$$\ln(L) = \ln\left[\lim_{x\to 0}(\cos(x))^{3/x^2}\right] = \lim_{x\to 0}\left[\ln\left(\cos(x)^{3/x^2}\right)\right]$$

ln(x) is continuous, so limit can be moved

$$= \lim_{x\to 0}\left[\underbrace{\frac{3}{x^2}\cdot\ln(\cos(x))}_{(\infty\cdot 0)}\right] = \lim_{x\to 0}\left[\underbrace{\frac{3\ln(\cos(x))}{x^2}}_{(0/0)}\right]$$

$$\overset{H}{=} \lim_{x\to 0}\left[\frac{3\cdot\frac{1}{\cos(x)}\cdot(-\sin(x))}{2x}\right] = \lim_{x\to 0}\left[\underbrace{\frac{-3\tan(x)}{2x}}_{(0/0)}\right]$$

$$\overset{H}{=} \lim_{x\to 0}\left[\frac{-3\sec(x)^2}{2}\right] = \frac{-3}{2}$$

So we have $\ln(L) = -\frac{3}{2}$, whence $L = e^{-3/2}$.

Ex. 6

Calculate $\displaystyle\lim_{x\to 0^-}\left(\frac{1}{\sin(x)} - \frac{1}{x}\right)$.

Solution:

We have the indeterminate form "$-\infty - (-\infty)$". We first rewrite the limit as a quotient.

$$\lim_{x\to 0^-}\underbrace{\left(\frac{1}{\sin(x)} - \frac{1}{x}\right)}_{(-\infty)-(-\infty)} = \lim_{x\to 0^-}\underbrace{\left(\frac{x-\sin(x)}{x\sin(x)}\right)}_{(0/0)} \overset{H}{=} \lim_{x\to 0^-}\underbrace{\left(\frac{1-\cos(x)}{\sin(x)+x\cos(x)}\right)}_{(0/0)}$$

$$\overset{H}{=} \lim_{x \to 0^-} \left(\frac{\sin(x)}{\cos(x) + \cos(x) - x\sin(x)} \right) = \frac{0}{1+1-0} = 0$$

Section 4.9: Antiderivatives

Def: We say F is an antiderivative of f on (a,b) if $F'=f$ on (a,b).

Ex:

Suppose $f(x) = \sin(x)$ on $(-\infty, \infty)$. Then what is an antiderivative of f?

$$F_1(x) = -\cos(x)$$

$$F_2(x) = -\cos(x) + C \qquad (C = \text{const.})$$

$$F_3(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \; ? \qquad \text{yes}$$

$$F_4(x) = \tan^{-1}\left(\frac{\sin(x)}{1-\cos(x)}\right) \; ? \qquad \text{no}$$

Thm: Suppose F and G are both antiderivatives of f on (a,b). Then there is a constant C such that $G(x) = F(x) + C$ for all x in (a,b).

Special Notation

$\int f(x)\,dx$ means the most general antiderivative of $f(x)$ wrt. x. (The interval (a,b) is understood.)

(Your textbook calls this an indefinite integral.)

Ex. 1

Calculate each antiderivative.

Solution:

(a) $\int x \, dx = \frac{1}{2}x^2 + C$

(b) $\int x^2 dx = \frac{1}{3}x^3 + C$

(c) $\int x^{17} dx = \frac{1}{18}x^{18} + C$

(d) $\int x^{-1/3} dx = \frac{3}{2}x^{2/3} + C$

Do we see a pattern?

(e) $\int \underbrace{x^{-1}}_{\substack{\text{domain:} \\ (-\infty,0)\cup(0,\infty)}} dx = \underbrace{\ln(x)}_{\substack{\text{domain:} \\ (0,\infty)}} + C$ ← This antiderivative works only on $(0,\infty)$

Thm: (Power Rule)

If $n \neq -1$, then $\int x^n dx = \frac{x^{n+1}}{n+1} + C$. If $n=-1$,

$$\int \frac{1}{x} dx = \begin{cases} \ln(x) + C_1 & \text{for } x > 0 \\ \ln(-x) + C_2 & \text{for } x < 0 \end{cases}$$

We usually write $\int \frac{1}{x} dx = \ln(|x|) + C$.

$\boxed{\text{Ex. 2}}$

Calculate each antiderivative:

Solution:

(a) $\int \frac{1}{x^3} dx = -\frac{1}{2}x^{-2} + C$

(b) $\int \left(3x^4 - 5x^{2/3} - x^{-1}\right) dx =$

$$\frac{3}{5}x^5 - 3x^{5/3} - \ln(|x|) + C$$

(c) $\int \frac{x^3 + \sqrt{2x} + x}{x^3}\, dx = \int \left(1 + \sqrt{2}\, x^{-5/2} + x^{-2}\right) dx$

$$= x + \sqrt{2} \cdot \frac{x^{-3/2}}{-3/2} + \frac{x^{-1}}{-1} + C$$

(d) $\int (x^2 - 4)^2\, dx = \int \left(x^4 - 8x^2 + 16\right) dx$

$$= \frac{x^5}{5} - \frac{8x^3}{3} + 16x + C$$

(e) $\int \sec(\theta)\left(\sec(\theta) + \tan(\theta)\right) d\theta$

$$= \int \left(\sec(\theta)^2 + \sec(\theta)\tan(\theta)\right) d\theta = \tan(\theta) + \sec(\theta) + C$$

(f) $\int \left(e^w + 2\cos(w) - 3\sin(w)\right) dw$

$$= e^w + 2\sin(w) + 3\cos(w) + C$$

Initial Value Problems (IVP's)

Goal: find an unknown function $f(x)$ given two pieces of information:

- $f'(x)$ ← gives $f(x) + C$
- $f(a) = b$ ← gives values of C

Summary of Particle Motion

$$\begin{array}{lcl} \text{Position:} & & s(t) \\ & \frac{d}{dt}\downarrow \quad \uparrow \int(\ldots)\,dt & \\ \text{Velocity:} & & v(t) \\ & \frac{d}{dt}\downarrow \quad \uparrow \int(\ldots)\,dt & \\ \text{Acceleration:} & & a(t) \end{array}$$

Ex. 3

A particle moves along the x-axis with velocity $v(t) = 9t^2 - 4t$. If the particle is at $x = 4$ when $t = 2$, find the position of the particle when $t = 3$.

Solution:

This is an IVP for the position. We are given:

- $\underbrace{\frac{dx}{dt} = 9t^2 - 4t}_{1}$
- $\underbrace{x(2) = 4}_{2}$

Antidifferentiating 1 gives

$$x(t) = \int \frac{dx}{dt}\,dt = \int (9t^2 - 4t)\,dt = 3t^3 - 2t^2 + C$$

Now use 2 to find C.

$$\left.(3t^3 - 2t^2 + C)\right|_{t=2} = x(2) = 4$$

$$16 + C = 4$$

$$C = -12$$

So we have:

$$x(t) = 3t^3 - 2t^2 - 12$$

Hence $x(3) = 51$.

$\boxed{\text{Ex.4}}$

Suppose the marginal revenue is

$$R'(x) = -3x^2 + 4x + 84$$

Assume $R(0) = 0$.

(a) Find the demand function $p(x)$.

(b) What is the market price if revenue is at a maximum?

Solution:

(a) Recall: $R(x) = x \cdot p(x)$

First find $R(x)$ by solving an IVP.

- $\underbrace{R'(x) = -3x^2 + 4x + 84}_{1}$ • $\underbrace{R(0) = 0}_{2}$

Antidifferentiate 1 to get:

$$R(x) = \int (-3x^2 + 4x + 84)\,dx = -x^3 + 2x^2 + 84x + K$$

Now use 2 to find the value of C.

$$\left.(-x^3 + 2x^2 + 84x + K)\right|_{x=0} = R(0) = 0$$

$$0 + K = 0 \implies K = 0$$

So our revenue and demand functions are:

$$R(x) = -x^3 + 2x^2 + 84x \Rightarrow p(x) = \frac{R(x)}{x} = -x^2 + 2x + 84$$

(b) The maximum of $R(x)$ occurs where $R' = 0$.

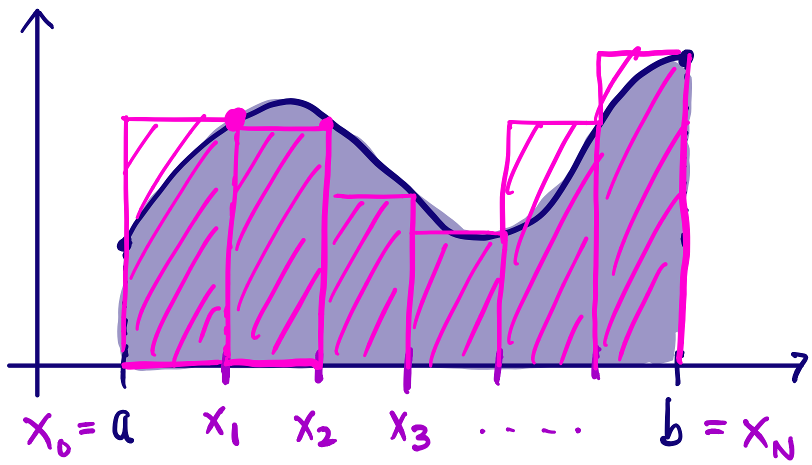
$$R'(x) = -3x^2 + 4x + 84 = 0$$

$$\cancel{x = -14/3} \quad \text{or} \quad x = 6$$

So the price is $p(6) = 60$.

Sections 5.1/5.2: The Integral

We want to approximate the area under the graph of $y=f(x)$ and above the interval $[a,b]$ on x-axis. (We assume f is continuous and non-negative.)



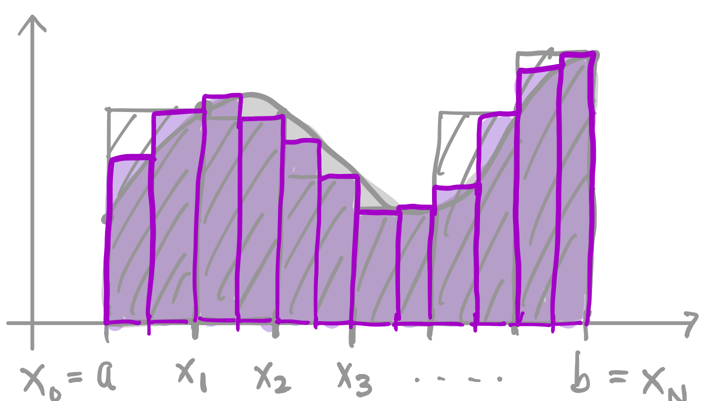
① Use rectangles whose bases lie on the x-axis to estimate the area.

② Divide $[a,b]$ into N equal-length subintervals. These subintervals are the bases of the N rectangles.

③ Choose height of rectangle so top intersects graph. We will choose the height of each rectangle to be the function value at the <u>right endpoints</u> of each subinterval.

④ Total area of rectangles estimates area under graph.

* If we use N rectangles with right-endpoint heights, the total area of rectangles is R_N and call it the right-endpoint Riemann sum.



If we increase N, we seem to get a better estimate of the area under the graph.

$$\boxed{A_{exact} = \lim_{N\to\infty} R_N}$$

Notation and Terminology

We define the exact area under the graph of $y=f(x)$ and above the interval $[a,b]$ on the x-axis to be the limit of the associated Riemann Sums as $N \to \infty$. This exact area is denoted:

$$\int_a^b f(x)\,dx$$

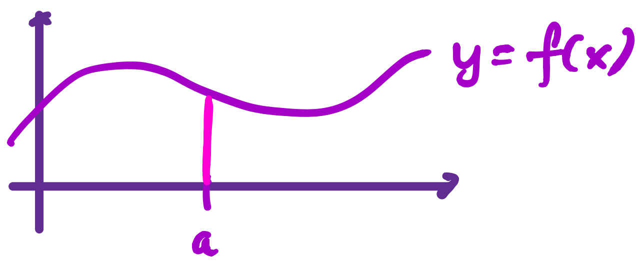
(Your textbook calls this a definite integral.)

Note: The limit $\lim_{N\to\infty} R_N$ exists for all functions that are continuous on $[a,b]$, except possibly at finitely many jump discontinuities.

Properties of Integrals

- $\int_a^a f(x)\,dx = 0$

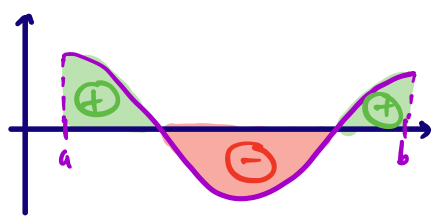
line segments have 0 area



- Interpretation of integral if $f(x)$ has negative values.

$$\int_a^b f(x)\,dx = \begin{pmatrix}\text{area above} \\ x\text{-axis}\end{pmatrix} - \begin{pmatrix}\text{area below} \\ x\text{-axis}\end{pmatrix}$$

Ex:



Ex: $\displaystyle\int_0^{2\pi} \sin(x)\,dx = 0$

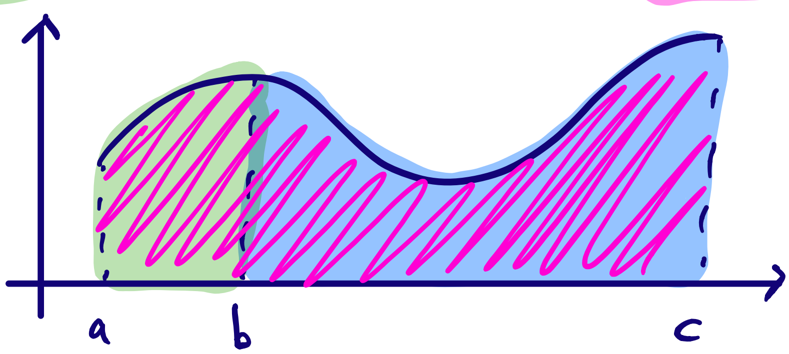
- Linearity

$$\rightarrow \int_a^b \left(f(x)+g(x)\right)dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\rightarrow \int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$$

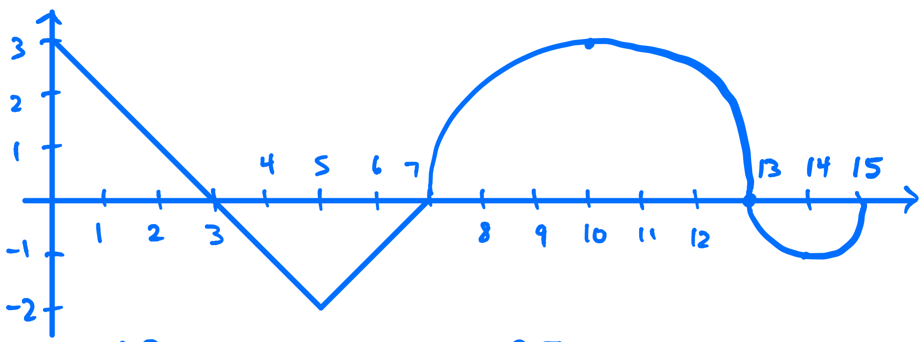
- Additivity

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



Ex. 1

Use the graph of $y=f(x)$ to calculate each integral.

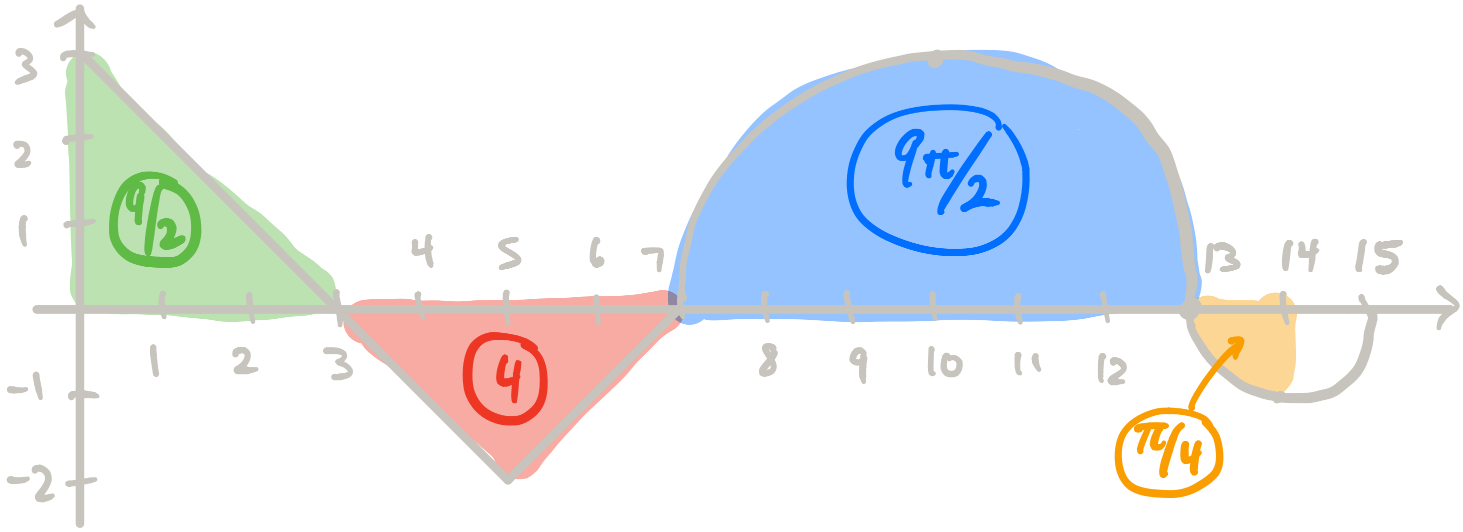


Note: graph consists of line segments and semicircles.

(a) $\int_0^3 f(x)\,dx$ (b) $\int_0^7 f(x)\,dx$ (c) $\int_7^{13} f(x)\,dx$

(d) $\int_7^{14} f(x)\,dx$ (e) $\int_3^{13} |f(x)|\,dx$

Solution:



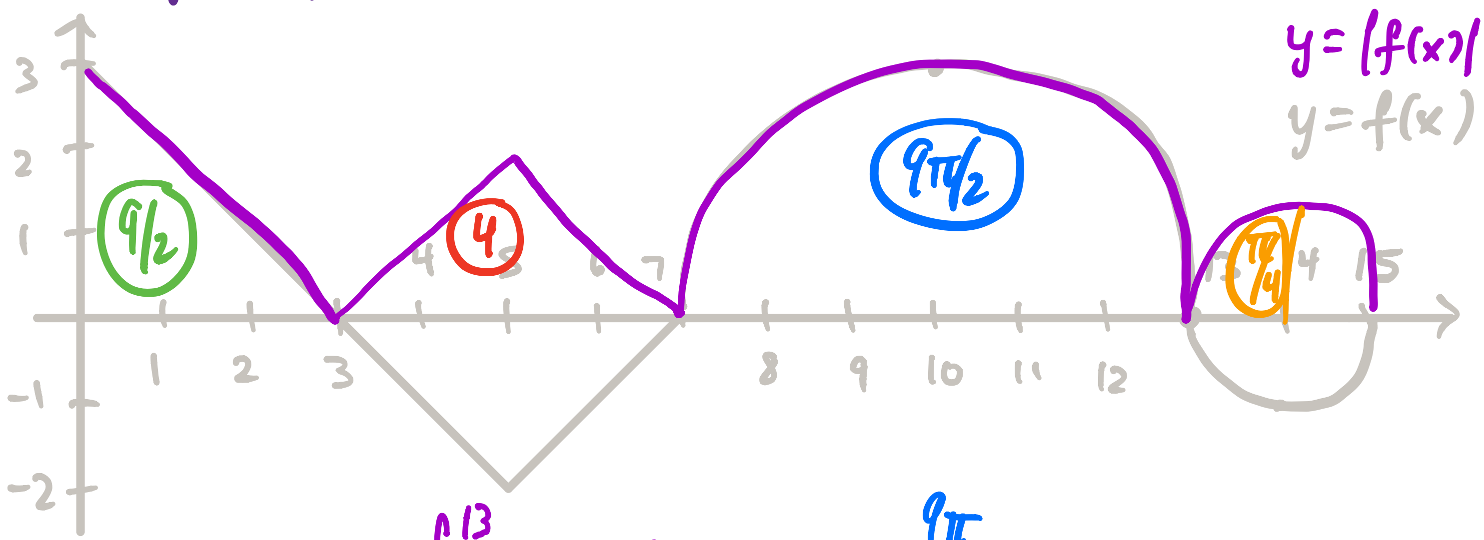
(a) $\int_0^3 f(x)\,dx = \frac{9}{2}$

(b) $\int_0^7 f(x)\,dx = \frac{9}{2} - 4 = \frac{1}{2}$

(c) $\int_7^{13} f(x)\,dx = \frac{9\pi}{2}$

(d) $\int_7^{14} f(x)\,dx = \frac{9\pi}{2} - \frac{\pi}{4} = \frac{17\pi}{4}$

(e) If you are given the graph of $y = f(x)$, how do you get the graph of $y = |f(x)|$?



$$\int_3^{13} |f(x)|\,dx = 4 + \frac{9\pi}{2}$$

Ex. 2

Calculate each integral.

(a) $\int_0^4 (8-2x)\,dx$

(d) $\int_{-5}^3 (|x|-2)\,dx$

(b) $\int_{-4}^{2} (2x+4)\,dx$

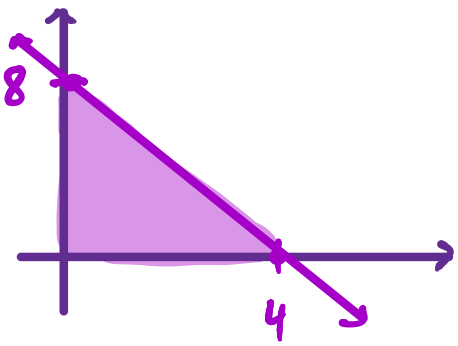
(c) $\int_{-1}^{5} \sqrt{9-(x-2)^2}\,dx$

(e) $\int_{1}^{10} g(x)\,dx$ where

$$g(x) = \begin{cases} 3x & , 0 \le x \le 2 \\ -5x+10 & , 2 < x \le 3 \\ -5 & , x > 3 \end{cases}$$

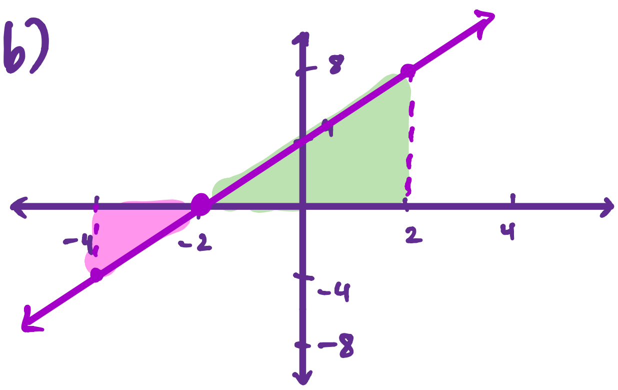
Solution:

(a)



$$\int_0^4 (8-2x)\,dx = \frac{1}{2}(4)(8) = 16$$

(b)

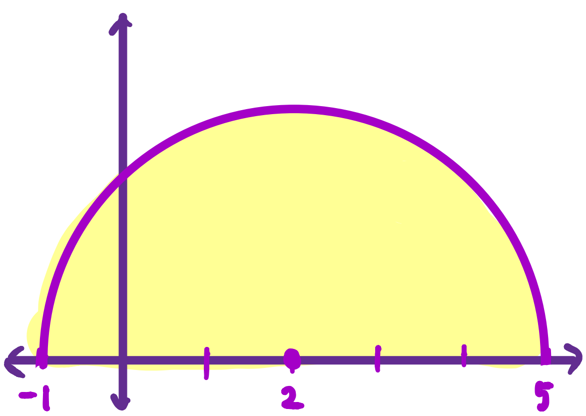


$$\int_{-4}^{2} (2x+4)\,dx =$$

$$= -\frac{1}{2}(2)(4) + \frac{1}{2}(4)(8) = 12$$

(c) If $y = \sqrt{9-(x-2)^2}$, then

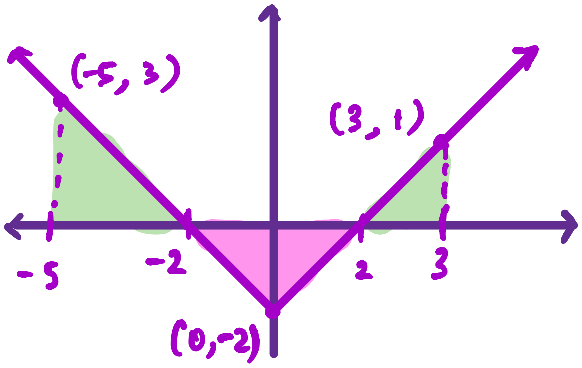
$(x-2)^2 + y^2 = 9$ $\begin{cases} \text{Radius: } 3 \\ \text{Center: } (2,0) \end{cases}$



$$\int_{-1}^{5} \sqrt{9-(x-2)^2}\,dx = \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$$

$(A_{disk} = \pi R^2)$

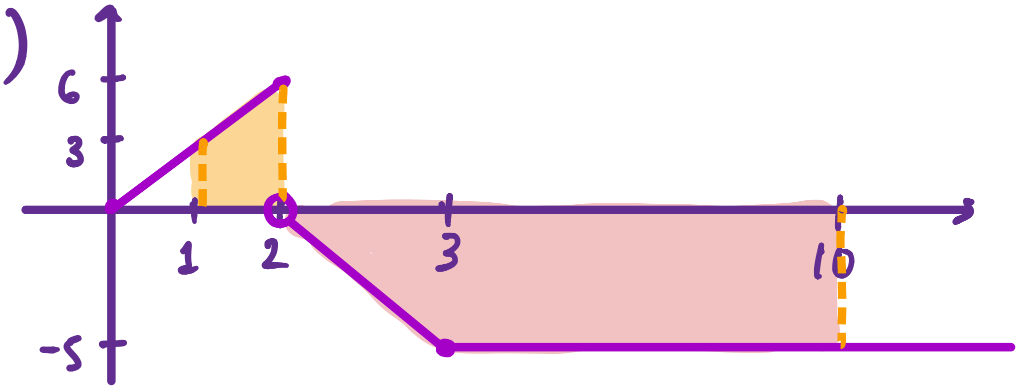
(d)



$$\int_{-5}^{3} (|x|-2)\,dx =$$

$$= \frac{1}{2}(3)(3) - \frac{1}{2}(4)(2) + \frac{1}{2}(1)(1) = 1$$

(e)



$$\int_{1}^{10} g(x)\,dx = \frac{1}{2}(3+6)\cdot 1 - \frac{1}{2}(8+7)\cdot 5 = -33$$

$$A_{trap} = \frac{1}{2}(b_1 + b_2)h$$

Section 5.3: Fundamental Theorem of Calculus

LIMIT

slope of tangent line

area under graph

DERIVATIVE $\xleftrightarrow{\text{FTOC}}$ **INTEGRAL**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\int_a^b f(x)\,dx = \lim_{N \to \infty} R_N$$

<u>Theorem</u>: Fundamental Theorem (Part 1)

Suppose f is continuous on $[a,b]$ and F is an antiderivative of f on $[a,b]$. Then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Recall Notation:</u>

$$\int_a^b f(x)\,dx$$

- integral of f on $[a,b]$
- area under graph
- a number

$$\int f(x)\,dx$$

- The most general antiderivative of f
- a family of functions

So FTC1 tells us that if we have an antiderivative of f, then we can use it to find integrals of f.

Q: Does a general function have an antiderivative?

A: No. (Continuous functions do.)

Q: If the antiderivative exists, how do you find a useful formula for it?

A: Very difficult. (Calculus II.)

Theorem: Fundamental Theorem (Part 2)

Suppose f is continuous on $[a,b]$. Let $x \in [a,b]$ and define

$$A(x) = \int_a^x f(t)\,dt$$

Then $A'(x) = f(x)$. That is,

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

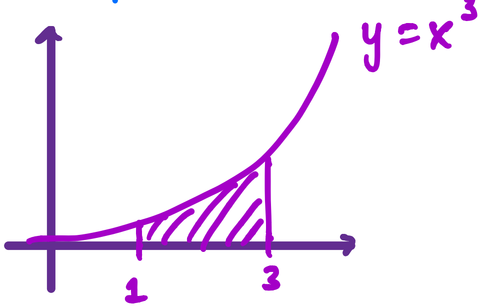
Notation: $g(x)\Big|_a^b = g(b) - g(a)$

Ex. 1

Calculate the following integrals.

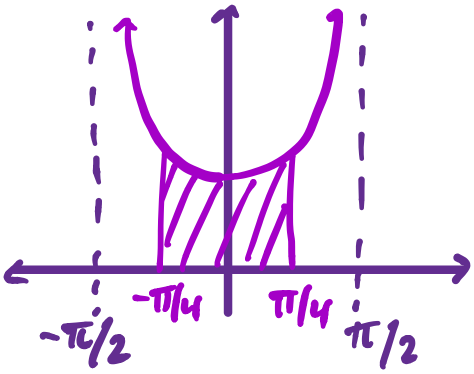
Solution:

(a) $\int_1^3 x^3\,dx$



$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = 20$$

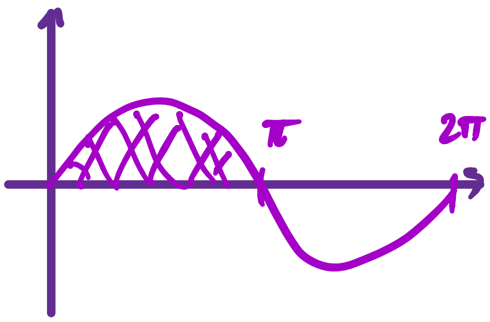
(b) $\int_{-\pi/4}^{\pi/4} \sec(\theta)^2 \, d\theta$



$$\int_{-\pi/4}^{\pi/4} \sec(\theta)^2 \, d\theta = \tan(\theta) \Big|_{-\pi/4}^{\pi/4} =$$

$$= \tan(\pi/4) - \tan(-\pi/4) = 2$$

(c) $\int_{0}^{\pi} \sin(\theta) \, d\theta$

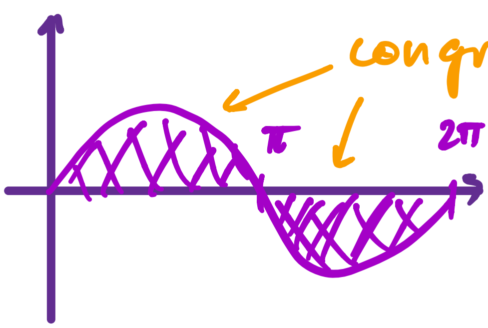


$$\int_{0}^{\pi} \sin(\theta) \, d\theta = -\cos(\theta) \Big|_{0}^{\pi} =$$

$$= (-\cos(\pi)) - (-\cos(0)) = 2$$

(d) $\int_{0}^{2\pi} \sin(\theta) \, d\theta$

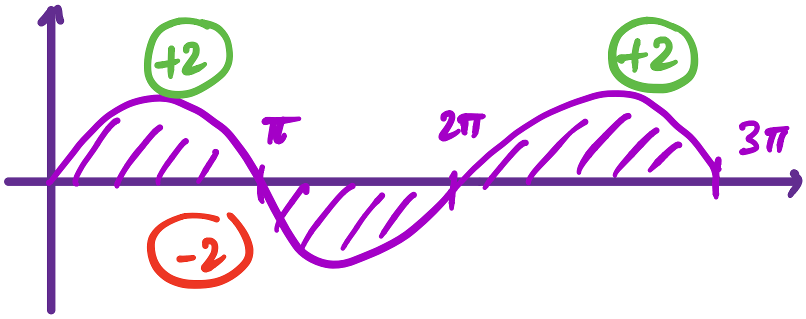
congruent regions, so area cancels



$$\int_{0}^{\pi} \sin(\theta) \, d\theta = -\cos(\theta) \Big|_{0}^{2\pi}$$

$$= -\cos(2\pi) - (-\cos(0)) = 0$$

(e) $\int_{0}^{3\pi} \sin(\theta) \, d\theta$



$$\int_{0}^{3\pi} \sin(\theta) \, d\theta = 2$$

(f) $\int_1^2 \frac{1}{x}\,dx = \ln(x)\Big|_1^2 = \ln(2) - \ln(1) = \ln(2)$

(g) $\int_0^2 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^2 = \ln(3) - \ln(1) = \ln(3)$

(h) $\int_0^2 \frac{1}{2x+1}\,dx = \frac{1}{2}\ln(2x+1)\Big|_0^2 = \frac{1}{2}\ln(5)$

Check: $\frac{d}{dx}\left(\ln(2x+1)\right) = \frac{1}{2x+1}\cdot 2$ ✗

$\frac{d}{dx}\left(\frac{1}{2}\ln(2x+1)\right) = \frac{1}{\cancel{2}}\cdot\frac{1}{2x+1}\cdot\cancel{2} = \frac{1}{2x+1}$ ✓

(i) $\int_0^1 (3x+5)^{18}\,dx = \frac{(3x+5)^{19}}{19}\cdot\frac{1}{3}\Big|_0^1 = \frac{8^{19}}{57} - \frac{5^{19}}{57}$

(j) $\int_{8/27}^1 \frac{4t^{4/3} - 10t^{1/3}}{t^2}\,dt = \int_{8/27}^1 \left(4t^{-2/3} - 10t^{-5/3}\right)dt$

$= \left(4\cdot\frac{t^{1/3}}{1/3} - 10\cdot\frac{t^{-2/3}}{-2/3}\right)\Big|_{8/27}^1 = \left(12t^{1/3} + 15t^{-2/3}\right)\Big|_{8/27}^1$

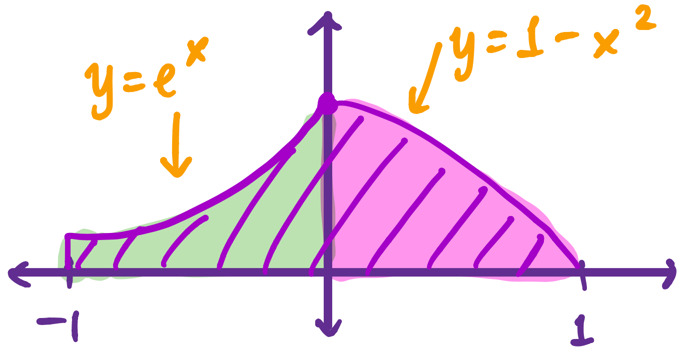
$= (12+15) - \left(12\cdot\frac{2}{3} + 15\cdot\frac{9}{4}\right) = -\frac{59}{4}$

$\left(\frac{8}{27}\right)^{1/3} = \frac{2}{3}, \quad \left(\frac{8}{27}\right)^{-2/3} = \frac{9}{4}$

$\boxed{\text{Ex. 2}}$

Let $f(x) = \begin{cases} e^x, & x < 0 \\ 1-x^2, & x \geq 0 \end{cases}$. Calculate $\int_{-1}^1 f(x)\,dx$.

Solution:



$$\int_{-1}^{1} f(x)\,dx = $$

$$= \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

Compute each integral separately.

- $\int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} e^x\,dx = e^x \Big|_{-1}^{0} = e^0 - e^{-1} = 1 - e^{-1}$

- $\int_{0}^{1} f(x)\,dx = \int_{0}^{1} (1-x^2)\,dx = \left(x - \frac{1}{3}x^3\right)\Big|_{0}^{1} = \left(1 - \frac{1}{3}\right) - (0-0)$

So we have:

$$\int_{-1}^{1} f(x)\,dx = 1 - e^{-1} + 1 - \frac{1}{3} = \frac{5}{3} - e^{-1}$$

$\boxed{\text{Ex. 3}}$

Let $G(x) = 5 + \int_{1}^{x} \sqrt{t^3+1}\,dt$.

(a) Compute $G(1)$. (b) Compute $G'(2)$.

Solution:

(a) $G(1) = 5 + \int_{1}^{1} \sqrt{t^3+1}\,dt = 5$

(b) Use FTC2 to get $G'(x)$. Recall:

top limit is x

$$\frac{d}{dx}\left(\int_{a}^{x} f(t)\,dt\right) = f(x)$$

bottom limit is constant

So we have:

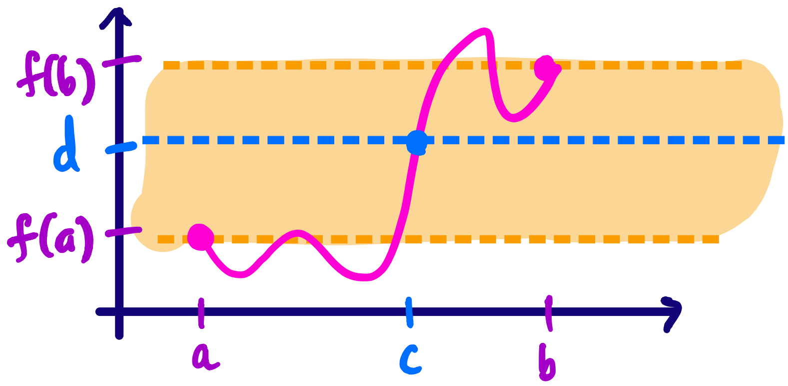
$$G'(x) = \frac{d}{dx}\left(5 + \int_1^x \sqrt{t^3+1}\,dt\right) = \sqrt{x^3+1}$$

Hence $G'(2) = \sqrt{8+1} = 3$.

Optional: Intermediate Value Theorem & Applications

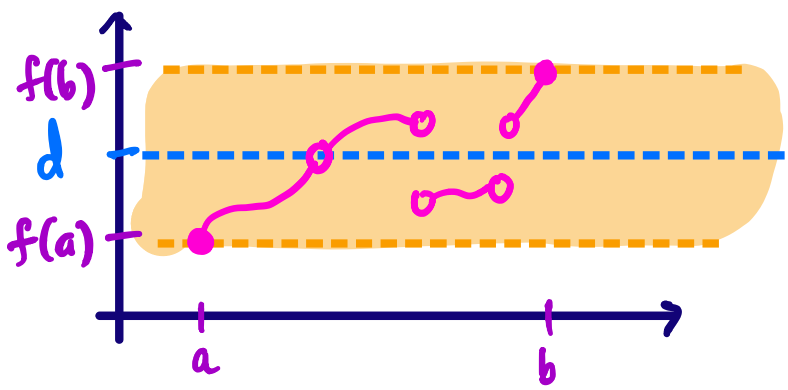
Thm: (IVT) Suppose f is continuous on $[a,b]$. Then given any number d between $f(a)$ and $f(b)$, there exists a number c in (a,b) such that $f(c)=d$.

What does this mean graphically?



Our intuitive notion of continuity implies that continuous functions cannot skip any y-values.

Challenge: Given $(a,f(a))$ and $(b,f(b))$, can we draw the graph of f so that some y-values between $f(a)$ and $f(b)$ are skipped?



Easy! As long as we are allowed to make f not continuous on $[a,b]$!

The IVT says this challenge is impossible if f has to be continuous.

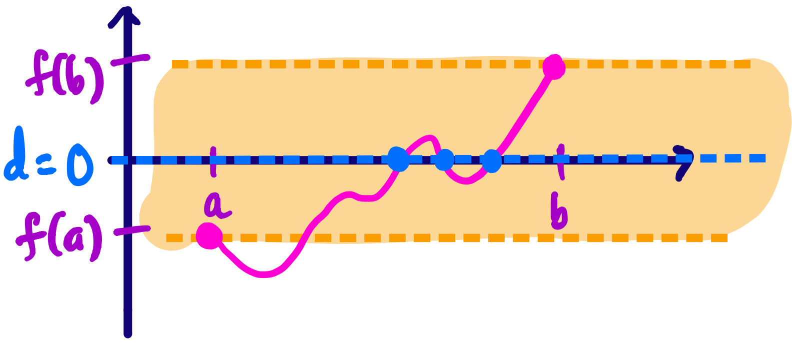
Proving an equation has a solution

Suppose we knew the following:

- f is continuous on $[a,b]$

- $f(a)$ and $f(b)$ have opposite signs (one is positive and the other is negative)

What can we say about the equation "$f(x)=0$"?



The IVT tells us that there must be at least one solution to "$f(x)=0$" in the interval (a,b)!

Ex. 1

Prove that the equation $\cos(x) = x^3 - x$ has a solution in $[\pi/4, \pi/2]$.

Solution:

Equivalently, we show $\cos(x) - x^3 + x = 0$ has a solution.

Let $f(x) = \cos(x) - x^3 + x$. Then observe:

- f is continuous on $[\pi/4, \pi/2]$

$$\left.\begin{aligned} &\bullet\ f(\pi/4) = \frac{1}{\sqrt{2}} - \frac{\pi^3}{64} + \frac{\pi}{4} \approx 1.008 \\ &\bullet\ f(\pi/2) = 0 - \frac{\pi^3}{8} + \frac{\pi}{2} \approx -2.305 \end{aligned}\right\} \text{opposite signs}$$

So by the IVT, the equation $f(x)=0$ has a solution in $(\pi/4, \pi/2)$.

Solving nonlinear inequalities

Suppose we want to solve "$f(x) > 0$" where f is a rational function (ratio of two polynomials, with no common factors). Recall the general method.

Ex. 2

Solve $\dfrac{2x-1}{x+2} > 1$.

Solution:

$\dfrac{2x-1}{x+2} - 1 > 0$ } move all terms to one side

$\dfrac{x-3}{x+2} > 0$ } simplify left side; write as quotient

$(x=3,\ x=-2)$ } determine "cut points" by setting each of numerator and denominator to 0

} draw a number line and mark cut points

$-3 \quad 0 \quad 4$ } choose one test point from each interval

$x=-3$: $\dfrac{x-3}{x+2} = 6 > 0$

$x=0$: $\dfrac{x-3}{x+2} = -\dfrac{3}{2} < 0$

$x=4$: $\dfrac{x-3}{x+2} = \dfrac{1}{6} > 0$

} in each interval, test the truth of the inequality using <u>one test point only</u>.

$x \in (-\infty,-2) \cup (3,\infty)$ } final answer is union of intervals for which inequality is true

---

Why does this work? Specifically, why can we test only one point per interval?

Q: At what values of x can a function f change sign?

A: The IVT tells us $f(x)$ can change sign at $x=c$ only if either $f(c)=0$ or f is not continuous at c. Otherwise, $f(x)$ has a single sign in each interval determined by these values of c.

When we find cut points for rational functions, we are finding these values of c where f can change sign.

- $f(c) = 0$
  ↪ set numerator to 0, solve for x
- f is not continuous at c
  ↪ set denominator to 0, solve for x

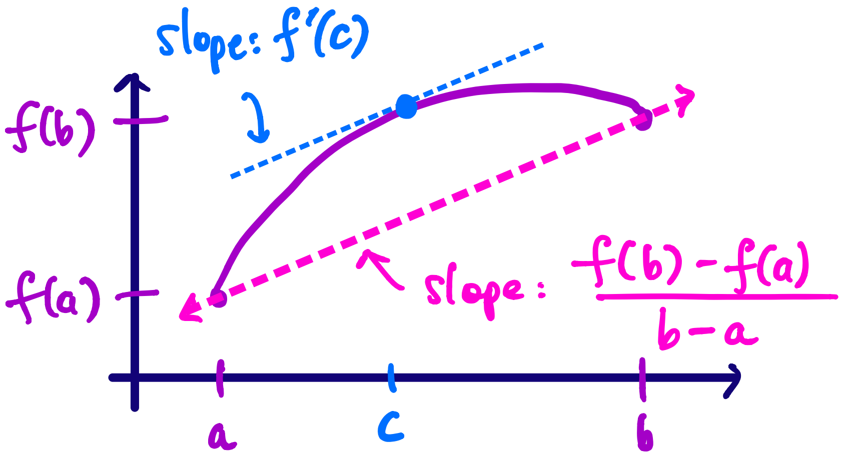
So we only test <u>one</u> point per interval because IVT tells us $f(x)$ has just one sign for the entire interval!

Optional: Mean Value Theorem

Thm: (MVT) Suppose f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists c in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Graphical interpretation of MVT:



MVT says there is a tangent line parallel to the secant line

Ex. 1

Let $f(x) = x^{2/3}$. For each interval, determine whether the hypotheses of the MVT are satisfied. If yes, find all values of c decribed by the MVT.

(a) $[-1, 1]$ (b) $[0, 8]$

Solution:

- Where is f continuous?

Power functions are continuous on their domain, so $f(x) = x^{2/3}$ is continuous on $(-\infty, \infty)$. So the continuity hypothesis of MVT is satisfied on all intervals.

- Where is f differentiable?

Since $0 < 2/3 < 1$, we know $f(x) = x^{2/3}$ is differentiable everywhere except $x = 0$. So any open interval with $x = 0$ does not satisfy the MVT hypotheses.

(a) Since $x=0$ is in $(-1,1)$, the MVT hypotheses are not satisfied.

(b) Since $x=0$ is not in $(0,8)$, the MVT hypotheses are satisfied. Hence there is some c in $(0,8)$ with:

$$f'(c) = \frac{f(8) - f(0)}{8-0}$$

$$\frac{2}{3}c^{-1/3} = \frac{8^{2/3} - 0}{8-0} = 8^{-1/3}$$

Solving for c gives

$$c = \left(\frac{3}{2}\right)^{-3} \cdot 8 = \frac{64}{27}$$

Important special case of MVT:

<u>Thm</u>: (Rolle's Theorem) Suppose f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$. Then there exists c in (a,b) with $f'(c) = 0$.
