WEAK INDESTRUCTIBILITY AND REFLECTION

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Abstract of the Dissertation

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We establish an equiconsistency between

1. weak indestructibility for all $\kappa+2$-degrees of strength for cardinals κ in the presence of a proper class of strong cardinals; and
2. a proper class of cardinals that are strong reflecting strongs.

This has a much higher consistency strength than weak indestructibility for all degrees of strength for a proper class of strong cardinals, even if we weaken (1) from a proper class to just two strong cardinals. (1) is also equivalent to weak indestructibility for all κ far beyond $\kappa+2$-strength; well beyond λ-strength for λ the next measurable limit of measurables above κ.

One direction of the equiconsistency of (1) and (2) is proven using forcing and the other using core model techniques from inner model theory. Additionally, connections between weak indestructibility and the reflection properties associated with Woodin cardinals are discussed, and similar results are derived for supercompacts and supercompacts reflecting supercompacts.

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1 Introduction

Theorems like compactness for first-order logic tell us there is a great degree of ambiguity for mathematical and set theoretic concepts. But much of this disappears if we restrict ourselves to only considering models that, in some sense, properly interpret membership (i.e. that are transitive). But this does not get rid of all ambiguity. The advent of *forcing* as a method of set theory has revealed that even with this restriction, there's a great amount of ambiguity not about the membership relation, but instead about what sets exist.

One response to this is to come up with axioms that help sharpen our conception of the universe so as to become immune to methods of forcing in some sense, and say more definitively what exists. For example, the axiom of "V = L" yields models that are immune to forcing. Under certain assumptions, the core model K has this property too. Forcing axioms can be considered in a similar way, essentially stating that we've already forced as much as we can.

The consistency of most of these axioms is unfortunately not something we can know,[1] as they often carry with them *large cardinal* hypotheses. These hypotheses assert the existence of some very nice cardinals. And such hypotheses are used as a standard measure of the strength of statements: if we want to know how strong a statement is, we show it is equiconsistent with some large cardinal axiom. The question then becomes to what extent can these large cardinals be immune to forcing?

Generally speaking, we cannot ensure large cardinal properties are immune to forcing—*indestructible*—because we may simply add via forcing a bijection between the cardinal κ and $\aleph_0$. But if we restrict our forcing to be, say, smaller than κ, we can get preservation of some properties, especially those involving elementary embeddings [8]. Large forcings can still pose a problem, but the answer isn't as clear if we restrict our attention to large posets that don't affect anything "small". These are the notions we will investigate here, and mostly we will consider $<\kappa$-strategically closed, $\leq\kappa$-distributive posets.

We investigate these posets because they form a broad class that includes Cohen forcing,

[1] Some are equiconsistent (in the sense that the consistency of one implies the consistency of the other) with ZFC: "V = L" and the forcing axiom MA for example. But most have a strictly stronger consistency strength than ZFC, like PFA or MM, and hence ZFC cannot prove their consistency. [8]

Add$(\lambda, 1)$ for $\lambda \geq \kappa^+$, which is a simple poset that (consistent relative to some large cardinal assumptions) can destroy large cardinal properties like being strong, as seen with Result 1.B • 1.

This topic starts with Laver's preparation for making a supercompact cardinal κ indestructible (by $< \kappa$-directed closed forcings) in [9]. Since then, there has been a great deal of literature about the limits of this for other cardinals and varying degrees of strength.[2] Ignoring for the moment what exactly a lot of this terminology means, a short selection of results in this area is the following.

- [2] explores making all degrees of supercompactness and strength indestructible while there is a supercompact.
- [6] explores the ways in which indestructible cardinals can be made destructible and subsequently resurrected.
- [3] explores a weaker version of indestructibility for strength while there is a strong cardinal, and gets an equiconsistency result: universal weak indestructibility (while there is a strong cardinal) is equiconsistent with a *hyperstrong* cardinal.
- [1] further explores this weaker version of indestructibility for strength to get (weak) indestructibility for lots of (strong) supercompact cardinals, and again establishes an equiconsistency for this indestructibility for many strongs from a hyperstrong cardinal (a proper class if we cut off the universe at the hyperstrong).

This work further explores [3] and [1]. Generally speaking, these works focus on exploring indestructibility for large degrees of strength and supercompactness in the presence of many strong or supercompact cardinals. Ostensibly, this is a harder task than showing weak indestructibility for smaller degrees of strength, simply because a strong cardinal's degrees of strength are arbitrarily large. But instead, this work shows that to get universal weak indestructibility for small degrees of strength and supercompactness, we actually need a large *increase* in consistency strength, much more than a proper class of hyperstrong or

[2]By a *degree of strength* of a caridnal κ, I mean an ordinal ρ such that κ is ρ-strong, defined with Definition 1.A • 9.

hypercompact cardinals, just one of which is used in [3] and [1] to get (many) weakly indestructible strongs and supercompacts.

There is a balance to the amount of indestructibility one can have, and generally speaking, indestructibility for large degrees of strength will conflict with indestructibility for small degrees of strength. The largest sense of "small" degrees is degrees of strength below the next cardinal κ that is $\kappa + 2$-strong, and the smallest sense of "large" degrees is degrees slightly above such a cardinal. So the universal indestructibility results of [3] and [2] critically proceed by making sure there is no measurable above the resulting strong or supercompact cardinal, thus being incompatible with the existence of multiple strong or supercompact cardinals. The work in [1] considers multiple strong and supercompact cardinals with indestructibility, but as a consequence, ignores indestructibility for smaller degrees of strength.

The following theorem was the main goal of [3].

1 • 1. Theorem (Apter–Sargsyan). *The following are equiconsistent with* ZFC*:*

1. *There is a hyperstrong cardinal.*
2. *There is a strong cardinal and universal weak indestructibility for all degrees of strength holds.*

Hence a hyperstrong cardinal is able to yield indestructibility for all small degrees of strength, and produce a strong cardinal (with also indestructible strength). Considering the balance of indestructible degrees of strength, a natural strengthening of (2) is having a proper class of strong cardinals while still ensuring universal weak indestructibility for small degrees of strength. A natural guess at the consistency strength of this is the existence of many hyperstrong cardinals. But this is insufficient, and indeed the consistency strength of this is strictly larger than a proper class of hyperstrong cardinals. The following is one of the main results of this thesis that ensures this increase in strength, assuming for the moment that a single strong reflecting strongs cardinal (with a strong above it) gives the consistency of a proper class of hyperstrongs.

1 • 2. Theorem (Main Goal). *The following are equiconsistent with* ZFC*:*

1. *There is a proper class of strongs reflecting strongs.*

2. *There is a proper class of strong cardinals and weak indestructibility for any cardinal κ's $\kappa+2$-strength.*

3. *There is a proper class of strong cardinals and weak indestructibility for any cardinal κ's λ-strength where λ is below the next measurable limit of measurables above κ.*

We also establish a similar result for supercompactness, and the following just by continuing the preparation from Theorem 1 • 2 through to a Woodin cardinal.

1 • 3. Corollary (Side Result). *The following are equiconsistent with* ZFC*:*

1. *There is a Woodin cardinal.*

2. *There is a Woodin cardinal δ such that weak indestructibility for any κ's $\kappa+2$-strength holds below δ.*

There are then several small side results related to Woodincardinals and indestructibility, including the following.

1 • 4. Result (Another Side Result). *There is no cardinal δ that is Woodin by weak indestructibility in the sense that that for every $A \subseteq \delta$, there is a $< \delta$-strong cardinal reflecting A such that this strength and reflection are weakly indestructible. Nevertheless, from a Woodin cardinal δ it's possible to preserve δ's Woodin-ness while forcing that every $< \delta$-strong cardinal has its strength weakly indestructible.*

This gives the consistency of a large number of weakly indestructible strong cardinals from a Woodin cardinal. Moreover, it tells us that weak indestructibility for large degrees of strength and for small degrees of strength below a Woodin cardinal are equiconsistent. So while weak indestructibility for small degrees (with, say, two strong cardinals) is strictly stronger than for a proper class of weakly indestructible strong cardinals, this difference levels out as we approach a Woodin.

1.A Background

Throughout we will use the following notation.

1.A • 1. Definition.

- *For* $\delta \in \mathrm{Ord}$, *write* $\delta^{+\sharp}$ *for the least measurable cardinal* $\kappa > \delta$.
- *Write* $\delta^{+\sharp\sharp}$ *for the least cardinal* $\kappa > \delta$ *that is* $\kappa + 2$*-strong.*
- *Write* $\delta^{+\P}$ *for the least strong cardinal* $\kappa > \delta$.
- *For a transitive model of set theory,* M, *and* $\lambda \in \mathrm{Ord}^{\mathrm{M}}$, *we write* $\mathrm{M} \mid \lambda$ *for* $\mathrm{V}^{\mathrm{M}}_{\lambda}$, *the* λ*-th stage of the cumulative hierarchy in* M.
- *For a poset* $\mathbb{R} \in \mathrm{V}$, *we write* $\mathrm{V}^{\mathbb{R}}$ *for an arbitrary generic extension of* V *by* $\mathbb{R}$, *occasionally working below some particular condition in* $\mathbb{R}$.

Let me be a little more precise about the notions of indestructibility used here.

1.A • 2. Definition. *Let* κ *be a cardinal,* φ *a property of cardinals, and* ψ *a property of posets.*

- κ *has* φ *as* indestructible *by posets with* ψ *iff*

$$\forall \mathbb{P} \text{ a poset } (\psi(\mathbb{P}) \to \mathbb{P} \Vdash \text{“}\varphi(\check{\kappa})\text{”}).$$

- κ *has* φ *as* weakly indestructibile *by posets with* ψ *iff it's indestructible by* $\leq \kappa$*-distributive posets with* ψ.

So the usual Laver indestructibility refers to indestructibility for all degrees of supercompactness of a single (supercompact) cardinal κ by $< \kappa$-directed closed posets. For strength, we usually consider weak indestructibility by $< \kappa$-strategically closed posets, and thus indestructibility $< \kappa$-strategically closed, $\leq \kappa$-distributive posets. $< \kappa$-strategic closure is a significant weakening of $< \kappa$-directed closure and is defined as follows.

1.A • 3. Definition. *Let* $\mathbb{P}$ *be a poset and* κ, λ *ordinals. The game* $\mathcal{G}^{\lambda}_{\mathbb{P}}$ *is the two person game of length* $\leq \lambda$

$$\begin{array}{lccccccccc} \mathbf{I}: & p_0 = \mathbb{1}^{\mathbb{P}} & & p_2 & & \cdots & p_\omega & & p_{\omega+2} & \cdots \\ \mathbf{II}: & & p_1 & & p_3 & \cdots & & p_{\omega+1} & & \cdots, \end{array}$$

where $p_\alpha \leq^{\mathbb{P}} p_\beta$ *for* $\alpha \geq \beta$*, and assuming such a* p_α *exists,* **I** *plays* p_α *for even* $\alpha < \lambda$ *(including limit* α*) and* **II** *plays* $p_\alpha \in \mathbb{P}$ *for odd* $\alpha < \lambda$*. We say* **II** *wins iff either player has no available move.* **I** *wins iff the resulting sequence of* p_α*s has length* λ*.*

- $\mathbb{P}$ *is* $\leq \kappa$-strategically closed *iff* **I** *has a winning strategy in* $\mathcal{G}_{\mathbb{P}}^{\kappa+1}$*.*
- $\mathbb{P}$ *is* κ-strategically closed *iff* **I** *has a winning strategy in* $\mathcal{G}_{\mathbb{P}}^{\kappa}$*.*
- $\mathbb{P}$ *is* $< \kappa$-strategically closed *iff* $\mathbb{P}$ *is* $\leq \alpha$*-strategically closed for all ordinals* $\alpha < \kappa$*.*
- $\mathbb{P}$ *is* $\ll \kappa$-strategically closed *iff* $\mathbb{P}$ *is* $\leq \lambda$*-strategically closed for all cardinals* $\lambda < \kappa$*.*

Strategic closure is a weakening of the usual closure of preorders:

- $\leq \kappa$-strategic closure corresponds to $\leq \kappa$-closure.
- $< \kappa$, $\ll \kappa$, and κ-strategic closed correspond to $< \kappa$-closure in slightly different senses.

Basically whereas $< \kappa$-closure ensures we can always extend a $\leqslant^{\mathbb{P}}$-decreasing sequence $\langle p_\alpha : \alpha < \lambda \rangle$ whenever $\lambda < \kappa$, κ-strategic closure only allows us to state this whenever the even entries of the sequence conform to a certain strategy by player **I**. Put in another sense, $< \kappa$-closure gives total freedom to choose a decreasing sequence and find something below it. κ-strategic closure only gives control over half of the sequence, relying on **I**'s strategy to extend at limits. Usually, in proving things with strategic closure, we—the theorem provers—play the role of **II**, relying on **I**'s strategy to clean up our mess, especially at limit stages. Note that Indestructibility for $< \kappa$-strategically closed posets is much stronger than indestructibility for $< \kappa$-directed closed posets, a notion that will be defined when discussing supercompactness.

As a side note, $\leq |\alpha|$-strategic closure is ostensibly weaker than α-strategic closure for $\alpha > |\alpha|$ since the game $\mathcal{G}_{\mathbb{P}}^{\alpha}$ could be much longer than $\mathcal{G}_{\mathbb{P}}^{|\alpha|+1}$. It's clear that we can play the game multiple times, which gives some additional strength: $\leq \kappa$-strategic closure implies $\leq \kappa + \kappa$-strategic closure and $< \kappa \cdot \omega$-strategic closure. Going beyond this is more difficult, and it's unclear to me whether $\leq \kappa$-strategic closure more generally implies $< \kappa^+$-strategic closure.

Despite their differences, the similarities between $<\kappa$-closed and κ-strategically closed preorders are enough to show to allow some arguments about $<\kappa$-closure to go through about κ-strategic closure. For example, κ-strategically closed preorders are $\leq\kappa$-distributive by basically the same proof as with full closure. There are many other common results in iterated forcing where using strategic closure vs. full closure makes no difference [4].

1.A • 4. Definition. *Let κ be a cardinal. A preorder $\mathbb{P}$ is* $<\kappa$-distributive *iff for every collection $\mathcal{D}$ of open, dense sets of $\mathbb{P}$, if $|\mathcal{D}|<\kappa$ then $\bigcap\mathcal{D}\neq\emptyset$ is open, dense. We may similarly define $\leq\kappa$-distributive.*

1.A • 5. Corollary. *For any infinite cardinal κ, if $\mathbb{P}$ is $\ll\kappa$-strategically closed, then $\mathbb{P}$ is $<\kappa$-distributive.*

In paticular, considering $\kappa=\delta^+$, $\leq\delta$-strategic closure implies $\leq\delta$-distributivity.[3]

$\leq\kappa$-distributive preorders are important for a variety of reasons, mostly the following.

1.A • 6. Result. *Let κ be a cardinal. Let $\mathbb{P}$ be $\leq\kappa$-distributive. Suppose $f:\kappa\to\mathrm{V}$ is in $\mathrm{V}^{\mathbb{P}}$. Therefore $f\in\mathrm{V}$. More succinctly, $\mathrm{V}^{\mathbb{P}}\vDash$ "${}^{\kappa}\mathrm{V}\subseteq\mathrm{V}$", and in particular, $\mathrm{V}^{\mathbb{P}}\mid\kappa+1=\mathrm{V}\mid\kappa+1$.*

So if κ is measurable, its $\kappa+1$-strength is weakly indestructible. So easy models of universal indestructibility for small degrees of strength are those with no large cardinals, or those with only measurables. Obviously if a poset *isn't* $<\kappa$-distributive, it can collapse the strength of κ, e.g. through a trivial collapse $\mathrm{Col}(\omega,\kappa)$.

1.A • 7. Corollary. *Let κ be measurable. Therefore κ's measurability (i.e. its $\kappa+1$-strength) is weakly indestructible.*

Proof. If $U\in\mathrm{V}$ is a measure and $\mathbb{P}\in\mathrm{V}$ is a $\leq\kappa$-distributive poset, then $U\in\mathrm{V}\subseteq\mathrm{V}^{\mathbb{P}}$ is still a measure in $\mathrm{V}^{\mathbb{P}}$. ⊣

On the topic of forcing, we also make use of lifting embeddings to show large cardinal properties in the generic extension.

[3] $<\kappa$-distributivity is indeed distinct from κ-strategic closure. For example, shooting a club in (a stationary, co-stationary subset of) ω_1 is $<\aleph_1$-distributive, but not $\aleph_1$-strategically closed. Indeed, shooting a club isn't even $\leq\aleph_0$-strategically closed: $\leq\aleph_0$-strategically closed preorders preserve stationary subsets of $\aleph_1$ with nearly the same proof as with $\leq\aleph_0$-closed preorders.

1.A • 8. Theorem. *Let $j : \mathrm{V} \to \mathrm{M}$ be elementary as a class of* V*. Let G be $\mathbb{P}$-generic over* V *for $\mathbb{P} \in \mathrm{V}$ a poset. Therefore for any $H \subseteq j(\mathbb{P})$, the following are equivalent:*

- *j"$G \subseteq H$ with H being $j(\mathbb{P})$-generic over* M*.*
- *$j^+ : \mathrm{V}[G] \to \mathrm{M}[H]$ is elementary as a class of $\mathrm{V}[G]$ with $j^+(G) = H$ and $j^+ \restriction \mathrm{V} = j$.*

The rest of this document will assume familiarity with strong cardinals and their fundamental properties.

1.A • 9. Definition.

- *A cardinal κ is* λ-strong *iff there is an elementary embedding $j : \mathrm{V} \to \mathrm{M}$ (as a class of* V*) with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $\mathrm{V} \mid \lambda = \mathrm{M} \mid \lambda$.*
- *A cardinal is* strong *iff it is λ-strong for every $\lambda \in \mathrm{Ord}$.*
- *We call embeddings (extenders) witnessing λ-strength* λ-strong embeddings (extenders)*.*

Beyond their definition, strong cardinals are useful for the following reflection principle.

1.A • 10. Lemma ($\mathbf{\Sigma_2}$-reflection). *Let κ be a strong cardinal. Suppose $\mathrm{V} \mid \delta \vDash$ "$\varphi(\vec{x})$" for some formula φ and parameters $\vec{x} \in \mathrm{V} \mid \kappa$. Therefore there are unboundedly many $\alpha < \kappa$ such that $\mathrm{V} \mid \alpha \vDash$ "$\varphi(\vec{x})$".*

Proof. Let $j : \mathrm{V} \to \mathrm{M}$ be a δ-strong embedding so that $j(\vec{x}) = \vec{x}$, $\delta < j(\kappa)$, and $\mathrm{M} \mid \delta = \mathrm{V} \mid \delta \vDash$ "$\varphi(\vec{x})$". In M, there is then some level of the cumulative hierarchy below $j(\kappa)$ that satisfies $\varphi(j(\vec{x}))$ and so by elementarity, in V there's a level $\mathrm{V} \mid \alpha$ of the cumulative hierarchy with $\mathrm{rank}(\vec{x}) < \alpha < \kappa$ satisfying $\varphi(\vec{x})$. Because we could have thrown in any fixed ordinal $\beta < \kappa$ as a useless parameter into $\vec{x}$, the idea above gives an $\alpha > \beta$ with this property. ⊣

We now establish some definitions related to [3], explaining Theorem 1 • 1.

1.A • 11. Definition. *Let κ and $\alpha > 0$ be ordinals. We define by transfinite recursion what it means for κ to be* α-hyperstrong*.*

- *κ is* 0-hyperstrong *iff κ is strong.*

- κ *is* $<\alpha$-hyperstrong *iff* κ *is* ξ*-hyperstrong for every* $\xi<\alpha$.

- κ *is* α-hyperstrong *iff for any* $\lambda>\kappa$, *there is an extender* E *giving an ultrapower embedding* $j_E:\mathrm{V}\to\mathrm{M}$ *with* $\mathrm{cp}(j_E)=\kappa$, $j_E(\kappa)>|\mathrm{V}\restriction\lambda|$, $\mathrm{V}\restriction\lambda\subseteq\mathrm{M}$, *and* $\mathrm{M}\vDash$ *"*κ *is* $<\alpha$*-hyperstrong".*

- κ *is* hyperstrong *iff* κ *is* ξ*-hyperstrong for every ordinal* ξ.

We will work with hyperstrongs mostly to show that they are insufficient in establishing the main result of Theorem 1 • 2. We will also make use of strongs reflecting strongs. We will discuss the interaction between these two definitions in the next subsection.

1.A • 12. Definition. *Let* κ *be a cardinal and* λ *an ordinal.*

- *We say* κ *is* λ-strong reflecting strongs (λ*-srs*) *iff there's a* λ*-strong* (κ,λ)*-extender* E *giving an elementary* $j_E:\mathrm{V}\to\mathrm{M}=\mathrm{Ult}_E(\mathrm{V})$ *with*

$$\{\xi<\lambda:\xi\text{ is strong}\}=\{\xi<\lambda:\mathrm{M}\vDash\text{“}\xi\text{ is strong”}\}.$$

- *We call such an embedding (extender) a* λ-srs embedding (*extender*).

- *We say* κ *is* srs *iff* κ *is* λ*-srs for every* $\lambda>\kappa$*; and we say* κ *is* $<\lambda$*-srs if* κ *is* α*-srs for each* $\alpha<\lambda$.

For the most part, we only care about strongs reflecting strongs when they have strongs above them since if there is no strong above κ, κ is srs iff κ is hyperstrong (as we will show with Corollary 1.B • 8). But under the not-uncommon assumption of a strong above, this is a strengthening of hyperstrong cardinals both in the sense of being hyperstrong and in having a (much) higher consistency strength. The (2) implies (1) direction of Theorem 1 • 2 then tells us that a proper class of hyperstrong cardinals is insufficient to ensure universal weak indestructibility for very small degrees of strength with a proper class of strongs.

For now, we end this subsection with the following definition.

1.A • 13. Definition. Universal weak indestructibility for small degrees of strength (UWISS) *is the proposition that for all* κ, *if* κ *is* ρ*-strong for* $\rho<\kappa^{+\sharp}$, *then* κ*'s* ρ*-strength is weakly indestructible by* $<\kappa$*-strategically closed posets.*

1.B Limiting results

The limit imposed on universal indestructibility was explored in [2] for full indestructibility, but we may consider weak indestructibility in a similar way. The main limiting result is the following with the proof adapted from [2] and the Superdestruction Theorem III from [6].

1.B • 1. Result. *Assume* GCH.[4] *Suppose* κ *is strong and this is not destroyed by* $\mathrm{Add}(\kappa^+, 1)$. *Suppose* $\kappa^{+\sharp\sharp}$ *exists. Therefore, there are arbitrarily large* $\delta < \kappa$ *whose* $\delta + 2$*-strength is destroyed by* $\mathrm{Add}(\delta^+, 1)$.

Proof. Let A be $\mathrm{Add}(\kappa^+, 1)$-generic over V. By hypothesis, κ is still strong in $\mathrm{V}[A]$ and so we get a sufficiently strong embedding $j : \mathrm{V}[A] \to \mathrm{M}[j(A)]$ with $\mathrm{cp}(j) = \kappa$ so that $\lambda = \kappa^{+\sharp\sharp}$ is still $\lambda + 2$-strong in $\mathrm{M}[j(A)]$.

Claim 1. *λ's $\lambda + 2$-strength is destroyed by* $\mathrm{Add}(\lambda^+, 1)$ *in* $\mathrm{V}[A]$.

Proof. Suppose not. Let G be $\mathrm{Add}(\lambda^+, 1)$-generic over $\mathrm{V}[A]$ and let $i : \mathrm{V}[A*G] \to \mathrm{N}[i(A*G)]$ be $\lambda + 2$-strong with $\mathrm{cp}(i) = \lambda$. Since $A \subseteq \kappa < \mathrm{cp}(i)$, $i(A * G) = A * i(G)$. Without loss of generality, i is generated by extenders such that $\mathrm{N}[A * i(G)]$ is closed under λ-sequences of $\mathrm{V}[A * G]$.

By gap forcing [5], $i \restriction \mathrm{V} : \mathrm{V} \to \mathrm{N}$ is a class of V, is closed under λ-sequences, and is still $\lambda + 2$-strong. It follows that $G \subseteq \lambda^+$ is in $\mathrm{H}_{\lambda^{++}}^{\mathrm{V}[A*G]} = \mathrm{H}_{\lambda^{++}}^{\mathrm{N}[A*i(G)]}$ and hence in $\mathrm{N}[A * i(G)]$, but wasn't added by $i(\mathrm{Add}(\lambda^+, 1))$ by $\leq i(\lambda^+)$-distributivity in $\mathrm{N}[A]$. Thus $G = \dot{G}_A \in \mathrm{N}[A]$ for some $\mathrm{Add}(\kappa^+, 1)$-name $\dot{G}$ which can be thought of as a function from $(\lambda^+)^{\mathrm{V}} = (\lambda^+)^{\mathrm{N}}$ to antichains of $\mathrm{Add}(\kappa^+, 1)$, and is therefore in the hereditarily $< (\lambda^{++})^{\mathrm{N}} = (\lambda^{++})^{\mathrm{V}}$-sized sets $\mathrm{H}_{\lambda^{++}}^{\mathrm{N}} = \mathrm{H}_{\lambda^{++}}^{\mathrm{V}}$. The ability to consider all of this in V means $G \in \mathrm{V}[A]$, a contradiction. $\dashv$

Without loss of generality, $\mathrm{M}[j(A)]$ has enough agreement with $\mathrm{V}[A]$ to witness this fact as well. Since $\kappa < \lambda < j(\kappa)$, this is reflected down into $\mathrm{V}[A]$: there are arbitrarily large $\delta < \kappa$ such that δ's $\delta + 2$-strength is destroyed by $\mathrm{Add}(\delta^+, 1)$ in $\mathrm{V}[A]$. It follows that $\mathrm{Add}(\kappa^+, 1) * \mathrm{Add}(\delta^+, 1)$ is $< \delta$-directed closed and $\leq \delta$-distributive in V but destroys δ's

[4] Note that the assumption of GCH here isn't much of a problem, since in the context of indestructibility by a large class of posets, one can just force the appropriate instances with Cohen forcing. More precisely, for the theorem to go through, we merely need $\lambda = \kappa^{+\sharp\sharp}$ to have $2^{2^\lambda} = \lambda^{++}$ or else we must consider a $\lambda > \kappa$ that is δ-strong where $2^{2^\lambda} = \lambda^{+\delta}$.

$\delta+2$-strength. Since $\text{Add}(\kappa^+,1)$ is sufficiently distributive, it can't add to nor destroy δ's $\delta+2$-strength. So it must be that $\text{Add}(\delta^+,1)$ destroyed δ's $\delta+2$-strong ⊣

And this result generalizes to larger degrees of strength.

1.B • 2. Corollary. *Assume* GCH. *Let* $\mu \in \text{Ord}$. *Suppose* κ *is strong, and this is not destroyed by* $\text{Add}(\kappa^{+\mu},1)$. *Suppose there is a* $\lambda > \kappa^{+\mu+1}$ *that is* $\lambda+\mu+1$*-strong. Therefore*

1. *There are unboundedly many* $\delta < \kappa$ *such that* δ *is* $\delta+\rho$*-strong for some* ρ*, but this is destructible by* $\text{Add}(\delta^{+\rho},1)$.
2. *If* $\mu < \kappa$*, then there are arbitrarily large* $\delta < \kappa$ *such that* δ*'s* $\delta+\mu+1$*-strength is destructible by* $\text{Add}(\delta^{+\mu},1)$.

Proof. The exact same proof as with Result 1.B • 1 tells us that after forcing with $\text{Add}(\kappa^{+\mu},1)$ to get $A \subseteq \kappa^{+\mu}$, a sufficiently strong embedding $j : \text{V}[A] \to \text{M}$ with $\text{cp}(j) = \kappa$ yields that λ's $\lambda+\mu+1$-strength in $\text{V}[A]$ and $\text{M}[j(A)]$ is destroyed by $\text{Add}(\lambda^{+\mu},1)$. If $\mu = j(\mu) < \kappa$, then we can reflect the following statement: for each $\alpha = j(\alpha) < \kappa$,

$$\begin{aligned}\text{M}[j(A)] \vDash \text{“}\exists\delta\ (j(\alpha) < \delta < j(\kappa) \wedge \delta \text{ is } \delta+j(\mu)+1\text{-strong}\\ \wedge\ \delta\text{'s } \delta+j(\mu)+1\text{-strength is destroyed by } \text{Add}(\delta^{+j(\mu)},1)\text{”},\end{aligned}$$

to get (2) in $\text{V}[A]$. If $\mu \geq \kappa$, we at least can reflect, for each $\alpha < \kappa$,

$$\begin{aligned}\text{M}[j(A)] \vDash \text{“}\exists\delta\ \exists\rho\ (j(\alpha) < \delta < j(\kappa) \wedge \delta \text{ is } \rho\text{-strong}\\ \wedge\ \delta\text{'s } \rho\text{-strength is destroyed by } \text{Add}(\delta^{+\rho},1)\text{”},\end{aligned}$$

to get (1) in $\text{V}[A]$. Note that $\delta+\rho < \kappa$ (or $\delta+\mu$ if $\mu < \kappa$) since otherwise δ would be $<\kappa$-strong where κ is strong. By Lemma 1.A • 10, this would mean that δ is strong since the lack of any extenders would be reflected down below κ. Because we destroyed strength, δ wouldn't be strong, and so we'd have a contradiction. So since $\delta+\rho<\kappa$, distributivity of $\text{Add}(\kappa^{+\mu},1)$ doesn't affect δ's strength, and hence δ would have been destructible in V by $\text{Add}(\delta^{+\rho},1)$. ⊣

In particular, one cannot have universal weak indestructibility for *all* degrees of strength with two strong cardinals. This also easily generalizes to supercompactness in place of strength.

The contrapositive to Result 1.B • 1 details one aspect of what goes wrong if one naïvely tries to continue the preparation from either [2] or [3].

1.B • 3. Corollary. *Assume* GCH*. Suppose κ is strong and $\kappa^{+\sharp\sharp}$ exists. Suppose every $\delta < \kappa$ that is $\delta + 2$-strong has this indestructible by* $\mathrm{Add}(\delta^+, 1)$*. Therefore κ's strength is destroyed by* $\mathrm{Add}(\kappa^+, 1)$.

All of this is just to justify that one simultaneously cannot have UWISS with any reasonable amount of indestructibility for any large degrees of strength. Moreover, Result 1.B • 1 and Corollary 1.B • 3 aren't unique to weak indestructibility as noted in the original Theorems 10, 11, and 12 of [2], which hold for full indestructibility. Really what Corollary 1.B • 3 and Result 1.B • 1 tell us is that the only way to ensure universal indestructibility for *all* degrees of strength with a strong cardinal κ is to ensure $\kappa^{+\sharp\sharp}$ doesn't exist. And this is exactly what [2] and [3] do.

Let us now show why—assuming the equiconsistency of Theorem 1 • 2—hyperstrongs are insufficient to establish UWISS with a proper class of (weakly destructible) strongs. To do this, we need to unfortunately look at the fine details of calculating hyperstrongs and strongs reflecting strongs inside the cumulative hierarchy. A straightforward, somewhat tedious induction gives the following, recalling Definition 1.A • 11 for the definition of a hyperstrong cardinal.

1.B • 4. Corollary. *Fix $\kappa, \alpha \in \mathrm{Ord}$ with κ infinite. Therefore the following are equivalent.*

1. *κ is $< \alpha$-hyperstrong.*

2. *for all limit $\delta \geq |\kappa + \alpha|^+$, $\mathrm{V} \mid \delta \vDash$ "κ is $< \alpha$-hyperstrong".*

Proof. (2) always implies (1) since any failure of hyperstrength is reflected to $\mathrm{V} \mid \delta$ for arbitrarily large δ. So we assume (1) and aim to show (2) by induction. For $\alpha = 1$—i.e. 0-hyperstrength or just being strong—this follows by the absoluteness of strength of extenders between levels of the cumulative hierarchy. All extenders witnessing the λ-strength of κ for $\lambda < \delta$ are in $\mathrm{V} \mid \delta$. So κ is strong implies $\mathrm{V} \mid \delta$ has such extenders for sufficiently large δ.

For limit α, this follows inductively: for each $\xi < \alpha$, κ is $< \xi + 1$-hyperstrong iff $\mathrm{V} \mid \delta \vDash$ "κ is $< \xi + 1$-hyperstrong" for all $\delta > |\kappa + \xi|^+$. Hence κ is $< \alpha$-hyperstrong implies $\mathrm{V} \mid \delta$ satisfies this for δ greater than $\sup_{\xi<\alpha} |\kappa + \xi|^+ \leq |\kappa + \alpha|^+$.

For successor α, assume the results for $< \alpha$-hyperstrength. In fact, we take the inductive hypothesis as this holding for any transitive model of ZFC containing $|\kappa+\alpha|^+$. But without loss of generality, and for the sake of notation, take our model as V. As notation, for E an extender, let $j_E : \mathrm{V} \to \mathrm{Ult}_E(\mathrm{V})$ be the canonical embedding.

Suppose κ is α-hyperstrong and let $\delta \geq |\kappa + \alpha|^+$ be arbitrary. Inductively, $\mathrm{V} \mid \delta \vDash$ "κ is $< \alpha$-hyperstrong". Let $\lambda < \delta$ be arbitrary. In V, there's a λ-strong extender E on κ such that $\mathrm{Ult}_E(\mathrm{V}) \vDash$ "κ is $< \alpha$-hyperstrong". Inductively, $\mathrm{Ult}_E(\mathrm{V}) \mid \gamma \vDash$ "κ is $< \alpha$-hyperstrong" for $\gamma = \sup_{\beta<\delta} j_E(\beta)$ because $\gamma \geq \delta \geq |\kappa+\alpha|^+ \geq (|\kappa+\alpha|^+)^{\mathrm{Ult}_E(\mathrm{V})}$ and the inductive hypothesis holds. But $E \in \mathrm{V} \mid \delta$ and taking the ultrapower there yields $\mathrm{Ult}_E^{\mathrm{V}|\delta}(\mathrm{V} \mid \delta) = \mathrm{Ult}_E(\mathrm{V}) \mid \gamma$. Hence E witnesses the α-hyperstrength of κ in $\mathrm{V} \mid \delta$. ⊣

We also make use of the following lemma.

1.B • 5. Lemma. *Let κ be strong such that $\kappa^{+¶}$ exists. Therefore,*

1. *If κ is $< \kappa^{+¶}$-hyperstrong, then κ is hyperstrong.*
2. *If $\mathrm{V} \mid \kappa^{+¶} \vDash$ "κ is hyperstrong", then κ is $< \kappa^{+¶}$-hyperstrong and hence hyperstrong by (1).*

Proof. 1. Suppose κ is not hyperstrong. By reflection,

$$\mathrm{V} \mid \delta \vDash \text{“}\exists\gamma\ (\kappa \text{ is not } \gamma\text{-hyperstrong}) \wedge \text{there is no largest cardinal”} \qquad (*)$$

for some $\delta > \alpha$. By Lemma 1.A • 10, we get arbitrarily large $\delta < \kappa^{+¶}$ such that $(*)$ holds. Since there is no largest cardinal in $\mathrm{V} \mid \delta$, such δ satisfy $\delta > |\kappa + \gamma|^+$ for any γ as in $(*)$. Corollary 1.B • 4 therefore implies κ is not γ-hyperstrong for some $\gamma < \kappa^{+¶}$, a contradiction.

2. Suppose κ is not α-hyperstrong for some $\alpha < \kappa^{+¶}$. This fact is reflected to some level $\mathrm{V} \mid \delta$, and by Lemma 1.A • 10, to arbitrarily large initial segments $\mathrm{V} \mid \delta$ for $\delta < \kappa^{+¶}$. In particular, we get some δ_0 with $|\kappa + \alpha|^+ < \delta_0 < \kappa^{+¶}$ such that $\mathrm{V} \mid \delta_0 \vDash$ "κ is not

α-hyperstrong". But applying Corollary 1.B • 4 in $V \mid \kappa^{+\P} \models \mathsf{ZFC}$, by the hypothesis, $(V \mid \kappa^{+\P}) \mid \delta = V \mid \delta \models$ "κ is α-hyperstrong" for *all* $\delta > |\kappa + \alpha|^{+}$, a contradiction with the hypothesis on δ_0. Hence κ must be α-hyperstrong for each $\alpha < \kappa^{+\P}$, and so hyperstrong by (1). ⊣

This lemma allows us to show that an srs (with a strong above it) is strictly stronger than a proper class of hyperstrongs.

1.B • 6. Result. *Let κ be srs such that $\kappa^{+\P}$ exists. Therefore κ is hyperstrong. In fact, any single $\kappa^{+\P}$-srs embedding witnesses all degrees of hyperstrength of κ. Moreover, in $V \mid \kappa$, there is a proper class of hyperstrongs.*

Proof. Let κ be srs. κ is already 0-hyperstrong. There is then a $\kappa^{+\P} + 1$-srs embedding $j : V \to M$ with $\mathrm{cp}(j) = \kappa$. We proceed by induction on α to show κ is $< \alpha$-hyperstrong in V and M, with the base case of $\alpha = 1$ already true. The limit case is trivial, so we consider only the successor case: showing α-hyperstrength in V and M. By Lemma 1.B • 5 (1), we may assume $\alpha < \kappa^{+\P}$.

Suppose κ is $< \alpha$-hyperstrong in V and M so that κ is α-hyperstrong in V. If there is some λ with no λ-strong extender in M witnessing α-hyperstrength of κ, then by reflection, some $M \mid \delta$ (correctly) satisfies there's no such extender. By Lemma 1.A • 10, we can assume $\delta, \lambda < \kappa^{+\P}$ so that $M \mid \delta = V \mid \delta$ also has no such extender. Corollary 1.B • 4 tells us V must satisfy κ isn't α-hyperstrong, a contradiction.

We also can show that κ is the limit of hyperstrongs, giving $V \mid \kappa$ as a model of a proper class of hyperstrongs. Suppose not: κ is the least hyperstrong above some λ. By elementarity, $j(\kappa)$ is still the least hyperstrong above $j(\lambda) = \lambda$ in M but $V \mid \kappa^{+\P} = M \mid \kappa^{+\P} \models$ "κ is hyperstrong". So by Lemma 1.B • 5 (2), κ is hyperstrong in M. But $\lambda < \kappa < j(\kappa)$ contradicts that $j(\kappa)$ is the least hyperstrong above λ in M. ⊣

Note also that being srs has a related property to hyperstrength with regard to embeddings $j : V \to M$. More precisely, if κ is λ-srs by a λ-srs embedding $j : V \to M$, then in M, κ is strong and $< \lambda$-srs, similar to a condition in Definition 1.A • 11. This is just due to the restriction on rank of ξ-srs extenders for $\xi < \lambda$.

1.B • 7. Result. *Let $j : \mathrm{V} \to \mathrm{M}$ be a λ-srs embedding. Therefore* $\mathrm{cp}(j)$ *is* $< \lambda$*-srs in* M.

This also gives an alternative way of proving a weaker version of Result 1.B • 6.

1.B • 8. Corollary. *Let κ be strong and λ-srs. Therefore κ is λ-hyperstrong. In particular, any srs is hyperstrong.*

Proof. Proceed by induction on $\lambda \geq \kappa$. For $\lambda = \kappa$, this is clear. For $\lambda > \kappa$, let $j : \mathrm{V} \to \mathrm{M}$ be λ-srs with $\mathrm{cp}(j) = \kappa$. It follows that κ is $< \lambda$-srs in M. Since κ is strong in M, inductively κ is $< \lambda$-hyperstrong in M and thus λ-hyperstrong in V. Limit stages are obvious. $\dashv$

This shows that if there is no strong above κ, κ is srs iff κ is hyperstrong. In this way, by cutting off the universe at the least measurable above a hyperstrong κ, the original result of [3] made use of a single srs cardinal.

In a similar vein to Result 1.B • 6, it will be useful much later in section 4 to understand calculations of srs cardinals inside the cumulative hierarchy.

1.B • 9. Lemma. *Let $\kappa < \delta \in$* Ord *with δ strong. Therefore the following are equivalent:*

1. $\mathrm{V} \mid \delta \vDash$ *"κ is srs".*

2. *κ is $< \delta$-srs.*

If δ is srs, these two are equivant to κ simply being srs.

Proof. Really this is just about the absoluteness of an extender E being λ-srs in $\mathrm{V} \mid \delta$ versus V. We show this instead as the equivalence between (1) and (2) follows easily. So suppose E is a λ-srs extender on κ in $\mathrm{V} \mid \delta$ with $\lambda < \delta$. We have two maps $j_E : \mathrm{V} \to \mathrm{Ult}_E(\mathrm{V})$ and $i_E : \mathrm{V} \mid \delta \to \mathrm{Ult}_E^{\mathrm{V}|\delta}(\mathrm{V} \mid \delta)$ where $\mathrm{Ult}_E^{\mathrm{V}|\delta}(\mathrm{V} \mid \delta) = \mathrm{Ult}_E(\mathrm{V}) \mid j_E(\delta)$. Since E is λ-srs in $\mathrm{V} \mid \delta$, i_E agrees with $\mathrm{V} \mid \delta$ on $\mathrm{V} \mid \delta$-strongs below λ. Since δ is strong, $\mathrm{V} \mid \delta$ and V agree on which ordinals are strong cardinals. As a result, the ultrapowers of each agree on which ordinals are strong cardinals below λ. Hence V agrees with $\mathrm{Ult}_E(\mathrm{V})$ on strong cardinals below λ and so E is λ-srs in V. This shows (1) implies (2): any λ-srs extender on κ in $\mathrm{V} \mid \delta$ is λ-srs in V whenever $\lambda < \delta$.

For the reverse, suppose E is λ-srs in V for some $\lambda < \delta$. By restricting E down to $E \upharpoonright \lambda'$ for some $\lambda < \lambda' < \delta$ if necessary, we may assume $E \in \mathrm{V} \mid \delta$ (due to the factor embedding

$k : \mathrm{Ult}_{E\restriction\lambda'}(\mathrm{V}) \to \mathrm{Ult}_E(\mathrm{V})$ having $\mathrm{cp}(k) \geq \lambda'$ so by elementarity the two ultrapowers agree on strongs below λ'). But then the similar reasoning as above shows that E is λ-srs in $\mathrm{V} \mid \delta$. More precisely, let $j_E : \mathrm{V} \to \mathrm{Ult}_E(\mathrm{V})$ and $i_E : \mathrm{V} \mid \delta \to \mathrm{Ult}_E^{\mathrm{V}|\delta}(\mathrm{V} \mid \delta) = \mathrm{Ult}_E(\mathrm{V}) \mid j_E(\delta)$ be the canonical embeddings. We have that $\mathrm{V} \mid \delta$ and V agree on strongs below λ, and V and $\mathrm{Ult}_E(\mathrm{V})$ agree on strongs below λ. Because $j_E(\delta)$ is strong in $\mathrm{Ult}_E(\mathrm{V})$, $\mathrm{Ult}_E(\mathrm{V})$ and $\mathrm{Ult}_E(\mathrm{V}) \mid j_E(\delta)$ agree on strongs below $j_E(\delta) > \delta > \lambda$. Hence $\mathrm{V} \mid \delta$ and $\mathrm{Ult}_E(\mathrm{V}) \mid j_E(\delta) = \mathrm{Ult}_E^{\mathrm{V}|\delta}(\mathrm{V} \mid \delta)$ agree on strongs below λ, meaning E is λ-srs in $\mathrm{V} \mid \delta$.

Now suppose δ is srs and κ is $< \delta$-srs. Let $\lambda > \delta$ be arbitrary, aiming to show κ is λ-srs. Let $j : \mathrm{V} \to \mathrm{M}$ witness that δ is λ-srs: j is λ-srs with $\mathrm{cp}(j) = \delta$ so that M and V agree on strongs below λ. In M, $j(\kappa) = \kappa$ is $< j(\delta)$-srs and hence λ-srs as witnessed by some $i : \mathrm{M} \to \mathrm{N}$ with $\mathrm{cp}(i) = \kappa$ where M and N agree on strongs below λ. Thus V, M, and N all agree on strongs below λ. Note that $\mathrm{cp}(i \circ j) = \kappa$ so that $i \circ j : \mathrm{V} \to \mathrm{N}$ is a λ-srs embedding of V so that κ is λ-srs. $\dashv$

The hypothesis that δ is strong ensures that V and $\mathrm{V} \mid \delta$ agree on their strongs. Otherwise, we have that $\mathrm{V} \mid \delta \vDash$ "κ is srs" merely implies that κ has λ-strong embeddings that agree on $< \delta$-strong cardinals for each $\lambda < \delta$.

2 Strongs Reflecting Strongs in the Core Model

We begin with proving the "easy" direction of Theorem 1 • 2 through core model techniques. We show (2) implies (1), meaning that UWISS with a proper class of strongs results in a proper class of srs cardinals. Firstly, note that a Woodin cardinal implies the existence of a proper class of srs cardinals by an easy proof.

2 • 1. Definition. *A strongly inaccessible cardinal* δ *is* Woodin *iff for every* $A \subseteq \mathrm{V} \mid \delta$*, there is a* $\kappa < \delta$ *that is* $< \delta$*-strong reflecting* A*, meaning for every* $\lambda < \delta$*, there is an elementary* $j : \mathrm{V} \to \mathrm{M}$ *such that*

1. $\mathrm{cp}(j) = \kappa$ *and* $j(\kappa) > \lambda$*;*
2. $\mathrm{V} \mid \lambda = \mathrm{M} \mid \lambda$*; and*
3. $j(A) \cap \mathrm{V} \mid \lambda = A \cap \mathrm{V} \mid \lambda$.

In this way, an srs cardinal can really be thought of as a strong cardinal reflecting the set of strongs. So it should be easy to see why Woodins imply lots of srs cardinals.

2 • 2. Lemma. *The existence of a Woodin cardinal implies* $\mathrm{Con}(\mathsf{ZFC} +$ *"There is a proper class of srs cardinals"*$)$.

Proof. Let δ be Woodin and $A = \{\kappa < \delta : \kappa \text{ is strong}\}$. By Woodin-ness and [8], there is an unbounded (in fact, stationary) set

$$\{\kappa < \delta : \kappa \text{ is } \lambda\text{-strong reflecting } A \text{ for all } \lambda < \delta\} \subseteq \mathrm{V} \mid \delta.$$

But this just means we get λ-strong extenders $E_\lambda \in \mathrm{V} \mid \delta$ for $\lambda < \delta$ on each κ in this set such that the resulting embedding $j_\lambda : \mathrm{V} \to \mathrm{M}_\lambda$ has $j_\lambda(A) \cap \mathrm{V} \mid \lambda = A \cap \mathrm{V} \mid \lambda$, i.e. $\{\alpha < \lambda : \mathrm{M} \models \text{"}\alpha \text{ is strong"}\} = \{\alpha < \lambda : \alpha \text{ is strong}\}$. Hence $\mathrm{V} \mid \delta$ witnesses the consistency statement. ⊣

Hence we may assume without loss of generality that there is no inner model with a Woodin and thus may work with the core model K below a Woodin as presented in [10] and [13]. There are a few standard facts about K that we will use.

2 • 3. Lemma. *Suppose there is no inner model with a Woodin cardinal. Therefore the core model* K *is such that*

1. *(Local definability) For every regular* $\kappa > \aleph_1$, $\mathrm{K}^{\mathrm{H}_\kappa} = \mathrm{K} \cap \mathrm{H}_\kappa$.

2. *(Generic absoluteness) For every poset* $\mathbb{P} \in \mathrm{V}$, *and every* $\mathbb{P}$*-generic* G *over* V, $\mathrm{K}^{\mathrm{V}} = \mathrm{K}^{\mathrm{V}[G]}$.

3. *(Initial segment condition) If* E *is an extender on the sequence of* K, *then for every* $\alpha < \mathrm{lh}(E)$, $E \restriction \alpha \in \mathrm{K}$ *is on the sequence of* K.

When working with K, it will be useful to know when an extender is on the sequence of K, so we make use of the following lemma.

2 • 4. Lemma. *Let* E *be a* (κ, λ)*-extender with strength* $\lambda = \kappa + \delta$ *such that* $(\kappa^{+\delta})^{\mathrm{V}}$ *is regular. Therefore* $F = E \cap \mathrm{K} \in \mathrm{K}$ *is on the sequence of* K *and in fact,* $\mathrm{K} \vDash$ *"*F *is a* (κ, λ)*-extender with* $\mathrm{str}(F) = \lambda$*". Moreover, for each regular* $\mu < (\kappa^{+\delta})^{\mathrm{V}}$ *in* K, *there is an extender* F_μ *such that* $\mathrm{Ult}_{F_\mu}(\mathrm{K})$ *and* K *agree on* H_μ.

Proof. Let $\mathrm{M} = \mathrm{Ult}_E(\mathrm{V})$ so that ${}^{\omega}M \subseteq M$. Let $j : \mathrm{V} \to \mathrm{M}$ be the canonical ultrapower map. The hypotheses of [12] are satisfied and hence K^{M} (i.e. $j(\mathrm{K})$) is an iterate of K. Moreover $j \restriction \mathrm{K} = \pi^{\mathcal{T}}$ for some normal iteration tree $\mathcal{T}$ on K of successor length where $\pi^{\mathcal{T}}$ is the iteration map for the main branch of $\mathcal{T}$, and $\mathcal{T}$ has last model K^{M}. More explicitly, let $\mathrm{lh}(\mathcal{T}) = \gamma + 1$. Let ξ be such that the $\mathcal{T}$-predecessor of $\xi + 1$ is 0 and $\xi + 1$ lies on the branch of γ. We have models $\langle \mathcal{M}^{\mathcal{T}}_\alpha : \alpha < \mathrm{lh}(\mathcal{T}) \rangle$ with extenders $\langle E_\alpha : \alpha < \mathrm{lh}(\mathcal{T}) - 1 \rangle$, iteration maps $i_{\alpha,\beta} : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{T}}_\beta$ for $\alpha \leq_{\mathcal{T}} \beta$, and the following branch

$$\mathrm{K} = \mathcal{M}^{\mathcal{T}}_0 \to_{E_\xi} \mathcal{M}^{\mathcal{T}}_{\xi+1} \to \cdots \to \mathcal{M}^{\mathcal{T}}_\gamma = \mathrm{K}^{\mathrm{M}}.$$

We now show that $E_\xi \restriction \lambda = F$. [12] tells us that $i_{0,\gamma} = j \restriction \mathrm{K}$. Since $\mathrm{V} \mid \lambda = \mathrm{M} \mid \lambda$ and $\lambda = \kappa + \delta$, $\mathrm{H}^{\mathrm{V}}_{\kappa^{+\delta}} = \mathrm{H}^{\mathrm{M}}_{\kappa^{+\delta}}$ so by local definability, $\mathrm{K}^{\mathrm{M}} \cap \mathrm{H}^{\mathrm{M}}_{\kappa^{+\delta}} = \mathrm{K} \cap \mathrm{H}_{\kappa^{+\delta}}$. It follows that $\mathrm{str}(E_\xi) = \mathrm{lh}(E_\xi) \geq \lambda$. By normality of $\mathcal{T}$, $\mathrm{cp}(i_{\xi+1,\gamma}) > \mathrm{lh}(E_\xi) \geq \lambda$. So for any given

$\langle a, A\rangle \in [\lambda]^{<\omega} \times [\kappa]^{|a|}$ in K,

$$\begin{aligned} \langle a, A\rangle \in F \quad &\text{iff} \quad a \in j(A) = i_{0,\gamma}(A) \\ &\text{iff} \quad i_{\xi+1,\gamma}(a) \in i_{\xi+1,\gamma}(i_{0,\xi+1}(A)) \\ &\text{iff} \quad a \in i_{0,\xi+1}(A) \quad \text{iff} \quad \langle a, A\rangle \in E_\xi \end{aligned}$$

Hence E_ξ and F agree up to λ. By the initial segment condition, $E_\xi \restriction \lambda = F$ ensures F is on the sequence of K. The strength of F being λ in K follows from $\text{str}(E_\xi) = \text{lh}(E_\xi) \geq \lambda$ from the normality of $\mathcal{T}$. Moreover, restricting E_ξ appropriately yields F_μ as in the statement, with these in K again by the initial segment condition. ⊣

Simply put, if κ is $\kappa + \delta$-strong in V, then κ is $< \kappa^{+\delta}$-strong in K according to the H-hierarchy: K $\vDash$ "there are extenders F_μ such that $\text{H}_\mu \in \text{Ult}_F(\text{K})$ for each regular $\mu < \kappa^{+\delta}$".

In particular, any κ strong in V is strong in K: to get κ as λ-strong in K, we just need a sufficiently large μ such that $\text{K} \mid \lambda \subseteq \text{H}^{\text{K}}_\mu$. Then κ being μ^+-strong in V implies being λ-strong in K.

But we can actually say much more V-strongs being K-strong assuming UWISS: any cardinal stronger than a measurable in V will be strong in K, as we will prove below. As a result, any V-strong cardinal is a limit of K-strong cardinals, for example. Indeed, we will show that any V-strong cardinal is srs in K. These ideas will be central for the (2) implies (1) direction of Theorem 1 • 2.

2 • 5. Theorem. *(2) implies (1) where these stand for*

1. Con(ZFC + *"There is a proper class of strongs reflecting strongs"*);
2. Con(ZFC + *"There is a proper class of strong cardinals"* + UWISS).

Proof. We're done if there is an inner model with a Woodin cardinal, so assume otherwise. Let V $\vDash$ ZFC+UWISS have a proper class of strong cardinals. It will be useful to understand what cardinals are strong in the core model K.

Claim 1. *Suppose κ is $\kappa + 2$-strong in* V*. Therefore κ is strong in* K*.*

Proof. Let $\delta > \kappa^+$ be arbitrarily large. Write $\lambda = \beth^+_\delta \geq \delta$. Consider $\mathbb{P} = \text{Col}(\kappa^+, \lambda)$. By indestructibility, κ is still $\kappa+2$-strong in $\mathrm{V}^{\mathbb{P}}$ as witnessed by some $(\kappa,\kappa+2)$-extender E and notice $\lambda < (\lambda^{+})^{\mathrm{V}} = (\kappa^{++})^{\mathrm{V}^{\mathbb{P}}}$. By generic absoluteness, $\mathrm{K}^{\mathrm{V}^{\mathbb{P}}} = \mathrm{K}^{\mathrm{V}}$ and so λ is regular in K, $\mathrm{Ult}_F(\mathrm{K})$. By Lemma 2 • 4, there is an extender $F = F_\lambda \in \mathrm{K}$ such that K and $\mathrm{Ult}_F(\mathrm{K})$ agree on H_λ. As $\beth_\delta < \lambda$ is a strong limit in V, this also holds in K, so that $\mathrm{K}, \mathrm{Ult}_F(\mathrm{K}) \vDash$ "$\mathrm{H}_\lambda \supseteq \mathrm{V}_\delta$". This means $\mathrm{K} \mid \delta = \mathrm{Ult}_F(\mathrm{K}) \mid \delta$ so that κ is δ-strong in K. ⊣

We now aim to show any κ strong in V is actually srs in K. Let κ be strong in V and let the λ-strength of κ, for some strong $\lambda \geq \kappa^{+\P}$, be witnessed by a (κ,λ)-extender $E \in \mathrm{V}$ and embedding $j_E : \mathrm{V} \to \mathrm{Ult}_E(\mathrm{V})$. Note that j_E restricts down to $j_E \upharpoonright \mathrm{K} : \mathrm{K} \to \mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$. Consider $F = E \cap \mathrm{K}$ which is in K and on the K-sequence by Lemma 2 • 4. So consider the resulting ultrapower $\mathrm{Ult}_F(\mathrm{K})$ via $j_F : \mathrm{K} \to \mathrm{Ult}_F(\mathrm{K})$ which then factors $j_E = k_F \circ j_F$ via $k_F : \mathrm{Ult}_F(\mathrm{K}) \to \mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$ as seen below.

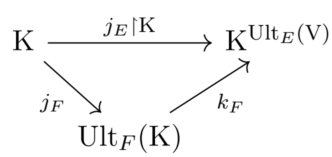

Figure 1: Factoring ultrapower embeddings with K

Note that $\mathrm{cp}(k_F) \geq \lambda$ so that for any cardinal $\xi < \lambda$,

$$\mathrm{Ult}_F(\mathrm{K}) \vDash \text{“}\xi \text{ is strong”} \quad \text{iff} \quad \mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})} \vDash \text{“}k_F(\xi) = \xi \text{ is strong”}.$$

Thus it suffices to show K and $\mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$ agree on strongs below λ, because then all three would agree and so F would witness a λ-srs embedding for κ in K. By local definability, $\mathrm{K} \mid \lambda = \mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})} \mid \lambda$ and hence we get that K and $\mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$ agree on "ξ is $<\lambda$-strong" whenever $\xi < \lambda$. So it suffices to show

$$\mathrm{K}, \mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})} \vDash \text{“}\forall \xi < \lambda\ (\xi \text{ is } <\lambda\text{-strong} \to \xi \text{ is strong})\text{”}. \qquad (*)$$

Since λ is strong (in V), λ is a limit of cardinals ξ that are $\xi+2$-strong. By the strength of E, if ξ is $\xi+2$-strong in V, then ξ is $\xi+2$-strong in $\mathrm{Ult}_E(\mathrm{V})$, and hence λ is a limit of cardinals ξ that are $\xi+2$-strong in $\mathrm{Ult}_E(\mathrm{V})$. So by Claim 1, λ is a limit of K-strongs and $\mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$-strongs. As a result, if $\xi < \lambda$ is not β-strong in either K or $\mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$ for some β, then by Lemma 1.A • 10, this is reflected by some strong below λ in K or $\mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$, showing $(*)$ holds. It follows that K and $\mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$ agree on strongs below λ, and since $\mathrm{K}^{\mathrm{Ult}_E(\mathrm{V})}$ and $\mathrm{Ult}_F(\mathrm{K})$ agree on strongs below λ, we get that $F \in \mathrm{K}$ witnesses that κ is λ-srs. Since we have a proper class of strongs in V, this gives a proper class of srs cardinals in K. $\dashv$

3 The Forcing Direction

Now we show the harder direction of Theorem 1 • 2. We show we can force a proper class of strongs with UWISS from a proper class of srs cardinals. The general idea behind the poset, as with most indestructibility results, is a trial by fire to kill all degrees of strength.

The result will be that the srs cardinals remain strong after forcing with the preparation, and small degrees of strength are, by virtue of surviving the trial by fire, weakly indestructible. We do not get universal indestructibility for *all* degrees of strength both because that's impossible by Result 1.B • 1 and because it's possible for the tail poset to resurrect degrees of strength that were destroyed, something avoided in [3] and [2] by cutting off the universe and declaring success anytime this might happen.

Our trial proceeds with appropriate posets via a lottery in an Easton iteration, that is to say "reverse" Easton in the sense of many of Hamkins' papers and in [3].[1] What posets are appropriate? Well, the ones that destroy the strength of a cardinal κ and also are simultaneously $<\kappa$-strategically closed and $\leq\kappa$-distributive, basically violating UWISS.

3 • 1. Definition. *For δ a cardinal of strength $\geq\rho$, we say a poset $\mathbb{Q}$ is δ,ρ-*appropriate *iff*

1. *$\mathbb{Q}$ is $<\delta$-strategically closed;*

2. *$\mathbb{Q}$ is $\leq\delta$-distributive; and*

3. *$\mathbb{Q}$ destroys the ρ-strength of δ.*

*If just (1) and (2) hold, we say $\mathbb{Q}$ is δ-*appropriate*.*

So the *destructibility* of a cardinal δ's degree of strength ρ by a $<\delta$-strategically closed and $\leq\delta$-distributive poset is obviously equivalent to the existence of a nearly δ,ρ-appropriate poset. Hence we can restate UWISS as the lack of any $\delta,\delta+2$-appropriate posets for any δ. We will show that the existence of appropriate posets is equivalent to the existence of "small" appropriate posets, basically meaning that we can bound the rank of $\mathbb{Q}$ and ρ by $\delta^{+¶}$.

In examining Definition 3 • 1, we have the following easy, useful result.

[1] I call an iteration's support *Easton* iff direct limits are taken at (weakly) inaccessible stages and inverse limits elsewhere.

3 • 2. Corollary. *Let* $\mathbb{P} \in \mathrm{V} \mid \alpha$ *be a poset. Therefore*

- $\mathbb{P}$ *is* δ, ρ*-appropriate iff for some (any)* $\lambda \geq |\alpha + \rho|^{+}$, $\mathrm{V} \mid \lambda \vDash$ *"*$\mathbb{P}$ *is* δ, ρ*-appropriate".*
- *For* $\dot{\mathbb{Q}}$ *a* $\mathbb{P}$*-name for a poset,* $\mathbb{P} \Vdash$ *"*$\mathbb{Q}$ *is* $\check{\delta}, \check{\rho}$*-appropriate with rank* $\check{\beta}$*" iff* $\mathrm{V} \mid \lambda$ *satisfies this for* λ *greater than* $|\max(\alpha, \rho, \beta)|^{+}$.

Proof. It suffices to show the second since the first follows from it in the case that $\mathbb{P}$ is trivial. Let $\kappa = |\max(\alpha, \rho, \beta)|^{+}$. $\mathbb{Q}$ is forced to be a subset of $(\mathrm{V} \mid \beta)^{\mathbb{P}}$. So without loss of generality, $\dot{\mathbb{Q}}$ is a nice $\mathbb{P}$-name for a subset of $(\mathrm{V} \mid \beta)^{\mathbb{P}}$ which therefore has rank $< \kappa$. $< \delta$-strategic closure and δ-distributivity only make claims about $\mathcal{P}(\mathbb{Q})$. Nice names again mean that we only need access to $\mathbb{P}$-names of rank $< \kappa$, meaning these concepts are absolute between V and $\mathrm{V} \mid \lambda$ for $\lambda \geq \kappa$. So the remaining concepts we need absoluteness for are related to the (non)-existence of extenders and inaccessibles: we need

$$\mathbb{P} \Vdash \text{"}\dot{\mathbb{Q}} \Vdash \text{"}\check{\delta} \text{ is no longer } \check{\rho}\text{-strong""}.$$

Working in $\mathrm{V}^{\mathbb{P}}$, nice $\mathbb{Q}$-names for (δ, ρ')-extenders will have rank $< \rho' + \beta + \omega < \kappa$. So in V, we only need to consider nice names for subsets of $(\mathrm{V} \mid \rho + \beta + \omega)^{\mathbb{P}}$, which all have rank $< \kappa$. ⊣

The search for a δ, ρ-appropriate poset—equivalently the weak destructibility of δ's strength—can be bounded in the presence of lots of strong cardinals: if there is some δ, ρ-appropriate poset, then we can choose ρ and the rank of the poset to be less than $\delta^{+¶}$ just as in [3]. This isn't too difficult to show, and helps us in defining our forcing preparation later.

3 • 3. Lemma. *Let* $\delta \in \mathrm{Ord}$. *Let* $\mathbb{R} \in \mathrm{V} \mid \alpha$ *be a poset. Suppose there's a nearly* δ, ρ^{**}*-appropriate* $\mathbb{Q}^{**} \in \mathrm{V}^{\mathbb{R}}$. *Therefore, there's a* δ, ρ*-appropriate* $\mathbb{Q} \in \mathrm{V}^{\mathbb{R}} \mid (\max(\delta, \alpha)^{+¶})^{\mathrm{V}}$ *where* $\rho < (\max(\delta, \alpha)^{+¶})^{\mathrm{V}}$ *and* $\rho \leq \rho^{**}$.

Proof. Let $\mathbb{Q}^{**}$ have rank greater than $\gamma = (\max(\delta, \alpha)^{+¶})^{\mathrm{V}}$ in $\mathrm{V}^{\mathbb{P}_{\delta}}$. Corollary 3 • 2 tells us for sufficiently large λ, $\mathbb{Q}^{**}$ is δ, ρ^{**}-appropriate in $\mathrm{V}^{\mathbb{R}} \mid \lambda$. More precisely, we get a name $\dot{\mathbb{Q}}^{**} \in \mathrm{V} \mid \lambda$ such that

$$\mathrm{V} \mid \lambda \vDash \text{"}\mathbb{R} \Vdash \text{"}\dot{\mathbb{Q}}^{**} \text{ is } \check{\delta}, \check{\rho}^{**}\text{-appropriate""}.$$

Let $j : \mathrm{V} \to \mathrm{M}$ witness the λ-strength of γ in V. It follows that

$$\mathrm{V} \mid \lambda = \mathrm{M} \mid \lambda \vDash \text{“}\mathbb{R} \Vdash \text{“}\dot{\mathbb{Q}}^{**} \text{ is } \check{\delta}, \check{\rho}^{**}\text{-appropriate””}.$$

Since ρ^{**} and the rank of $\mathbb{Q}^{**}$ are below $\lambda \leq j(\gamma)$, M believes there's an $\mathbb{R}$-name for a poset $\dot{\mathbb{Q}}^{*}$ in $\mathrm{M} \mid j(\gamma)$ that is δ, ρ^{*}-appropriate for some $\rho^{*} < j(\gamma)$ and in particular, $\rho^{*} = \rho^{**} \leq j(\rho^{**})$. Elementarity then gives a name $\dot{\mathbb{Q}}$ for a nearly δ, ρ-appropriate poset in $\mathrm{V} \mid \gamma$ with $\rho < \gamma$ and $\rho \leq \rho^{**}$. The rank of this poset in $\mathrm{V}^{\mathbb{R}}$ is therefore below γ. ⊣

In particular, if $\alpha < \delta^{+¶}$ then $\dot{\mathbb{Q}}$ will have rank $< \delta^{+¶}$.

3.A Forcing UWISS

We now attempt to prove the following, one of the directions from Theorem 1 • 2.

3.A • 1. Theorem. *(1) implies (2) where these stand for*

1. Con(ZFC + *“There is a proper class of srs cardinals”*).
2. Con(ZFC + *“There is a proper class of strongs”* + UWISS).

Suppose $\mathrm{V} \vDash$ ZFC+“there is a proper class of srs cardinals”. Without loss of generality, also assume that $\mathrm{V} \vDash$ GCH.[2]

The basic idea is a trial by fire where anything that emerges with some degree of strength should have its degree of strength weakly indestructible. A slight hiccup with this is that we don't actually have much control over what the resulting strength is, so rather than all of its degrees of strength being ewakly indestructible, just its small ones are. A broad outline of the proof is as follows. Here $\mathbb{P}_\kappa = \ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$, the whole preparation is $\mathbb{P} = \mathbb{P}_{\mathrm{Ord}}$, and the tail forcings from δ up to γ are $\dot{\mathbb{R}}_{[\delta,\gamma)} \cong \ast_{\delta \leq \alpha < \gamma} \dot{\mathbb{Q}}_\alpha$.

i. Show that if κ is $\kappa+2$-strong in $\mathrm{V}^{\mathbb{P}}$, then this is weakly indestructible, showing $\mathrm{V}^{\mathbb{P}} \vDash$ UWISS. This is done by showing
   - if we are able to destroy κ's $\kappa+2$-strength, then $\dot{\mathbb{Q}}_\kappa$ should be non-empty and would have already destroyed this.

[2]This can be done if necessary by forcing with the Easton support iteration that adds a cohen subset to each successor cardinal. This will preserve srs cardinals (and many other large cardinal properties).

ii. Show that if κ is λ-srs in V then κ is λ-strong in $V^{\mathbb{P}}$, implying that any V-srs cardinal is $V^{\mathbb{P}}$-strong, and there is a proper class of both. This is done by showing κ is λ-strong in $V^{\mathbb{P}_\lambda}$ for large λ as follows.

- Take a λ-srs embedding $j : V \to M$ and in $V[G] = V^{\mathbb{P}_\lambda}$, find a generic H for the tail forcing $\dot{\mathbb{R}}_{[\lambda, j(\lambda))}$ over a specific hull $N[G] \preccurlyeq M[G]$.
- Then we show this is generic over M[G] using the strategic closure of the tail *within* M[G], not V[G], to find representations for dense sets of M[G] inside N[G].
- Then we lift j to the λ-strong $j^+ : V[G] \to M[G * H]$ within V[G], showing κ is still λ strong in V[G].

A necessary consequence of Result 1.B • 1 is that srs cardinals have their strengths destroyed by $\mathbb{P}_{\kappa+1}$, but will have their λ-strength resurrected by stage λ (whenever λ is a non-measurable, inaccessible limit of strongs).

More generally, showing that degrees of strength aren't resurrected is not as simple a task as it might seem, and disregarding Result 1.B • 1, this is partly why we can't use this preparation to get indestructible strength for large strengths. Let me describe the situation in a little more detail to show what goes wrong. The basic idea is that these cardinals are not going through this "trial by fire" alone, and a later cardinal can act like a medic.

Naïvely, one will proceed destroying as much strength as possible: if κ is ρ-strong in $V^{\mathbb{P}_\kappa}$, we try to destroy this with κ-appropriate posets if we can, and this constitutes the forcing done at stage κ. If we destroy κ's strength down to be $< \rho$, but some non-trivial forcing is done at stage $\delta < \rho$, we might accidentally resurrect κ's ρ-strength, as below with Figure 2.

Moreover, there is a problem if a cardinal's strength is merely playing dead. For suppose κ is λ-strong in V. As we approach κ, κ might reduce its strength so that κ is merely ρ-strong in $V^{\mathbb{P}_\kappa}$ and weakly indestructible there. In that case, we would do nothing, for we don't see any strength to destroy. But then, it may be that ρ is large enough that the next non-trivial stage of forcing occurs at some $\delta < \rho$, in which case, κ might wake up and we accidentally resurrect κ's λ-strength. Because we never think to return to previously dealt

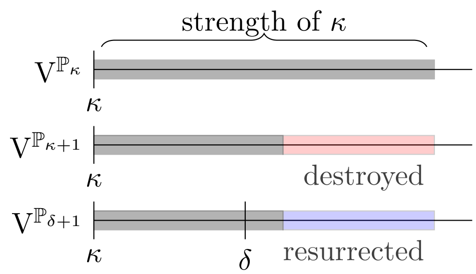

Figure 2: Interaction resurrecting strength

with stages, κ might remain λ strong in $V^{\mathbb{P}}$ but not be weakly indestructible.

Any attempted solution to this will ultimately either result in no resulting large strength or further forcing that might again resurrect strength by Result 1.B • 1. For example, we might try to collapse destroyed degrees of strength below the next stage of forcing. This is due to the general fact that if κ's strength is $\rho < \kappa^{+\sharp\sharp}$, then the subsequent stages won't resurrect degrees of strength by distributivity with Corollary 3.A • 5 as with Figure 3 below.

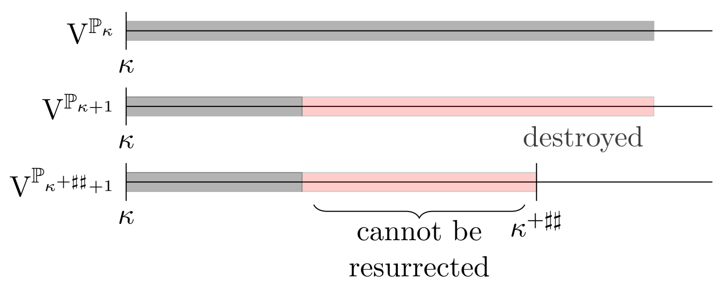

Figure 3: Non-interaction leaving strength destroyed

The forcing at $\kappa^{+\sharp\sharp}$ shouldn't affect smaller degrees of strength by distributivity. But the collapse we're considering done at stage κ *could* then resurrect degrees of strength. So this gives the following result, the proof of which is given later with Result 3.B • 6.

3.A • 2. Result. *It's consistent (relative to two srs cardinals and proper class of strong cardinals) that we can force a κ that is not ρ-strong to be ρ-strong by a collapse* $\mathrm{Col}(\kappa^+, \rho)$

This problem isn't an issue when we only care about finding *one* strong or supercompact cardinal with indestructibility properties: if we ever have a measurable between a cardinal and its remaining strength, we could have just cut off the universe and end up in the desired model; no need to deal with the resurrection. This is essentially the argument given in [3] and [2]. Unfortunately for us, we can't just stop our preparation and declare success, and this is partially why resurrected degrees of strength remain a fact of life.

Our partial ordering is a modification of the one found in the proof of [3]. Specifically, we define an iteration (of proper class length) $\ast_{\xi\in\mathrm{Ord}}\dot{\mathbb{Q}}_\xi$ which begins by adding a Cohen subset of ω to make use of gap forcing arguments. In other words, $\mathbb{P}_{\omega+1}\cong\mathrm{Add}(\omega,1)$. All other nontrivial stages of forcing can only occur at inaccessible δ that are $\delta+2$-strong in V.

3.A • 3. Definition (The Preparation for Strongs). *Suppose $\mathbb{P}_\delta=\ast_{\xi<\delta}\dot{\mathbb{Q}}_\xi$ has been defined for $\delta>\omega$. We aim to define $\dot{\mathbb{Q}}_\delta$.*

1. *If no $p\in\mathbb{P}_\delta$ forces that δ is inaccessible, then $\dot{\mathbb{Q}}_\delta=\dot{\mathbb{1}}$ is trivial.*
2. *Otherwise, suppose δ is forced by some $p\in\mathbb{P}_\delta$ to be $<\lambda$-strong for some (maximal) λ (allowing $\lambda=0$ if δ isn't measurable, and $\lambda=\mathrm{Ord}$ if strong), and work below p.*
   - *(a) If the $<\lambda$-strength of δ is weakly indestructible via posets of rank $<(\delta^{+¶})^{\mathrm{V}}$, then define $\dot{\mathbb{Q}}_\delta=\dot{\mathbb{1}}$.*
   - *(b) Otherwise, we let $\rho<\lambda$ be the minimal degree of δ's strength that is weakly destructible and let $\dot{\mathbb{Q}}_\delta$ be the lottery sum of all (what are forced to be) posets that take the form $\dot{\mathbb{B}}\ast\dot{\mathbb{C}}$ where the following happen.*
   - *(c) $\dot{\mathbb{B}}$ is an δ,ρ-appropriate poset of rank $<(\delta^{+¶})^{\mathrm{V}}$, and*
   - *(d) In $\mathrm{V}^{\mathbb{P}_\delta\ast\dot{\mathbb{B}}}$, if $\rho<\delta^{+\sharp\sharp}\leq|\dot{\mathbb{B}}|$, then $\dot{\mathbb{C}}$ is a name for $\mathrm{Col}(\rho^{+\sharp},|\dot{\mathbb{B}}|)$. Otherwise $\dot{\mathbb{C}}$ is trivial.*

Using Easton support for limit stages, we write $\mathbb{P}=\ast_{\xi\in\mathrm{Ord}}\dot{\mathbb{Q}}_\xi$ for the class iteration and $\dot{\mathbb{R}}_{[\delta,\lambda)}\cong\ast_{\delta\leq\xi<\lambda}\dot{\mathbb{Q}}_\xi$ for the tail forcing of $\mathbb{P}_\lambda$ whenever $\delta<\lambda$.

Two remarks about this preparation: firstly, note that each $\dot{\mathbb{Q}}_\delta$ is nearly δ,ρ-appropriate for some ρ whenever it's non-trivial. Secondly, it's not hard to see that we can regard $\mathbb{P}_\delta\subseteq\mathrm{V}\restriction\delta^{+¶}$ for any δ. In fact, we can regard $\mathbb{P}_\delta\subseteq\mathrm{V}\restriction\delta$ whenever $\dot{\mathbb{Q}}_\delta$ is non-trivial, since δ is therefore inaccessible and not collapsed by any previous stage, meaning every previous stage had a smaller cardinality and thus smaller rank.

The collapsing poset used for each poset in the lottery is used to give better control over the tail forcing, and in particular to show $\mathbb{P}_\delta$ is δ-cc whenever δ is still inaccessible there.

Moreover, the collapse allows us to ensure that once we collapse a *small* degree of strength with a small poset, the strength stays collapsed and won't be resurrected. Larger degrees, however, we make no promises about.

There are several standard facts of tail forcings we need to use in order to prove that the tail forcings of the preparation are strategically closed. The use of this will be in showing that $\dot{\mathbb{R}}_{[\delta,\lambda)}$ is δ-appropriate and thus can be used in arguments with δ, ρ-appropriate posets since it is close to being one. One background result used is the following [4].

3.A • 4. Theorem. *Let $\mathbb{P}'_\lambda = \ast_{\xi<\lambda} \dot{\mathbb{Q}}'_\xi$ be a λ-stage iteration such that for some $\delta < \lambda$ and some μ,*

1. *inverse limits or direct limits are taken at every limit stage;*
2. *inverse limits are taken at every limit stage $\geq \delta$ of cofinality $< \mu$;*
3. *$\dot{\mathbb{Q}}'_\xi$ is (forced to be) $< \mu$-strategically closed for every ξ with $\delta \leq \xi < \lambda$; and*
4. *$\mathbb{P}'_\delta$ is μ-cc.*

Therefore the tail forcing $\dot{\mathbb{R}}_{[\delta,\lambda)}$ is (forced to be) $< \mu$-strategically closed.

3.A • 5. Corollary. *Let $\delta < \lambda$. Therefore, the tail forcing of our preparation, $\dot{\mathbb{R}}_{[\delta,\lambda)}$ is (forced to be) $< \delta$-strategically closed and $\leq \delta$-distributive.*

Proof. If $\dot{\mathbb{Q}}_\delta$ is non-trivial, then below the appropriate conditions, δ is inaccessible in $V^{\mathbb{P}_\delta}$ so we take a direct limit at stage δ. It follows that δ was not collapsed at some previous stage by a κ, ρ-appropriate poset $\mathbb{Q}$ where $\kappa < \delta \leq \max(\rho, |\mathbb{Q}|)$. So for each $\kappa < \delta$, $\dot{\mathbb{Q}}_\kappa$ is equivalent to a poset with rank $< \delta$ and in fact we can regard $\mathbb{P}_\delta \subseteq V \mid \delta$ and so $\mathbb{P}_\delta$ is δ-cc. Using Theorem 3.A • 4, Easton support iterations clearly take inverse or direct limits everywhere, and only direct limits at certain regular stages. So at any $\xi \geq \delta > \mathrm{cof}(\xi)$, we take an inverse limit. Thus (1) and (2) hold from Theorem 3.A • 4. By hypothesis, (3) holds since lottery sums of $< \kappa$-strategically closed posets are $< \kappa$-strategically closed. (4) holds since $\mathbb{P}_\delta$ is δ-cc and thus $\dot{\mathbb{R}}_{[\delta,\lambda)}$ is $< \delta$-strategically closed.

If $\dot{\mathbb{Q}}_\delta$ is trivial, then below the appropriate conditions, the next non-trivial stage (if there is one) $\dot{\mathbb{Q}}_\mu$ has by the above argument that $\dot{\mathbb{R}}_{[\mu,\lambda)} \cong \dot{\mathbb{R}}_{[\delta,\lambda)}$ is $< \mu$-strategically closed and hence $< \delta$-strategically closed. If there is no non-trivial stage above δ, then the tail forcing is trivial and hence $< \delta$-strategically closed.

For $\leq \delta$-distributivity, note that $\dot{\mathbb{R}}_{[\delta,\lambda)} \cong \dot{\mathbb{Q}}_\delta * \dot{\mathbb{R}}_{(\delta,\lambda)}$ is the two-step iteration of two $\leq \delta$-distributive posets. ⊣

Given that measurability is already indestructible, the next non-trivial stage of forcing after δ occurs at stage $\delta^{+\sharp\sharp}$ at the earliest.

3.A • 6. Lemma. *The first non-trivial stage of forcing after any* $\kappa \in \mathrm{Ord}$ *is at least* $(\kappa^{+\sharp\sharp})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$

Proof. To have any degree of δ's strength weakly destructible over $\mathrm{V}^{\mathbb{P}_\delta}$, we require δ to be at least $\delta+2$-strong. Hence the least δ such that $\mathbb{P}_\delta \cong \mathbb{P}_{\kappa+1}$ but $\dot{\mathbb{Q}}_\delta$ is non-trivial must have that δ is $\delta+2$-strong in $\mathrm{V}^{\mathbb{P}_\delta} = \mathrm{V}^{\mathbb{P}_{\kappa+1}}$ and hence $\delta \geq (\kappa^{+\sharp\sharp})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$. ⊣

Hence all the degrees of κ's strength below $(\kappa^{+\sharp\sharp})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$ remain indestructible even if larger degrees are accidentally resurrected. The basic proof of this is that if we *could* destroy something, we would have destroyed it already.

3.A • 7. Result. *Let* κ *be such that* $\mathbb{P}_\kappa$ *and* $\mathbb{P}_{\kappa+1} \Vdash$ *"$\check{\kappa}$ is $\check{\rho}$-strong for $\check{\rho} < \kappa^{+\sharp\sharp}$". Therefore* $\mathbb{P} \Vdash$ *"$\check{\kappa}$'s ρ-strength is weakly indestructible".*

Proof. By downward absoluteness, in $\mathrm{V}^{\mathbb{P}_\kappa}$, κ is still inaccessible. Hence we are in case (2) of Definition 3.A • 3. As "small" posets and by gap forcing [5], $\delta^{+\sharp}$, $\delta^{+\sharp\sharp}$, and $\delta^{+\P}$ are all the same in V and $\mathrm{V}^{\mathbb{P}_\kappa}$. (Such cardinals retain their large cardinal status by small forcing, and no new such cardinals are added above δ by [5].)

As a result, because $\rho < \kappa^{+\sharp\sharp}$ in $\mathrm{V}^{\mathbb{P}_{\kappa+1}}$, the tail forcing after κ is sufficiently distributive such that the ρ-strong extender on κ in $\mathrm{V}^{\mathbb{P}_{\kappa+1}}$ is still ρ-strong in $\mathrm{V}^{\mathbb{P}}$. Thus it suffices to show weak indestructibility for this degree of strength. So suppose $\dot{\mathbb{Q}}$ is κ,ρ-appropriate in $\mathrm{V}^{\mathbb{P}}$. By distributivity of the tail forcing, $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}_\lambda}$ for some $\lambda > \kappa$. The tail forcing $\dot{\mathbb{R}}_{(\kappa,\lambda)}$ is therefore κ-appropriate by Corollary 3.A • 5.

This tells us that $\dot{\mathbb{Q}}_\kappa$ must be non-trivial. To see this, otherwise κ must be ρ-strong in $\mathrm{V}^{\mathbb{P}_{\kappa+1}} = \mathrm{V}^{\mathbb{P}_\kappa}$. It follows that $\dot{\mathbb{R}}_{[\kappa,\lambda)} * \dot{\mathbb{Q}}$ is κ,ρ-appropriate in $\mathrm{V}^{\mathbb{P}_\kappa}$ so that by Lemma 3 • 3, we can find a κ, ρ-appropriate poset of small size and so $\dot{\mathbb{Q}}_\kappa$ should be non-trivial, a contradiction.

Since $\dot{\mathbb{Q}}_\kappa$ is non-trivial, by Definition 3.A • 3, we forced with some poset $\dot{\mathbb{B}} * \dot{\mathbb{C}}$ at stage κ where $\dot{\mathbb{B}}$ is κ, ρ_κ-appropriate in $\mathrm{V}^{\mathbb{P}_\kappa}$ with minimal ρ_κ. If $\rho < \rho_\kappa$, the κ, ρ-appropriate $\dot{\mathbb{R}}_{[\kappa,\lambda)} * \dot{\mathbb{Q}}$ would violate minimality of ρ_κ (via Lemma 3 • 3 to ensure we stay below $(\delta^{+\P})^{\mathrm{V}}$ in $\mathrm{V}^{\mathbb{P}_\kappa}$). So $\rho \geq \rho_\kappa$. We now break into cases.

- Suppose $\dot{\mathbb{C}}$ is non-trivial. Thus $\rho_\kappa < \kappa^{+\sharp\sharp} \leq |\dot{\mathbb{B}}|$ and $\dot{\mathbb{C}}$ collapses $|\dot{\mathbb{B}}|$ to be $\rho^{+\sharp}$ (which is $< \kappa^{+\sharp\sharp}$) in a way that is $< \rho^{+\sharp}$-distributive. In particular, this preserves the lack of ρ_κ-strong extenders, and so the lack of ρ-strong extenders in $\mathrm{V}^{\mathbb{P}_\kappa * \dot{\mathbb{B}}}$ to $\mathrm{V}^{\mathbb{P}_\kappa * \dot{\mathbb{B}} * \dot{\mathbb{C}}}$, contradicting that κ is ρ-strong in $\mathrm{V}^{\mathbb{P}_{\kappa+1}}$
- Suppose $\dot{\mathbb{C}}$ is trivial so that because $\dot{\mathbb{B}}$ is κ, ρ_κ-appropriate, κ is not ρ_κ-strong in $\mathrm{V}^{\mathbb{P}_\kappa * \dot{\mathbb{B}}} = \mathrm{V}^{\mathbb{P}_{\kappa+1}}$ and hence not ρ-strong there, a contradiction. ⊣

Unfortunately, this does not guarantee weak indestructibility for κ's degrees of strength $< \kappa^{\sharp\sharp}$ in $\mathrm{V}^{\mathbb{P}}$, but merely for strength $< (\kappa^{\sharp\sharp})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$ and so in particular $\kappa + 2$-strength, $\kappa^{+\sharp}$-strength, and $< \lambda$-strength for the first λ a measurable limit of measurables in $\mathrm{V}^{\mathbb{P}}$.

3.A • 8. Corollary (Forcing UWISS). *Suppose* $\mathrm{V} \models$ *"there is a proper class of strongs". Therefore*

- *the preparation* $\mathbb{P}$ *is well defined and* $\mathbb{P} \Vdash \mathsf{UWISS}$.
- *In fact, if* κ *is* ρ*-strong for* $\rho < (\kappa^{+\sharp\sharp})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$ *in* $\mathrm{V}^{\mathbb{P}}$ *then this strength is weakly indestructible in* $\mathrm{V}^{\mathbb{P}}$.
- *In particular, any cardinal* κ*'s strength that is below the least measurable limit of measurables above* κ *is weakly indestructible, e.g.* $\kappa+2$*-strength,* $\kappa^{+\sharp}$*-strength,* $(\kappa^{+\sharp})^{+\sharp}$*-strength, etc.*

Proof. If κ is measurable in $\mathrm{V}^{\mathbb{P}}$, then this degree of strength is weakly indestructible. So suppose κ is stronger than a measurable, but there is some $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ that is κ, ρ-appropriate for some $\rho < (\kappa^{+\sharp\sharp})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$, which has some rank below an inaccessible $\gamma > \rho, \mathrm{rank}(\dot{\mathbb{Q}})$. The tail forcing $\dot{\mathbb{R}}_{[\gamma,\infty)}$ is $\leq \gamma$-distributive by Corollary 3.A • 5 and hence $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}_\gamma}$ and

$V^{\mathbb{P}_\gamma} \mid \gamma = V^{\mathbb{P}} \mid \gamma$. So by Corollary 3 • 2, $\dot{\mathbb{Q}}$ is κ, ρ-appropriate in $V^{\mathbb{P}_\gamma}$. The tail forcing $\dot{\mathbb{R}}_{(\kappa,\gamma)}$ is sufficiently distributive by Lemma 3.A • 6 to show κ was ρ-strong in $V^{\mathbb{P}_{\kappa+1}}$ and so $\dot{\mathbb{R}}_{(\kappa,\lambda)} * \dot{\mathbb{Q}}$ is κ, ρ-appropriate there, contradicting Result 3.A • 7.

Since measurables between κ and $(\kappa^{+\sharp\sharp})^{V^{\mathbb{P}_{\kappa+1}}}$ are preserved by the tail forcing by Lemma 3.A • 6, it follows that every cardinal below the least measurable limit of measurables above κ—being below $(\kappa^{+\sharp\sharp})^{V^{\mathbb{P}_{\kappa+1}}}$—is indestructible. ⊣

We may not get better than this, since we could have the following: in $V^{\mathbb{P}_{\kappa+1}}$, κ is ρ-strong for $\rho > \kappa^{+\sharp\sharp}$. Then at stage $\mu = (\kappa^{+\sharp\sharp})^{V^{\mathbb{P}_{\kappa+1}}}$ we do some non-trivial forcing that destroys μ's $\mu + 2$-strength, and subsequently resurrects κ's ρ'-strength for $\rho' > \mu$ in a way that the new $(\kappa^{+\sharp\sharp})^{V^{\mathbb{P}}} > \rho'$ but κ's ρ'-strength is now destructible. Nevertheless, this doesn't affect somewhat small degrees of strength as the above shows.

3.B A proper class of strongs

So all that remains is to show that srs cardinals of V are strong in $V^{\mathbb{P}}$. We do this by working with partial degrees of reflecting strongs. The usefulness of reflecting strongs allows us to properly calculate the preparation up to limits of strongs.

3.B • 1. Lemma. *Let λ be a limit of strongs. Let $j : V \to M$ be at least λ-strong such that V and M agree on strongs $< \lambda$. Therefore $\mathbb{P}^V_\lambda = j(\mathbb{P})_\lambda = \mathbb{P}^M_\lambda$.*

Proof. Let $\kappa = \mathrm{cp}(j)$. Recall that j being λ-strong means $V \mid \lambda = M \mid \lambda$. j will not change the Easton support below λ, so it suffices to show $\dot{\mathbb{Q}}^V_\delta = \dot{\mathbb{Q}}^M_\delta$ for all cardinals $\delta < \lambda$. Proceed by induction on δ. In case (1) of Definition 3.A • 3, δ's non-inaccessibility is easily absolute between V and M since inductively $\mathbb{P}_\delta = \mathbb{P}^M_\delta$.

Note that since λ is a limit of strongs, if $\delta < \lambda$, then $(\delta^{+\P})^V = (\delta^{+\P})^M$ and so we can unambiguously write $\delta^{+\P}$ in such cases. V and M also agree on partial strongs below λ since if δ is $< \lambda_0$-strong in V for some λ_0, then either

- $\lambda_0 = \delta^{+\P} < \lambda$ implies δ is strong in both since the models agree on strongs below λ, or
- else $\lambda_0 < \delta^{+\P} < \lambda$ and so the lack of λ_0-strong extenders in $V \mid \lambda$ matches with $M \mid \lambda$.

Note also that all (names for) collapses we consider will exist in $V \mid \delta^{+\P} = M \mid \delta^{+\P}$ and thus have the same interpretation in both models.

In case (2) of Definition 3.A • 3, by Corollary 3 • 2, the existence of a $\rho < \delta^{+\P} < \lambda$ and a $< \delta$-strategically closed, $\leq \delta$-distributive $\dot{\mathbb{B}} \in V \mid \delta^{+\P}$ such that δ isn't ρ-strong after forcing with $\mathbb{P}_\delta * \dot{\mathbb{B}}$ can be calculated in $V \mid \lambda = M \mid \lambda$. Hence the two share the same such posets, and moreover, the minimal ρ witnessing this is the same for both. The calculation of $\delta^{+\sharp\sharp}$ in both will be below $\delta^{+\P}$ and easily the same in both models. The collapse (also being below $\delta^{+\P}$) will also be the same, meaning $\dot{\mathbb{Q}}_\delta$ is the same in both. ⊣

In particular, if $j : V \to M$ is a λ-srs embedding, i.e. witnessing that $\mathrm{cp}(j)$ is λ-srs, then $\mathbb{P}^V_\lambda = \mathbb{P}^M_\lambda$ whenever λ is a limit of strongs. This gives us the edge over hyperstrength in generalizing [3] which generally only gets agreement for $\mathbb{P}_{\mathrm{cp}(j)}$, but the resulting argument is adapted from [3].

3.B • 2. Lemma. *Let κ be strong and λ-srs where λ is a limit of strongs. Therefore $\mathbb{P}_\lambda \Vdash$ "κ is λ-strong".*

Proof. Let $j : V \to M$ be λ-srs with $\mathrm{cp}(j) = \kappa$ so that $M = \mathrm{Ult}_E(V)$ for some (κ, λ)-extender E. We can factor by Lemma 3.B • 1

$$\begin{aligned} j(\mathbb{P}_\lambda) = \mathbb{P}^M_{j(\lambda)} &\cong (\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{R}}_{(\kappa,\lambda)})^M * \dot{\mathbb{R}}^M_{[\lambda,j(\kappa))} * \dot{\mathbb{R}}^M_{[j(\kappa),j(\lambda))} \\ &\cong (\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{R}}_{(\kappa,\lambda)})^V * \dot{\mathbb{R}}^M_{[\lambda,j(\kappa))} * \dot{\mathbb{R}}^M_{[j(\kappa),j(\lambda))} \\ &\cong \mathbb{P}^V_\lambda * \dot{\mathbb{R}}^M_{[\lambda,j(\kappa))} * \dot{\mathbb{R}}^M_{[j(\kappa),j(\lambda))}. \end{aligned}$$

Without loss of generality, λ isn't $\lambda+2$-strong in M (otherwise just take another ultrapower by the Mitchell-least measure on λ) so that in $M^{\mathbb{P}_\lambda}$, λ is at most measurable with therefore indestructible degrees of strength and its original strength already small: $\dot{\mathbb{Q}}^M_\lambda = \dot{\mathbb{1}}$. Thus $\dot{\mathbb{R}}^M_{[\lambda,j(\lambda))}$ is actually λ^+-strategically closed in M (and indeed much more).

So let $G = G_0 * G_1$ be $\mathbb{P}_\kappa * \dot{\mathbb{R}}_{[\kappa,\lambda)}$-generic over V such that $V[G] \vDash$ "κ is not λ-strong". Our goal is to lift j to $j^+ : V[G] \to M[G * H]$ in $V[G]$ for some $H = H_0 * H_1 \in V[G]$ that is $(\dot{\mathbb{R}}^M_{[\lambda,j(\kappa))} * \dot{\mathbb{R}}^M_{[j(\kappa),j(\lambda))})_G$-generic over $M[G]$ such that j^+ remains λ-strong. This requires examining $j``G$, and beyond this there are only a couple steps in this lift-up argument: first building H_0 arbitrarily and then generating H_1 from $j"G_1$.

Claim 1. *For $j(p) \in j"G$, $j(p) \restriction [\lambda, j(\kappa))$—and in fact $j(p) \restriction [\kappa, j(\kappa))$—is trivial. Hence the only non-trivial information $j"G$ encodes occurs before λ and after $j(\kappa)$.*

Proof. Let $p \in G$ be arbitrary. Since we take a direct limit at stage κ, there is some $\alpha < \kappa$ where $p \restriction [\alpha, \kappa)$ is just a sequence of $\dot{\mathbb{1}}$s: $p \restriction [\alpha, \kappa)$ is trivial in $\dot{\mathbb{R}}_{[\alpha,\kappa)}$. By elementarity, $j(p)$ is similarly trivial from $j(\alpha) = \alpha < \kappa$ to $j(\kappa)$. In particular, $j(p) \restriction [\kappa, j(\kappa))$ is trivial. $\dashv$

As a result, *any* H_0 that is $\mathbb{R}^{\mathrm{M}[G]}_{[\lambda,j(\kappa))} = (\dot{\mathbb{R}}^{\mathrm{M}}_{[\lambda,j(\kappa))})_G$-generic over $\mathrm{M}[G]$ has $j"G \restriction j(\kappa)$ contained in $G * H_0$. So we merely need to find such a generic over $\mathrm{M}[G]$ in $\mathrm{V}[G]$ to get H_0. Then we find H_1. Claim 1 is partly why we need to break up the iteration as we do. The general idea is that $j"G$ might have conditions with potentially unbounded support in $j(\lambda)$. So we must generate the generic over the end tail. Claim 1 ensures that the middle is left unaffected since $j"G$ has no real information about it.

As an ultrapower, we can regard $\mathrm{M} = \{j(f)(r) : r \in [\lambda]^{<\omega} \wedge f : [\kappa]^{<\omega} \to \mathrm{V}\}$ so that there are elements of $r_0, r_1 \in [\lambda]^{<\omega}$ and functions from $[\kappa]^{<\omega}$ to V that represent $\mathbb{P}^{\mathrm{V}}_{\kappa}$, $\dot{\mathbb{R}}^{\mathrm{M}}_{(\kappa,j(\lambda))}$. So now we consider the elementary submodel $\mathrm{N} \preccurlyeq \mathrm{M}$ defined by

$$\mathrm{N} = \{j(f)(r_0 \cup r_1, \kappa, \lambda) : f \colon [\kappa]^{<\omega} \to \mathrm{V}\} \ni \mathbb{P}^{\mathrm{V}}_{\kappa}, \mathbb{P}^{\mathrm{V}}_{\lambda}, \dot{\mathbb{Q}}_{\kappa}, \dot{\mathbb{R}}^{\mathrm{M}}_{(\lambda,j(\kappa))}, \kappa, \lambda,$$

We write $\mathrm{N}[G]$ for $\{\tau_G : \tau \in \mathrm{N}\}$. Note that $\mathrm{N}^{\kappa} \cap \mathrm{V} \subseteq \mathrm{N}$ and $\mathrm{N}[G]^{\kappa} \cap \mathrm{V}[G] \subseteq \mathrm{N}[G]$ just because if $\langle j(f_\alpha)(r_0 \cup r_1, \kappa, \lambda) : \alpha < \kappa \rangle \in \mathrm{N}^{\kappa}$ then $j(\mathrm{const}_{\langle f_\alpha : \alpha < \kappa \rangle})(r_0 \cup r_1, \kappa, \lambda) \restriction \kappa = \langle j(f_\alpha) : \alpha < \kappa \rangle \in \mathrm{N}$—where const_x is the constant function that outputs x—and so evaluation at $(r_0 \cup r_1, \kappa, \lambda)$ yields the desired sequence in N.

Claim 2. *In $\mathrm{V}[G]$, there is an H_0 $\mathbb{R}^{\mathrm{M}[G]}_{(\lambda,j(\kappa))}$-generic over $\mathrm{N}[G]$.*

Proof. $\mathbb{R}^{\mathrm{M}[G]}_{(\lambda,j(\lambda))}$ is λ^+-strategically closed in $\mathrm{M}[G]$ and this translates to being merely κ^+-strategically closed in $\mathrm{V}[G]$ by the closure conditions of M and N in V: $\mathrm{N}[G] \preccurlyeq \mathrm{M}[G]$ is closed under κ-sequences. Hence $\mathbb{R}^{\mathrm{M}[G]}_{(\lambda,j(\lambda))} \cap \mathrm{N}[G]$ is still κ^+-strategically closed in $\mathrm{V}[G]$. Since dense sets in $\mathrm{N}[G]$ can be identified with $f : [\kappa]^{<\omega} \to \mathrm{V}_{\kappa}$, a simple counting argument shows that in $\mathrm{V}[G]$, there are at most $2^{j(\kappa)} = \kappa^+$-many antichains of the poset in $\mathrm{N}[G]$. Thus we can find in $\mathrm{V}[G]$ an H that is $\mathbb{R}^{\mathrm{M}[G]}_{(\kappa,j(\kappa))} \cap \mathrm{N}[G]$-generic over $\mathrm{N}[G]$ by standard techniques (extending into dense open sets one by one and leaving the other player in the strategic closure game to clean up our mess at limit stages). ⊣

Now we must show that H_0 is actually generic over $M[G]$. Let D be a dense open set of $\mathbb{R}^{\mathrm{M}[G]}_{(\lambda,j(\kappa))}$. We can represent D by $j(f)(r)$ for some $f : [\kappa]^{<\omega} \to \mathbb{P}_\kappa$ and $r \in [\lambda]^{<\omega}$. So in M$[G]$, consider the set $\{j(f)(s) \text{ dense open} : s \in [\lambda]^{<\omega}\}$. Since M$[G]$ thinks that the tail $\mathbb{R}^{\mathrm{M}[G]}_{(\lambda,j(\kappa))}$ is λ^+-strategically closed, the intersection of all of these sets $E = \bigcap_{s\in[\lambda]^{<\omega}} j(f)(s)$ is non-empty. Moreover, we have that the map defined by $g(x, y, \alpha) = \bigcap_{s\in[\alpha]^{<\omega}} f(s)$ has $E = j(g)(r_0 \cup r_1, \kappa, \lambda) = \bigcap_{s\in[\lambda]^{<\omega}} j(f)(s)$ is in N$[G]$. Hence $E \cap H_0 \neq \emptyset$, and showing that H_0 is generic over M$[G]$.

Claim 3. *Let H_1 be the filter generated by j"G_1. Therefore H_1 is $\mathbb{R}^{\mathrm{M}[G*H_0]}_{[j(\kappa),j(\lambda))}$-generic over* M$[G * H_0]$ *and as a result of Claim 1, j"$G \subseteq G * H_0 * H_1$.*

Proof. Let $D \in \mathrm{M}[G * H_0]$ be open dense with name $\dot{D}$. It follows that we can represent D as $j(d)(r)_{G*H_0}$ for some $d : [\kappa]^{<\omega} \to \mathrm{V}$ and $r \in [\lambda]^{<\omega}$ where each $d(s)$ is a name for a dense open set in $\dot{\mathbb{R}}_{[\kappa,\lambda)}$. In V$[G_0]$, since the tail forcing is $\leq \kappa$-distributive by Corollary 3.A • 5, we can intersect all of these dense open sets $\bigcap_{s\in[\kappa]^{<\omega}} d(s)_{G_0}$ and get another dense open set that intersects G_1: there is a $p \in G_1 \cap d(s)_{G_0}$ for every $s \in [\kappa]^{<\omega}$. Thus some $q \in G_0$ has in M that $j(q) \Vdash$ "$j(p) \in j(d)(s)$ for every $s \in [j(\kappa)]^{<\omega}$" and in particular $j(q) \Vdash$ "$j(p) \in \dot{D}$". Since $q \in G_0 \subseteq \mathbb{P}_\kappa$, $j(q)$ is just q with a bunch of $\dot{\mathbb{1}}$s appended, meaning $j(q) \in G * H_0$ and so indeed $j(p) \in j$"$G_1 \cap D \subseteq H_1 \cap D$ in M$[G * H_0]$. ⊣

Thus j"$G \subseteq G * H_0 * H_1$ and so $j : \mathrm{V} \to \mathrm{M}$ lifts to $j^+ : \mathrm{V}[G] \to \mathrm{M}[G * H]$. It's not hard to see that j^+ is still λ-strong since $G \subseteq \mathrm{V} \mid \lambda = \mathrm{M} \mid \lambda$ and H adds no sets of rank $< \lambda$. ⊣

More generally what this shows is the following.

3.B • 3. Corollary. *If $j : \mathrm{V} \to \mathrm{M}$ is λ-srs for λ a limit of strongs such that λ isn't stronger than a measurable in* M, *then we can lift j to $j^+ : \mathrm{V}^{\mathbb{P}_\lambda} \to \mathrm{M}^{j(\mathbb{P}_\lambda)}$.*

And this gives the desired result.

3.B • 4. Result. *Assume there is a proper class of srs cardinals and* GCH *holds. Therefore* $\mathbb{P} \Vdash$ UWISS + *"there is a proper class of strongs"*

Proof. That $\mathbb{P} \Vdash \mathsf{UWISS}$ follows from Corollary 3.A • 8. For a proper class of strongs, let κ be srs and let $\lambda > \kappa$ be strong in V. By Lemma 3.B • 2, in $V^{\mathbb{P}_\lambda}$, κ is λ-strong and this degree of strength is weakly indestructible, meaning κ is still λ-strong in all later stages (since, again, the tail forcings will be $\leq \kappa$-distributive and $< \kappa$-strategically closed) and so λ-strong in $V^{\mathbb{P}}$. Since there are a proper class of V-strongs above κ, it follows that κ is strong in $V^{\mathbb{P}}$. Hence any κ srs in V is strong in $V^{\mathbb{P}}$, and we have a proper class of both. ⊣

This completes the proof of Theorem 3.A • 1, repeated below, and so in conjunction with Theorem 2 • 5, Theorem 1 • 2 holds.

3.B • 5. Corollary. *(1) implies (2) and (3) where these stand for*

1. Con(ZFC + *"There is a proper class of srs cardinals"*).
2. Con(ZFC + *"There is a proper class of strongs"* + UWISS).
3. Con(ZFC + *"There is a proper class of strongs"* + *"Every λ-strong κ has weakly indestructible λ-strength whenever λ is below the least measurable limit of measurable cardinals larger than κ"*.

Again, we can go further than even what (3) states; beyond the next measurable limit of measurable limits of measurables, and so on. But at some point, saying this just becomes silly. The point is that we get much more than weak indestructibility for $\kappa + 2$-strength, up to the next cardinal λ that has at least as many measurables below it as a $\lambda + 2$-strong cardinal should (since it *was* $\lambda + 2$-strong in $V^{\mathbb{P}_{\kappa+1}}$).

The above results allow us to also prove Result 3.A • 2, restated below.

3.B • 6. Result. *It's consistent (relative to two srs cardinals and proper class of strong cardinals) that we can force a κ that is not ρ-strong to be ρ-strong by a collapse* $\mathrm{Col}(\kappa^+, \rho)$

Proof. Consider the Easton support iteration $\mathbb{P}_\kappa = \ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ where $\dot{\mathbb{Q}}_\kappa$ is defined by the following.

1. At every stage κ that is (forced to be) $\kappa + 2$-strong, we force with the lottery of κ, ρ-appropriate posets for minimal ρ (if any exist) of rank $< \kappa^{+}$¶.
2. Then with $\mathrm{Col}(\kappa^+, \rho)$.

3. If there is no such ρ, and κ was originally $< \lambda$-strong in V where $\kappa^{+\sharp\sharp} \leq \lambda < \kappa^{+¶}$, then force with $\mathrm{Col}(\kappa^+, \lambda)$.

4. Otherwise force trivially.

If the collapses never resurrect degrees of strength, then the proof below goes through to show that the resulting srs cardinals are strong. Moreover, weak indestructibility will hold for *all* degrees of strength, simply because the tail will be appropriate (by Corollary 3.A • 5) and

- Anytime we destroy κ's ρ-strength, we make sure the situation in Figure 2 does not happen and Figure 3 does: the next stage of forcing occurs far beyond ρ, so the lack of ρ-strong extenders continues through to the end of the preparation.

- If we *don't* destroy κ's strength with a lottery at stage κ, then κ's strength in $V^{\mathbb{P}_\kappa}$ is indestructible at that stage. The collapse, if there is one, in (3) doesn't add any degrees of strength by hypothesis. Thus κ's strength would be weakly indestructible in $V^{\mathbb{P}_{\kappa+1}}$ (if there were some κ, ρ-appropriate $\dot{\mathbb{Q}}$ then in $V^{\mathbb{P}_\kappa}$, we could force with the κ, ρ-appropriate $\dot{\mathbb{Q}}_\kappa * \dot{\mathbb{Q}}$). The collapse will again ensure the next stage of forcing occurs far beyond λ as in Figure 3, so the lack of λ-strong extenders continues through to the end of the preparation.

The same proof as Corollary 3.B • 3 with the previous preparation in Definition 3.A • 3 tells us that the srs cardinals become strong in $V^{\mathbb{P}}$. But two strong cardinals and weak indestructibility for all degrees of strength contradicts Result 1.B • 1. $\dashv$

4 Small Side Results

It's not hard to see that the forcing preparation $\mathbb{P}$ from Definition 3.A • 3 can be generalized and restricted to the following theorem when combined with the original proof from [3]. The idea really is that [3] only gives UWISS and the only reason there is a weakly indestructible strong cardinal in the end is that all of its degrees of strength are small: there is no measurable cardinal above the strong.

4 • 1. Theorem. *Let* $\alpha \in \mathrm{Ord}$. *Therefore the following are equiconsistent.*

1. ZFC + *"there are (at least)* α *srs cardinals with a hyperstrong above them";*
2. ZFC + *"there are* $\alpha + 1$ *srs cardinals";*
3. $\mathsf{ZFC} + \mathsf{UWISS}$ + *"there are (at least)* $\alpha + 1$ *strong cardinals". In fact, the last of those* $\alpha + 1$*-strong cardinals is weakly indestructible for all of its degrees of strength.*

Proof. The equiconsistency between (1) and (2) can be noted if we let κ be the hyperstrong above the α srs cardinals. In this case, either $\kappa^{+¶}$ doesn't exist and κ is already srs by Result 1.B • 6, or $\kappa^{+¶}$ does exist and κ is srs in $\mathrm{V} \mid \kappa^{+¶}$ and so are the smaller srs cardinals due to Lemma 1.B • 9.

That (3) is relatively consistent relative to (1) follows from the techniques of [3]: forcing GCH, and cutting off the universe at $\kappa^{+\sharp}$ whenever κ has weakly indestructible $\kappa^{+\sharp}$-strength in $\mathrm{V}^{\mathbb{P}_\kappa}$. This always ends at or before the final hyperstrong. Such a κ is strong in the final model. The lower srs cardinals become strong after forcing with $\mathbb{P}_\kappa$ as noted with Lemma 3.B • 2, and so there are $\alpha + 1$-strong cardinals after forcing with $\mathbb{P}_\kappa$ and clearly $\mathbb{P}_\kappa \cong \mathbb{P}_\mu$ forces UWISS over $\mathrm{V} \mid \mu$.

That (1) is relatively consistent to (3) follows again from the techniques of [3]: if $\mathrm{V} \vDash \mathsf{ZFC} + \mathsf{UWISS}$, any V-strong cardinal κ will be hyperstrong in K and if there's a V-strong above κ, then κ will be srs in K. ⊣

We need to consider successor ordinals in our equiconsistency result since getting an srs cardinal in K required us to have a strong above it, whereas getting a hyperstrong doesn't.

Another small topic before getting into Woodin cardinals here is the mix of full and weak indestructibility. If we use the preparation from [2], it's not hard to see that the cardinals below still have weak indestructibility depending on the notion of indestructibility we consider.[1] If we modify the preparation to instead force with $<\kappa$-strategically closed posets we can mix weak and full indestructibility. The easiest way to do this is as follows with strong cardinals.

4 • 2. Result. *Suppose there is an srs cardinal and a high-jump cardinal above it. Therefore, there is a poset that forces*

- *there are (at least) two strong cardinals;*
- *there is strong cardinal fully indestructible by $<\kappa$-directed closed posets; and*
- *every strong and partially strong cardinal has its ρ-strength weakly indestructible by $<\kappa$-directed closed posets whenever $\rho<\kappa^{+\sharp}$.*

Proof. Consider the iteration that begins with $\mathbb{P}_\omega \cong \mathrm{Add}(\omega,1)$ and at stage $\alpha>1$, $\alpha\leq\kappa$, $\dot{\mathbb{Q}}_\alpha$ is as in Definition 3.A • 3 where κ is supercompact reflecting supercompacts. For $\alpha>\kappa$, $\dot{\mathbb{Q}}_\alpha$ is also as in Definition 3.A • 3 except the lottery also includes non-$\leq\kappa$-distributive posets (to ensure full indestructibility and not merely weak indestructibility), and we stop the construction whenever we reach a cardinal $\delta>\kappa$ such that its $<\gamma$-supercompactness is indestructible by $<\kappa$-directed closed posets where $\gamma>\delta$ is any measurable cardinal. The resulting preparation is $\mathbb{P}_\delta = \ast_{\alpha<\delta}\dot{\mathbb{Q}}_\alpha$ where we use Easton support.

It's not hard to see that the tail forcing $\dot{\mathbb{R}}_{[\alpha,\delta)}$ is still $\leq\alpha$-distributive and $<\alpha$-strategically closed whenever $\alpha\leq\kappa$. Hence the proof of Lemma 3.B • 2 and Corollary 3.A • 8 still apply to tell us that κ is strong in $\mathrm{V}^{\mathbb{P}}$ and UWISS holds for cardinals $\leq\kappa$. The rest of the poset also forces UWISS for cardinals above κ since any strength that remains is *de facto* indestructible (and hence weakly indestructible) by $<\kappa$-directed closed posets. It follows that δ is fully strong in $\mathrm{V}^{\mathbb{P}_\delta}\mid\gamma$ and its small degrees of strength are fully indestructible by $<\kappa$-directed closed posets. ⊣

This is, however, not that interesting, because the idea is fundamentally just glueing together two separate ideas. Nevertheless, we cannot go any further than this, because

[1] Namely, if we try to destroy everything with $<\kappa$-directed closed posets, then they will still be indestructible under $<\kappa$-directed closed, $\leq\kappa$-distributive posets

of the limitation from [2] and Result 1.B • 1: we cannot have two cardinals with (weakly) indestructible "large" degrees of strength while universal (weak) indestructibility holds for "small" degrees of strength.

A motivating question before considering Woodin cardinals is what cardinals are preserved after forcing with the preparation of Definition 3.A • 3? It's fairly straightforward to show that measurable limits of strong cardinals remain measurable.

4 • 3. Result. *Let κ be measurable and a limit of strongs. Let $\mathbb{P}$ be as in Definition 3.A • 3. Therefore $\mathbb{P} \Vdash$ "κ is measurable".*

Proof. Let U be the Mitchell-order least ultrafilter on κ meaning that the ultrapower map $j_U : \mathrm{V} \to \mathrm{M} = \mathrm{Ult}(\mathrm{V}, U)$ has $\mathrm{M} \vDash$ "κ isn't measurable". Since κ is a limit of strongs, Lemma 3.B • 1 applies with $\lambda = \kappa$. So we can factor $j_U(\mathbb{P}_\kappa)$ as $\mathbb{P}_\kappa * \dot{\mathbb{R}}^{\mathrm{M}}_{[\kappa, j_U(\kappa))}$. By standard facts of Easton support iterations:

- $\mathbb{P}_\kappa$ is κ-cc because $|\mathbb{P}_\alpha| < \kappa$ for each $\alpha < \kappa$.
- In M, the tail $\dot{\mathbb{R}}^{\mathrm{M}}_{[\kappa, j_U(\kappa))}$ is (forced to be) $< \delta$-strategically closed, where δ is next non-trivial stage of the iteration. In particular, $\dot{\mathbb{R}}_{[\kappa, j_U(\kappa))}$ is (forced to be) $< \kappa^+$-strategically closed in M and hence in V since ${}^{<\kappa^+}\mathrm{M} \cap \mathrm{V} \subseteq \mathrm{M}$.

We now lift j_U from V to $\mathrm{V}^{\mathbb{P}_\kappa}$ to witness the measurability of κ in $\mathrm{V}^{\mathbb{P}_\kappa}$. Let G be $\mathbb{P}_\kappa$-generic over V. By (1), $\mathrm{M}[G]$ is still closed under κ-sequences in $\mathrm{V}[G]$. As an ultrapower embedding, $\mathrm{V} \vDash$ "$|j_U(\kappa)| = 2^\kappa$" and since $\mathrm{V}, \mathrm{V}[G] \vDash \mathsf{GCH}_{\geq\kappa}$, these are just $(\kappa^+)^{\mathrm{V}}$. Moreover, in V, $(\dot{\mathbb{R}}^{\mathrm{M}}_{[\kappa, j(\kappa))})_G$ has at most $2^\kappa = \kappa^+$-many dense subsets in $\mathrm{M}[G]$ and therefore by (2) and standard results, we can lift. More precisely, there is an $H \in \mathrm{V}[G]$ that is $\mathbb{R}^{\mathrm{M}}_{[\kappa, j_U(\kappa))}$-generic over $\mathrm{M}[G]$ so that $j_U"G \subseteq H$ (because the support of $j(p)$ is bounded in κ for any $p \in G$). We can then lift j_U to $j_U^+ : \mathrm{V}[G] \to \mathrm{M}[G * H]$ in $\mathrm{V}[G]$ so that $\mathrm{V}[G] \vDash$ "κ is measurable". This implies κ is measurable in $\mathrm{V}^{\mathbb{P}}$ since the tail forcing adds no subsets of κ by Corollary 3.A • 5. ⊣

Other measurable cardinals, however, might be collapsed accidentally. But if $\dot{\mathbb{Q}}_\kappa$ is (forced to be) non-trivial, then κ's measurability will be preserved. Generally speaking, what large cardinals are preserved by $\mathbb{P}$ is a matter for future research. But another example of this is Woodin cardinals.

4.A Indestructibility and Woodin cardinals

As stated, reflecting strongs is just one kind of reflection, but in principle, we could also enforce that we reflect more. This essentially converges onto the idea of a Woodin cardinal, and one idea to consider is bringing Woodin cardinals into the mix, and ask what happens to them with the preparation from Definition 3.A • 3.

4.A • 1. Theorem. *Let* $\mathbb{P}$ *be as in Definition 3.A • 3. Let* δ *be a Woodin cardinal. Therefore* $\mathbb{P}_\delta \Vdash$ *"*δ *is Woodin". Hence the existence of a Woodin cardinal* δ *gives the consistency of a Woodin cardinal* δ *where* $\mathrm{V} \mid \delta \models \mathsf{UWISS}$*.*

Proof. Note that we take a direct limit at δ and as a limit of strongs, we can regard $\mathbb{P}_\delta \subseteq \mathrm{V} \mid \delta$. Let G be $\mathbb{P}_\delta$-generic over V so that, as notation, $G \restriction \lambda$ is $\mathbb{P}_\lambda$-generic over V whenever $\lambda < \delta$. Let $A \subseteq \mathrm{V}[G]$ be arbitrary. Note that for each $\lambda < \delta$, $\mathrm{V}[G] \mid \lambda + 1 = \mathrm{V}[G \restriction \lambda] \mid \lambda + 1$ due to Corollary 3.A • 5, meaning

$$A \cap \mathrm{V}[G] \mid \lambda = A \cap \mathrm{V}[G \restriction \lambda] \mid \lambda \in \mathrm{V}[G \restriction \lambda].$$

In this way, whenever we can regard $\mathbb{P}_\lambda \subseteq \mathrm{V} \mid \lambda$, e.g. at stages that are limits of strongs, we can let $\dot{A}_\lambda$ be a $\mathbb{P}_\lambda$-name for $A \cap \mathrm{V}[G] \mid \lambda$. By standard facts of iterated forcing, we can translate names with a function $T_{\alpha,\beta}$ from $\mathbb{P}_\alpha$-names to $\mathbb{P}_\beta$-names for $\alpha < \beta$ (just by adding $\dot{\mathbb{1}}$s to translate the conditions). So we can iteratively build a name $\dot{A}$ for A in this way while ensuring that for $\dot{A}_\alpha$ and $\dot{A}_\beta$ defined and $\alpha < \beta$:

1. $T_{\alpha,\beta}(\dot{A}_\alpha) \subseteq \dot{A}_\beta$; and
2. If $\tau \in \mathrm{dom}(\dot{A}_\beta \setminus T_{\alpha,\beta}(\dot{A}_\alpha))$ then $\mathbb{P}_\beta \Vdash$ "$\check{\alpha} \leq \mathrm{rank}(\tau) < \check{\beta}$" (i.e. we don't try to add sets to previous stages).

We can then translate $\dot{A}_\lambda$ to $\dot{A} \mid \lambda = T_{\lambda,\delta}(\dot{A}_\lambda)$, a $\mathbb{P}_\delta$-name for $A \cap \mathrm{V}[G] \mid \lambda$. In this way $\dot{A} = \bigcup_{\lambda<\delta} \dot{A} \mid \lambda$ is a $\mathbb{P}_\delta$-name for A. (1) and (2) tell us that stretching $\dot{A}_\lambda$ and restricting down to λ should just give us $\dot{A}_\lambda$ whenever we reflect $\dot{A}$. More precisely, from $\dot{A}$ and λ, we can define $\dot{A}_\lambda$ as the subset consisting of $\langle\tau,p\rangle\in\dot{A}$ where the support of p is a subset of λ and τ is forced to have rank below λ.

Now onto the meat of the proof, which is mostly just technical details. The idea is that we lift a λ-srs $j:\mathrm{V}\to\mathrm{M}$ with $\mathrm{cp}(j)=\kappa<\delta$ that reflects $\dot{A}$ to $j^+:\mathrm{V}[G\restriction\lambda]\to\mathrm{M}[j(G\restriction\lambda)]$ that reflects $A\cap\mathrm{V}[G\restriction\lambda]\mid\lambda$. The extender generating this embedding in $\mathrm{V}[G\restriction\lambda]$ also works in $\mathrm{V}[G]$. This tells us δ remains Woodin in $\mathrm{V}[G]$.

Writing out the details, since δ is Woodin, let $\kappa<\delta$ be $<\delta$-strong reflecting both $\dot{A}$ (regarded as a subset of $\mathrm{V}\mid\delta$) and the set of strongs below δ. It follows that κ is $<\delta$-srs. So consider Corollary 3.B • 3: for any $\lambda<\delta$ that's a non-measurable limit of strongs, if $j:\mathrm{V}\to\mathrm{M}$ is a λ-srs embedding with $\mathrm{cp}(j)=\kappa$, we can lift this to $j^+:\mathrm{V}^{\mathbb{P}_\lambda}\to\mathrm{M}^{j(\mathbb{P}_\lambda)}$ that is also λ-strong. This j^+ will also reflect $\dot{A}$ assuming j does, partially because $\dot{A}\subseteq\mathrm{V}$:

$$j^+(\dot{A})\cap\mathrm{V}^{\mathbb{P}_\delta}\mid\lambda=j(\dot{A})\cap\mathrm{V}\mid\lambda=\dot{A}\cap\mathrm{V}\mid\lambda=\dot{A}\cap\mathrm{V}^{\mathbb{P}_\delta}\mid\lambda.$$

This tells us $j^+(A)\cap\mathrm{V}[G\restriction\lambda]\mid\lambda=A\cap\mathrm{V}[G\restriction\lambda]\mid\lambda$ as follows. Reflecting $\dot{A}$ up to λ implies $j(\dot{A})\mid\lambda=\dot{A}\mid\lambda$ and thus $j(\dot{A}_\lambda)_\lambda=j(\dot{A})_\lambda=\dot{A}_\lambda$ by (1) and (2) above. It follows that, since $G\restriction\lambda=j^+(G)\restriction\lambda$ and $j^+(\tau_{G\restriction\lambda})=j(\tau)_{j^+(G\restriction\lambda)}$ for τ a $\mathbb{P}_\lambda$-name,

$$\begin{aligned}j^+(A\cap\mathrm{V}[G]\mid\lambda)\cap\mathrm{V}[G\restriction\lambda]\mid\lambda&=j^+((\dot{A}_\lambda)_{G\restriction\lambda})\cap\mathrm{V}[G\restriction\lambda]\mid\lambda\\&=j(\dot{A}_\lambda)_{j^+(G\restriction\lambda)}\cap\mathrm{V}[G\restriction\lambda]\mid\lambda\\&=(j(\dot{A}_\lambda)_\lambda)_{j^+(G\restriction\lambda)\restriction\lambda}\\&=(\dot{A}_\lambda)_{G\restriction\lambda}=A\cap\mathrm{V}[G]\mid\lambda.\end{aligned}$$

Since λ isn't measurable, the first non-trivial stage of forcing in $\dot{\mathbb{R}}_{[\lambda,\delta)}$ is well above λ. So the extender generating j^+ in $\mathrm{V}[G\restriction\lambda]$ remains an extender in $\mathrm{V}[G\restriction\delta]$ and all the functions needed to witness that

$$j^+(A\cap\mathrm{V}[G\restriction\lambda]\mid\lambda)\cap\mathrm{V}[G\restriction\lambda]\mid\lambda=A\cap\mathrm{V}[G\restriction\lambda]\mid\lambda$$

are in $\mathrm{V}[G\restriction\lambda]$. Basically, if we take the ultrapower by this extender in $\mathrm{V}[G]$ with $j^{++}:\mathrm{V}[G]\to\mathrm{N}$, we get a λ-strong embedding with

$$j^{++}(A)\cap\mathrm{V}[G]\mid\lambda=j^+(A\cap\mathrm{V}[G]\mid\lambda)\cap\mathrm{V}[G\restriction\lambda]\mid\lambda=A\cap\mathrm{V}[G]\mid\lambda,$$

as desired. This tells us δ is Woodin in $\mathrm{V}[G]$. By Corollary 3.A • 8, $\mathrm{V}[G] \mid \delta \vDash \mathsf{UWISS}$. $\dashv$

One corollary of Theorem 4.A • 1 is that the preparation $\mathbb{P}$ from Definition 3.A • 3 preserves Woodin cardinals more generally just as with measurable limits of strongs with Result 4 • 3.

4.A • 2. Corollary. *The following are equiconsistent with* ZFC*:*

1. *There is a Woodin cardinal.*

2. *There is a Woodin cardinal δ such that $\mathrm{V} \mid \delta \vDash \mathsf{UWISS}$.*

This gives the consistency of UWISS with a proper class of a large variety of large cardinal notions relative to the existence of a Woodin cardinal.

While on the topic of Woodins, consider the interaction of universal weak indestructibility for *large* degrees of strength with Woodin cardinals. For example, we used a cardinal that is strong and reflects strongs to get some weak indestructibility results for strength. Can we do something similar for reflection properties? A subsequent idea is a "Woodin by weak indestructibility" cardinal, which basically would be that there are so many strong cardinals with weak indestructibility for their large degrees of strength and their reflection that we could potentially witness Woodin-ness just with such weakly indestructible cardinals. This is impossible.

4.A • 3. Definition. *An inaccessible cardinal δ is* Woodin by weak indestructibility *iff for every $A \subseteq \mathrm{V} \mid \delta$, there is a $\kappa < \delta$ that is $< \delta$-strong reflecting A such that this strength and reflection is weakly indestructible by $< \kappa$-strategically closed posets.*

4.A • 4. Result. *It is not possible to have a cardinal that is Woodin by weak indestructibility.*

Proof. Suppose not: let δ be Woodin by weak indestructibility. Let W be the set of all cardinals with weakly indestructible $< \delta$-strength in V. There is therefore an (arbitrarily large) $\kappa < \delta$ that is $< \delta$-strong reflecting W such that this is weakly indestructible. In particular, after adding a Cohen subset $A \subseteq \mathrm{Add}(\kappa^+, 1)$, κ is still $< \delta$-strong reflecting W. Let $j : \mathrm{V}[A] \to \mathrm{M}[j(A)]$, $\mathrm{cp}(j) = \kappa$ witness this for some large $\lambda < \delta$. In $\mathrm{M}[j(A)]$, $j(W) \cap \lambda = W \cap \lambda \ni \kappa$. Consider the least element $\mu \in W$ above κ that is $< \delta$-strong, and assume without loss of generality that λ is large enough that $\mu < \lambda$. By [6] in $\mathrm{V}[A]$, since $\mathrm{Add}(\kappa^+, 1)$ is small relative to μ, μ's strength is weakly destructible by $\mathrm{Add}(\mu^+, 1)$. Reflected in $\mathrm{V}[A]$, this means arbitrarily large cardinals $\hat{\mu} \in W$ have their $< \delta$-strength destroyed by $\mathrm{Add}(\hat{\mu}^+, 1)$. Hence $\mathrm{Add}(\kappa^+, 1) * \mathrm{Add}(\hat{\mu}^+, 1)$ destroys $\hat{\mu}$'s $< \delta$-strength in V, contradicting that $\hat{\mu} \in W$. ⊣

Despite this result, we can still have a Woodin cardinal such that the cardinals witnessing this can always be chosen to have weakly indestructible strength. All Definition 4.A • 3 tells us is that their *reflection* need not be weakly indestructible. In fact reflection of the set of cardinals with weakly indestructible strength cannot be weakly indestructible by the above.

4.A • 5. Definition. *A cardinal δ is* Woodin witnessed by (weak) indestructibility *iff δ is Woodin, and every $< \delta$-strong cardinal $\kappa < \delta$ has (weakly) indestructible strength.*

Using strongs reflecting strongs again, it's possible to force any Woodin cardinal δ to be witnessed by weak indestructibility. We do this as follows.

4.A • 6. Definition (Woodin Preparation). *Let δ be Woodin. We define $\mathbb{P}^\delta_\omega \cong \mathrm{Add}(\omega, 1)$. For $\mathbb{P}^\delta_\kappa$ defined for $\omega < \kappa < \delta$ we aim to define $\dot{\mathbb{Q}}^\delta_\kappa$ as follows.*

1. *If no $p \in \mathbb{P}^\delta_\kappa$ forces that κ is strong, then $\dot{\mathbb{Q}}^\delta_\kappa$ is trivial.*
2. *Otherwise, suppose κ is forced by some $p \in \mathbb{P}^\delta_\kappa$ to be $< \delta$-strong.*
   (a) *If κ's strength is weakly indestructible by posets of rank $< (\kappa^{+¶})^{\mathrm{V}}$, then define $\dot{\mathbb{Q}}^\delta_\delta$ to be trivial.*
   (b) *Otherwise, let ρ be the minimal degree of κ's strength that is weakly destructible and let $\dot{\mathbb{Q}}^\delta_\kappa$ be the lottery sum of κ, ρ-appropriate posets of rank $< (\kappa^{+¶})^{\mathrm{V}}$.*

For limit stages, κ, $\mathbb{P}^\delta_\kappa = \mathop{\ast}_{\alpha<\kappa} \dot{\mathbb{Q}}^\delta_\alpha$ is an Easton support iteration. The full iteration is $\mathbb{P}^\delta = \mathbb{P}^\delta_\delta$. We write $\dot{\mathbb{R}}^\delta_{[\kappa,\lambda)}$ whenever $\kappa < \lambda < \delta$ for the tail forcing.

We have similar results to subsection 3.A. In particular, Corollary 3.A • 5 and Result 3.A • 7 still hold but not for large degrees of strength.

4.A • 7. Corollary. *Let δ be Woodin. Therefore for any $\kappa < \lambda \leq \delta$, the tail forcing $\dot{\mathbb{R}}^\delta_{[\kappa,\lambda)}$ is $< \kappa$-strategically closed and $\leq \kappa$-distributive.*

Because we only force at strong stages (equivalently, $<\delta$-strong stages), we don't need to care about measurables or small degrees of strength.

4.A • 8. Corollary. *Let δ be Woodin with $\kappa<\delta$. Suppose $\mathbb{P}^\delta_{\kappa+1}\Vdash$ "κ is not ρ-strong" for $\rho<\delta$. Therefore $\mathbb{P}^\delta\Vdash$ "κ is not ρ-strong".*

Proof. Suppose κ is ρ-strong in $\mathrm{V}^{\mathbb{P}^\delta}$. Since $\mathbb{P}_\delta$ admits a gap at $\aleph_1$, it follows that κ must have been ρ-strong in V. By Lemma 1.A • 10, any failure to ρ-strength would need to occur before the next $<\delta$-strong cardinal $\mu=(\kappa^{+¶})^{\mathrm{V}|\delta}$. So we may assume $\rho<\mu$. The next non-trivial stage of the tail forcing $\dot{\mathbb{R}}^\delta_{(\kappa,\delta)}$ is at least stage μ, meaning $\dot{\mathbb{R}}^\delta_{(\kappa,\delta)}\cong\dot{\mathbb{R}}^\delta_{[\mu,\delta)}$, which is $<\mu$-strategically closed and $\leq\mu$-distributive by Corollary 4.A • 7. In particular, the lack of ρ-strong extenders in $\mathrm{V}^{\mathbb{P}^\delta_{\kappa+1}}$ holds in $\mathrm{V}^{\mathbb{P}^\delta}$. ⊣

4.A • 9. Result. *Let δ be Woodin. Therefore $\mathbb{P}^\delta$ forces that every cardinal κ that is $<\delta$-strong has its degrees of strength as weakly indestructible.*

Proof. Suppose κ is $<\delta$-strong in $\mathrm{V}^{\mathbb{P}^\delta}$. It follows that κ must be $<\delta$-strong in $\mathrm{V}^{\mathbb{P}^\delta_{\kappa+1}}$ by Corollary 4.A • 8. But we only force with a non-trivial $\dot{\mathbb{Q}}^\delta_\kappa$ whenever we destroy κ's strength from $\mathrm{V}^{\mathbb{P}^\delta_\kappa}$ to be below the next $<\delta$-strong cardinal. But because κ is fully $<\delta$-strong in $\mathrm{V}^{\mathbb{P}^\delta_{\kappa+1}}$, we didn't destroy anything, meaning $\dot{\mathbb{Q}}^\delta_\kappa$ was trivial and κ was $<\delta$-strong in $\mathrm{V}^{\mathbb{P}^\delta_\kappa}$.

To see that this strength is indestructible, let $\dot{\mathbb{Q}}$ be κ,ρ-appropriate for some $\rho<\delta$ in $\mathrm{V}^{\mathbb{P}^\delta}$. The tail forcing $\dot{\mathbb{R}}^\delta_{[\kappa,\delta)}*\dot{\mathbb{Q}}$ is therefore κ,ρ-appropriate in $\mathrm{V}^{\mathbb{P}^\delta_\kappa}$ and so by Lemma 3 • 3, there is a κ,ρ'-appropriate poset in $\mathrm{V}^{\mathbb{P}^\delta_\kappa}$ of rank $<(\kappa^{+¶})^{\mathrm{V}|\delta}$ with ρ' also below this. In other words, $\dot{\mathbb{Q}}^\delta_\kappa$ should be non-trivial, meaning κ couldn't have ρ-strength in $\mathrm{V}^{\mathbb{P}^\delta}$, a contradiction. ⊣

We make use of reflecting strongs and subsets $A\subseteq\delta$ of $\mathrm{V}^{\mathbb{P}^\delta}$ to ensure there still are $\kappa<\delta$ that are $<\delta$-strong reflecting A in $\mathrm{V}^{\mathbb{P}^\delta}$. This is done in the exact same manner as Theorem 4.A • 1, proving the consistency of a Woodin witnessed by weak indestructibility from a Woodin cardinal.

4.A • 10. Lemma. *Let δ be Woodin. Let $A\subseteq\delta$ be arbitrary in $\mathrm{V}^{\mathbb{P}^\delta}$. Let $\kappa<\delta$ be $<\delta$-strong reflecting strongs and $\dot{A}$, a $\mathbb{P}^\delta$-name for A. Therefore $\mathbb{P}^\delta$ forces that κ is $<\delta$-strong reflecting A.*

Proof. Let $j : \mathrm{V} \to \mathrm{M}$ be λ-strong reflecting strongs and $\dot{A}$ with $\mathrm{cp}(j) = \kappa$ so that $\mathrm{M} = \mathrm{Ult}_E(\mathrm{V})$ for some (κ, λ)-extender E with $\lambda < \delta$ sufficiently large (e.g. an inaccessible limit of $< \delta$-strong cardinals). By the same argument as in Lemma 3.B • 1, we can factor

$$j(\mathbb{P}^\delta_\lambda) = (\mathbb{P}^\delta_{j(\lambda)})^{\mathrm{M}} \cong (\mathbb{P}^\delta_\lambda)^{\mathrm{M}} * (\dot{\mathbb{R}}^\delta_{[\lambda, j(\lambda))})^{\mathrm{M}} \cong (\mathbb{P}^\delta_\lambda)^{\mathrm{V}} * (\dot{\mathbb{R}}^\delta_{[\lambda, j(\lambda))})^{\mathrm{M}}.$$

Without loss of generality, λ isn't $< \delta$-strong in M. So by the exact same argument as with Lemma 3.B • 2, if G is $\mathbb{P}^\delta_\lambda$-generic over V, we can lift $j : \mathrm{V} \to \mathrm{M}$ to $j^+ : \mathrm{V}[G] \to \mathrm{M}[j(G)]$ such that j^+ is still λ-strong.

Moreover, by choosing the right name $\dot{A}$ for A as in Theorem 4.A • 1, we also get that j^+ will still reflect $\dot{A}$ up to λ just as j reflects $\dot{A}$ up to λ. So with access to G (since $j(G)$ takes the form $G * H$ for some H) in $\mathrm{M}[j(G)]$ we can reconstruct $A \cap \lambda$, as desired. ⊣

4.A • 11. Corollary. *Let δ be Woodin. Therefore $\mathbb{P}^\delta$ forces that δ is Woodin witnessed by weak indestructibility.*

Proof. For any $A \subseteq \delta$ in $\mathrm{V}^{\mathbb{P}^\delta}$, take a $< \delta$-strong cardinal κ reflecting strongs and a name for A. By Lemma 4.A • 10, κ is $< \delta$-strong reflecting A in $\mathrm{V}^{\mathbb{P}^\delta}$. Since A was arbitrary, δ is still Woodin. By Result 4.A • 9, δ is Woodin witnessed by weak indestructibility. ⊣

Stated differently, we get an equiconsistency, combined with Corollary 4.A • 2.

4.A • 12. Corollary. *The following are equiconsistent with* ZFC.

1. *There is a Woodin cardinal.*
2. *There is a Woodin witnessed by weak indestructibility.*
3. *There is a Woodin cardinal δ such that $\mathrm{V} \mid \delta \vDash$* UWISS.

This suggests there is a delicate balance between (weak) indestructibility and reflection: one can have weakly indestructible strength with relative ease, but too much of this precludes too much weakly indestructible reflection properties by Result 4.A • 4. What kinds of non-trivial reflection can be indestructible is a subject of further research. For example, is it possible to have a proper class of strongs, and a strong reflecting strongs with this strength and reflection weakly indestructible? Below a Woodin witnessed by weak indestructibility,

δ, the answer is no, because in V | δ, the weakly indestructible strong cardinals are just all of the strong cardinals. But in a more general setting the answer is not as obvious to me.

5 Supercompacts

We can also generalize the idea behind the forcing of section 3 to bypass the restriction of [2] of having no measurable above the desired supercompact cardinal at the expense of (weak) indestructibility for its large degrees of supercompactness. More precisely, with the stronger form of indestructibility there, one cannot have multiple supercompact cardinals. Nevertheless, with the forcing idea above, we can get multiple supercompact cardinals with weak indestructibility for all small degrees of supercompactness. All this requires is a supercompact reflecting supercompacts to lift the embeddings just as with Lemma 3.B • 2. The added reflection gives more agreement between V and M on what the preparation looks like, and so we can use similar techniques to lift the embedding.

Let me give the necessary background. Note that we need to change the sense of indestructibility here because of how elements of the ultrapowers are represented: we can't carry out exactly the same lifting argument where we regard a dense set as represented by an $f : [\kappa]^{<\omega} \to \mathrm{V}$ and $r \in [\lambda]^{<\omega}$. Instead, we can only represent dense sets in the ultrapower by functions $f : [\lambda]^{<\omega} \to \mathrm{V}$, of which there are (ostensibly) many more in V. The usual Laver indestructibility refers to indestructibility for all degrees of supercompactness of a single cardinal κ by $< \kappa$-directed closed posets. So the weakening we will consider for supercompactness will also consider $< \kappa$-directed closed posets. This is useful in the lift-up arguments with a standard master condition argument.

5 • 1. Definition. Universal weak indestructibility for small degrees of supercompactness (UWISSc) *is the statement that any cardinal κ, if κ is ρ-supercompact, then κ's ρ-supercompactness is weakly indestructible by $< \kappa$-directed closed posets whenever $\rho < \kappa^{+\sharp}$.*

As a small remark, $< \kappa$-directed closed posets are both $< \kappa$-closed and κ-strategically closed. And we also have much of the same properties and results, especially around iterations and their tails like Theorem 3.A • 4 [4].

5 • 2. Definition. *For $\kappa < \lambda \in \mathrm{Ord}$,*

- *κ is* λ-supercompact *iff there is an elementary $j : \mathrm{V} \to \mathrm{M}$ such that $j(\kappa) > \lambda$ and $\mathrm{V} \vDash$ "${}^{\lambda}\mathrm{M} \subseteq \mathrm{M}$".*

- κ *is* supercompact *iff* κ *is* λ*-supercompact for every* λ*.*
- κ *is* $<\lambda$-supercompact *iff* κ *is* ξ*-supercompact for every* $\xi < \lambda$*.*

So measurable cardinals κ are κ-supercompact. It should be clear that being 2^κ-supercompact cardinal is also $\kappa + 2$-strong. Generally speaking, if $\lambda > \kappa$ is a strong limit cardinal, being λ-supercompact implies being λ-strong. So any supercompact is therefore strong. In fact, κ being λ-supercompact is equivalent [7] to having a certain kind of measure over $[\lambda]^{<\kappa}$, which can be proparly calculated in $\mathrm{V} \mid \lambda + \omega$. In particular, Lemma 1.A • 10 gives that the least λ such that κ fails to be λ-supercompact occurs at some degree $\lambda < \kappa^{+¶}$ and since this is below the next supercompact cardinal, we get the following.

5 • 3. Definition. *For* $\kappa \in \mathrm{Ord}$*, if they exist,*

- *Let* $\kappa^{+\mathrm{SC}}$ *denote the least supercompact* $\lambda > \kappa$*.*
- *Let* $\kappa^{+\$}$ *denote the least* $\lambda > \kappa$ *that is* λ^+*-supercompact.*

5 • 4. Corollary. *Let* κ *be such that* $\kappa^{+\mathrm{SC}}$ *exists. Therefore,* κ *is supercompact iff* κ *is* $< \kappa^{+¶}$*-supercompact iff* κ *is* $< \kappa^{+\mathrm{SC}}$*-supercompact.*

The same unsurprisingly also holds when forcing.

5 • 5. Corollary. *Let* κ *be such that* $\kappa^{+¶}$ *exists. Let* $\mathbb{P}$ *be a poset of rank* $< \kappa^{+¶}$*. Therefore,* $\mathbb{P} \Vdash$ *"*κ *is supercompact" iff* $\mathbb{P} \Vdash$ *"*κ *is* $< (\kappa^{+¶})^{\mathrm{V}}$*-supercompact".*

Proof. Suppose κ is $< (\kappa^{+¶})^{\mathrm{V}}$-supercompact in $\mathrm{V}^{\mathbb{P}}$, but not λ-supercompact for some $\lambda \geq (\kappa^{+¶})^{\mathrm{V}}$. We thus have some $p \in \mathbb{P}$ that forces this:

$$p \Vdash \text{"}\exists\alpha\ (\check{\kappa} \text{ is not } \alpha\text{-supercompact})\text{"}.$$

By Lemma 1.A • 10, this is reflected down and holds in arbitrarily large $\mathrm{V} \mid \gamma$ for some $\gamma < (\kappa^{+¶})^{\mathrm{V}}$ with γ far above the rank of $\mathbb{P}$ and without loss of generality, γ is a limit cardinal and so well above any particular witness α. In particular, we may fix some particular $\gamma' < \gamma$ such that

$$p \Vdash \text{"}\exists\alpha < \check{\gamma}'\ (\check{\kappa} \text{ is not } \alpha\text{-supercompact})\text{"},$$

which also holds in V, contradicting the hypothesis. $\dashv$

We of course get the same idea as Result 1.B • 1 for supercompacts, now almost directly Superdestruction Theorem III from [6].

5 • 6. Theorem. *Assume* GCH*. Suppose κ is supercompact and this is not destroyed by* $\mathrm{Add}(\kappa^+, 1)$*. Suppose $\kappa^{+\$}$ exists. Therefore there are arbitrarily large $\delta < \kappa$ whose δ^+-supercompactness is destroyed by* $\mathrm{Add}(\kappa^+, 1) * \mathrm{Add}(\delta^+, 1)$*.*

So we have the same sort of balancing act as before with weak indestructibility for large vs. small degrees of supercompactness, and we are investigating small degrees. The inner model theory with supercompacts is not understood at this time, so using something like core model techniques to get an *equi*consistency is not a good idea. So we will only arrive at a relative consistency: supercompacts reflecting supercompacts become supercompact in the resulting model. We now present a forcing similar to section 3, starting off with a definition similar to Definition 3 • 1.

5 • 7. Definition. *For δ a cardinal of degree of supercompactness $\geq \rho$, we say a poset $\mathbb{Q}$ is* δ, ρ-sc-appropriate *iff*

1. *$\mathbb{Q}$ is $< \delta$-directed closed;*
2. *$\mathbb{Q}$ is $\leq \delta$-distributive; and*
3. *$\mathbb{Q}$ destroys the ρ-supercompactness of δ.*

If just (1) and (2) hold, we say $\mathbb{Q}$ is δ-sc-appropriate*.*

The same proof for Corollary 3 • 2 gives Lemma 3 • 3 for sc-appropriate posets.

5 • 8. Corollary.

1. *Let $\mathbb{P} \in \mathrm{V} \mid \alpha$ be a poset. Therefore $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\check{\delta}, \check{\rho}$-sc-appropriate with rank $\leq \check{\beta}$" iff $\mathrm{V} \mid \lambda$ satisfies this for any inaccessible $\lambda > \alpha, \rho, \beta$.*
2. *Let $\mathbb{P} \in \mathrm{V} \mid \kappa^{+\P}$ be such that $\mathbb{P}$ forces the existence of a κ, ρ^*-sc-appropriate poset. Therefore $\mathbb{P}$ forces that there is a κ, ρ-sc-appropriate poset of rank $< \kappa^{+\P}$ with $\rho \leq \rho^*$ and $\rho < \kappa^{+\P}$.*

We then define our preparation similar to Definition 3.A • 3, destroying supercompactness as much as possible, and collapsing if necessary to gain some control over the tail poset.

5 • 9. Definition (The Preparation for Supercompacts). *Suppose* $\mathbb{P}_\delta = \ast_{\xi<\delta} \dot{\mathbb{Q}}_\xi$ *has been defined for* $\delta > \omega$*. We aim to define* $\dot{\mathbb{Q}}_\delta$*.*

1. *If no* $p \in \mathbb{P}_\delta$ *forces that* δ *is inaccessible, then* $\dot{\mathbb{Q}}_\delta = \dot{\mathbb{1}}$ *is trivial.*

2. *Otherwise, suppose* δ *is forced by some* $p \in \mathbb{P}_\delta$ *to be* $<\lambda$*-supercompact for some (maximal)* λ *(allowing* $\lambda = 0$ *if* δ *isn't measurable, and* $\lambda = \mathrm{Ord}$ *if supercompact), and work below* p*.*

   (a) *If the* $<\lambda$*-supercompactness of* δ *is weakly indestructible via posets of rank* $< (\delta^{+\P})^{\mathrm{V}}$*, then define* $\dot{\mathbb{Q}}_\delta = \dot{\mathbb{1}}$*.*

   (b) *Otherwise, we let* $\rho < \lambda$ *be the minimal degree of* δ*'s supercompactness that is weakly destructible and let* $\dot{\mathbb{Q}}_\delta$ *be the lottery sum of all (what are forced to be) posets that take the form* $\dot{\mathbb{B}} * \dot{\mathbb{C}}$ *where the following happen.*

   (c) $\dot{\mathbb{B}}$ *is an* δ, ρ*-sc-appropriate poset of rank* $< (\delta^{+\P})^{\mathrm{V}}$*, and*

   (d) *In* $\mathrm{V}^{\mathbb{P}_\kappa * \dot{\mathbb{B}}}$*, if* $\rho < \delta^{+\$\$} \leq |\dot{\mathbb{B}}|$*, then* $\dot{\mathbb{C}}$ *is a name for* $\mathrm{Col}(\rho^{+\sharp}, |\dot{\mathbb{B}}|)$*. Otherwise* $\dot{\mathbb{C}}$ *is trivial.*

Using Easton support for limit stages, we write $\mathbb{P} = \ast_{\xi\in\mathrm{Ord}} \dot{\mathbb{Q}}_\xi$ *for the class iteration and* $\dot{\mathbb{R}}_{[\delta,\lambda)} \cong \ast_{\delta\leq\xi<\lambda} \dot{\mathbb{Q}}_\xi$ *for the tail forcing of* $\mathbb{P}_\lambda$ *whenever* $\delta < \lambda$*.*

The exact same proof as before tells us Corollary 3.A • 5 holds for this new preparation.

5 • 10. Corollary. *Let* $\delta < \lambda$*. Therefore, the tail forcing of our preparation* $\dot{\mathbb{R}}_{[\delta,\lambda)}$ *is (forced to be)* $<\delta$*-directed closed and* $\leq\delta$*-distributive.*

And from this, we again get weak indestructibility for small degrees of supercompactness in $\mathrm{V}^{\mathbb{P}_{\kappa+1}}$ by the almost exact same proof as Result 3.A • 7

5 • 11. Lemma. *The first non-trivial stage of forcing after any* $\kappa \in \mathrm{Ord}$ *is at least* $(\kappa^{+\$})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$

Proof. To have any degree of δ's supercompactness weakly destructible over $\mathrm{V}^{\mathbb{P}_\delta}$, we require δ to be at least δ^+-supercompact. Hence the least δ such that $\mathbb{P}_\delta \cong \mathbb{P}_{\kappa+1}$ but $\dot{\mathbb{Q}}_\delta$ is non-trivial must have that δ is δ^+-supercompact in $\mathrm{V}^{\mathbb{P}_\delta} = \mathrm{V}^{\mathbb{P}_{\kappa+1}}$ and hence $\delta \geq (\kappa^{+\S})^{\mathrm{V}^{\mathbb{P}_{\kappa+1}}}$. ⊣

5 • 12. Result. *Let κ be such that $\mathbb{P}_\kappa$ and $\mathbb{P}_{\kappa+1} \Vdash$ "κ is ρ-supercompact for $\rho < \kappa^{+\$}$". Therefore $\mathbb{P} \Vdash$ "κ's ρ-supercompactness is weakly indestructible".*

Proof. By downward absoluteness, in $V^{\mathbb{P}_\kappa}$, κ is still inaccessible. Hence we are in case (2) of Definition 5 • 9. As "small" posets and by gap forcing [5], $\delta^{+\sharp}$, $\delta^{+\$}$, and $\delta^{+\P}$ are all the same in V and $V^{\mathbb{P}_\kappa}$. Because $\rho < \kappa^{+\$}$ in $V^{\mathbb{P}_{\kappa+1}}$, the tail forcing after κ is sufficiently distributive such that κ is still ρ-supercompact in $V^{\mathbb{P}}$. Thus it suffices to show weak indestructibility for this degree of supercompactness. So suppose $\dot{\mathbb{Q}}$ is κ, ρ-sc-appropriate in $V^{\mathbb{P}}$. By distributivity of the tail forcing, $\dot{\mathbb{Q}} \in V^{\mathbb{P}_\lambda}$ for some $\lambda > \kappa$. The tail forcing $\dot{\mathbb{R}}_{(\kappa,\lambda)}$ is therefore κ-sc-appropriate by Corollary 5 • 10. This tells us that $\dot{\mathbb{Q}}_\kappa$ must be non-trivial since otherwise κ would be ρ-supercompact in $V^{\mathbb{P}_{\kappa+1}} = V^{\mathbb{P}_\kappa}$, and $\dot{\mathbb{R}}_{[\kappa,\lambda)} * \dot{\mathbb{Q}}$ would be κ, ρ-sc-appropriate in $V^{\mathbb{P}_\kappa}$. But then $\dot{\mathbb{Q}}_\kappa$ would be non-trivial.

Since $\dot{\mathbb{Q}}_\kappa$ is non-trivial, by Definition 5 • 9, we forced with some poset $\dot{\mathbb{B}} * \dot{\mathbb{C}}$ at stage κ where $\dot{\mathbb{B}}$ is κ, ρ_κ-sc-appropriate in $V^{\mathbb{P}_\kappa}$ with minimal ρ_κ so that $\rho_\kappa \leq \rho$. We now break into cases.

- Suppose $\dot{\mathbb{C}}$ is non-trivial. Thus $\rho_\kappa < \kappa^{+\$} \leq |\dot{\mathbb{B}}|$ and $\dot{\mathbb{C}}$ collapses $|\dot{\mathbb{B}}|$ to be $\rho^{+\sharp}$ (which is $< \kappa^{+\$}$) in a way that is $< \rho^{+\sharp}$-distributive. In particular, this preserves the lack of ρ_κ-supercompact measures—and so the lack of ρ-supercompact measures—in $V^{\mathbb{P}_\kappa * \dot{\mathbb{B}}}$ to $V^{\mathbb{P}_\kappa * \dot{\mathbb{B}} * \dot{\mathbb{C}}}$, contradicting that κ is ρ-supercompact in there.
- Suppose $\dot{\mathbb{C}}$ is trivial so that because $\dot{\mathbb{B}}$ is κ, ρ_κ-sc-appropriate, κ is not ρ_κ-supercompact in $V^{\mathbb{P}_\kappa * \dot{\mathbb{B}}} = V^{\mathbb{P}_{\kappa+1}}$ and hence not ρ-supercompact there, a contradiction. $\dashv$

Again, this gives universal weak indestructibility for small degrees of supercompactness, giving the following when $\mathbb{P}$ is well defined, just as with Corollary 3.A • 8.

5 • 13. Corollary (Forcing UWISSc). *Suppose V $\vDash$ "there is a proper class of strongs". Therefore*

- *the preparation $\mathbb{P}$ is well defined and $\mathbb{P} \Vdash$ UWISSc.*
- *In fact, if κ is ρ-supercomapct for $\rho < (\kappa^{+\$})^{V^{\mathbb{P}_{\kappa+1}}}$ in $V^{\mathbb{P}}$ then this degree of supercompactness is weakly indestructible in $V^{\mathbb{P}}$.*

- *In particular, any cardinal κ's degree of supercompactness that is below the least measurable limit of measurables above κ is weakly indestructible, e.g. $\kappa^{+\sharp}$-supercompactness, $(\kappa^{+\sharp})^{+\sharp}$-supercompactness, etc.*

Proof. If κ is measurable in $V^{\mathbb{P}}$, then this degree of supercompactness is weakly indestructible. So suppose κ is ρ-supercompact, $\rho \geq \kappa^+$, but there is some $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ that is κ, ρ-sc-appropriate for some $\rho < (\kappa^{+\$})^{V^{\mathbb{P}_{\kappa+1}}}$, which has some rank below an inaccessible $\gamma > \rho, \text{rank}(\dot{\mathbb{Q}})$. The tail forcing $\dot{\mathbb{R}}_{[\gamma,\infty)}$ is $\leq \gamma$-distributive by Corollary 5 • 10 and hence $\dot{\mathbb{Q}} \in V^{\mathbb{P}_\gamma}$ and $V^{\mathbb{P}_\gamma} \mid \gamma = V^{\mathbb{P}} \mid \gamma$. So by Corollary 5 • 8, $\dot{\mathbb{Q}}$ is κ, ρ-appropriate in $V^{\mathbb{P}_\gamma}$. The tail forcing $\dot{\mathbb{R}}_{(\kappa,\gamma)}$ is sufficiently distributive by Lemma 5 • 11 to show κ must have been ρ-supercompact in $V^{\mathbb{P}_{\kappa+1}}$ and so $\dot{\mathbb{R}}_{(\kappa,\lambda)} * \dot{\mathbb{Q}}$ is κ, ρ-sc-appropriate there, contradicting Result 5 • 12. ⊣

Getting a supercompact in the end can be accomplished with a supercompact that reflects supercompacts.

5 • 14. Definition. *For $\kappa, \lambda \in \text{Ord}$, and A a set of ordinals,*

- *κ is* λ-supercompact reflecting A *iff there is an elementary $j : V \to M$ generated from a measure over $[\lambda]^{<\kappa}$ such that $j(\kappa) > \lambda$, $V \models$ "${}^{\lambda}M \subseteq M$", and $j(A) \cap \lambda = A \cap \lambda$.*
- *κ is* $< \lambda$-supercompact reflecting A *iff κ is ξ-supercompact reflecting A for each $\xi < \lambda$.*
- *κ is* supercompact reflecting A *iff κ is λ-supercompact reflecting A for every $\lambda \geq \kappa$.*

We write "supercompact reflecting supercompacts" for A being the set of supercompacts.

The existence of such cardinals is obviously consistent relative to a Woodin-for-supercompactness cardinal, similar to Lemma 2 • 2. Woodin-for-supercompactness cardinals are the same as Vopěnka cardinals, and have a strictly smaller consistency strength than a high-jump cardinal [11]. An interesting side-note about supercompacts reflecting supercompacts is that such cardinals also reflect strong cardinals.

5 • 15. Result. *Suppose κ is supercompact reflecting supercompacts such that $\kappa^{+\mathrm{SC}}$ exists. Therefore κ is srs.*

Proof. Let $j : \mathrm{V} \to \mathrm{M}$ be a λ^+-supercompact reflecting supercompact embedding where $\lambda > \kappa$ is supercompact. Note that λ is therefore supercompact in M and so we only need to show any V-strong $\delta < \lambda$ is $< \lambda$-strong in M since λ itself is also strong. Note by the closure condition of M, strong extenders of length $< \lambda$ will be in M as will the relevant functions witnessing that they have the same strength as in V. Hence $\delta < \lambda$ is strong in V iff it is strong in M.

j is given by a measure U on $[\lambda^+]^{<\kappa}$. So project down to U^- on $[\lambda]^{<\kappa}$ and consider that the factor embedding $k : \mathrm{Ult}(\mathrm{V}, U^-) \to \mathrm{M}$ has critical point $\mathrm{cp}(k) \geq \lambda$ so that again $\delta < \lambda$ is strong in V iff $\delta = k(\delta)$ is strong in M, $\mathrm{Ult}(\mathrm{V}, U^-)$. Hence $j^- : \mathrm{V} \to \mathrm{Ult}(\mathrm{V}, U^-)$ is λ-strong reflecting strongs. ⊣

Just as with Lemma 3.B • 1, this reflection allows us to understand the modified preparation better, which will allow us to lift more as in Lemma 3.B • 2.

5 • 16. Lemma. *Let λ be a limit of supercompacts. Let $j : \mathrm{V} \to \mathrm{M}$ be λ-supercompact reflecting supercompacts. Therefore $\mathbb{P}^{\mathrm{V}}_\lambda = j(\mathbb{P})_\lambda = \mathbb{P}^{\mathrm{M}}_\lambda$.*

Proof. Let $\kappa = \mathrm{cp}(j)$. As with Lemma 3.B • 1, it suffices to show $\dot{\mathbb{Q}}^{\mathrm{V}}_\delta = \dot{\mathbb{Q}}^{\mathrm{M}}_\delta$ for all cardinals $\delta < \lambda$. Since λ is a limit of strongs, the same reasoning as in Result 5 • 15 applies to show that if $\delta < \lambda$, then $(\delta^{+¶})^{\mathrm{V}} = (\delta^{+¶})^{\mathrm{M}}$ and so we can unambiguously write $\delta^{+¶}$ in such cases. Note also if δ is $< \lambda_0$-supercompact in V, then either $\lambda_0 = \delta^{+¶} < \lambda$ implies δ is supercompact in both since the models agree on strongs below λ, or else $\lambda_0 < \delta^{+¶} < \lambda$ and so the lack of λ_0-supercompact measures in $\mathrm{V} \mid \lambda$ matches with $\mathrm{M} \mid \lambda$. Note also that all (names for) collapses we consider will exist in $\mathrm{V} \mid \delta^{+\$} = \mathrm{M} \mid \delta^{+\$}$ and thus have the same interpretation in both models.

In case (2), by Corollary 3 • 2, the existence of $\rho < \lambda_0 \leq \delta^{+¶} < \lambda$ and a $< \delta$-strategically closed, $\leq \delta$-distributive $\dot{\mathbb{B}} \in \mathrm{V} \mid \delta^{+¶}$ such that δ isn't ρ-supercompact after forcing with $\mathbb{P}_\delta * \dot{\mathbb{B}}$ can be calculated in $\mathrm{V} \mid \lambda = \mathrm{M} \mid \lambda$. Hence the two share the same such posets, and moreover, the minimal ρ witnessing this is the same for both. The calculation of $\delta^{+\$}$ in both will be below $\delta^{+\mathrm{SC}}$ and easily the same in both models. The collapse (also being below $\delta^{+\mathrm{SC}}$) will also be the same, meaning $\dot{\mathbb{Q}}_\delta$ is the same in both. ⊣

Lifting the embedding requires another piece of background in forcing [4].

5 • 17. Lemma.

- *Let* $\mathrm{V} \models$ "${}^{\lambda}\mathrm{M} \subseteq \mathrm{M}$" *for* $\mathrm{M} \subseteq \mathrm{V}$ *an inner model and* λ *regular.*
- *Let* $\mathbb{P} \in \mathrm{V} \cap \mathrm{M}$ *be* λ*-cc in* V*.*
- *Let* G *be* $\mathbb{P}$*-generic over* V*.*

Therefore $M[G]$ *is closed under* λ*-sequences of* $\mathrm{V}[G]$*.*

5 • 18. Result. *Assume* GCH*. Let* κ *be* λ*-supercompact reflecting supercompacts where* λ *is an inaccessible limit of supercompacts. Therefore the modified preparation* $\mathbb{P}_\lambda \Vdash$ "κ *is* λ*-supercompact".*

Proof. Let $j : \mathrm{V} \to \mathrm{M}$ witness that κ is λ-supercompact reflecting supercompacts. We can factor by Lemma 3.B • 1

$$j(\mathbb{P}_\lambda) = \mathbb{P}^{\mathrm{M}}_{j(\lambda)} \cong (\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{R}}_{(\kappa,\lambda)})^{\mathrm{M}} * \dot{\mathbb{R}}^{\mathrm{M}}_{[\lambda,j(\lambda))}$$
$$\cong (\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{R}}_{(\kappa,\lambda)})^{\mathrm{V}} * \dot{\mathbb{R}}^{\mathrm{M}}_{[\lambda,j(\lambda))} \cong \mathbb{P}^{\mathrm{V}}_\lambda * \dot{\mathbb{R}}^{\mathrm{M}}_{[\lambda,j(\lambda))}.$$

Without loss of generality, λ isn't λ^+-supercompact in M so that in $\mathrm{M}^{\mathbb{P}_\lambda}$, λ is at most measurable with therefore indestructible degrees of supercompactness and its original degree is already small: $\dot{\mathbb{Q}}^{\mathrm{M}}_\lambda = \dot{\mathbb{1}}$. Thus $\dot{\mathbb{R}}^{\mathrm{M}}_{[\lambda,j(\kappa))}$ is actually (forced to be) $\leq \lambda$-directed closed in M and (since ${}^{\lambda}\mathrm{M} \cap \mathrm{V} \subseteq \mathrm{M}$) also in V. Since λ is a limit of supercompacts, we can regard $\mathbb{P}_\lambda \subseteq \mathrm{V} \mid \lambda$.

Now let $G = G_0 * G_1$ be $\mathbb{P}_\kappa * \dot{\mathbb{R}}_{[\kappa,\lambda)}$-generic over V. As an ultrapower, each dense subset of $\mathbb{R}^{\mathrm{M}[G]}_{[\lambda,j(\lambda))} = (\dot{\mathbb{R}}^{\mathrm{M}}_{(\lambda,j(\lambda)]})_G$ can be regarded as given by a function $f : [\lambda]^{<\kappa} \to \mathbb{P}_\lambda$ in $\mathrm{V}[G]$. We know $|\mathbb{P}_\lambda| \leq \lambda$ so there are at most $\lambda^{\lambda^{<\kappa}} = 2^\lambda = \lambda^+$-many such functions and thus in V, M contains at most λ^+-many dense subsets of $\mathbb{R}^{\mathrm{M}[G]}_{[\lambda,j(\lambda))}$ in V.

By standard techniques, the $\leq \lambda$-directed closure and λ^+-many dense sets allows us to get a $\mathbb{R}_{[\lambda,j(\lambda))}$-generic H over $\mathrm{M}[G]$ such that j"$G \subseteq H$. More precisely, note that $|G| = \lambda$, so we can enumerate j"$G = \{p_\alpha : \alpha < \lambda\}$ which is a directed set and by the $\leq \lambda$-directed closure in $\mathrm{V}[G]$, get a $p^* \in \mathbb{R}^{\mathrm{M}[G]}_{[\lambda,j(\lambda))}$ beneath all of these. Then we enumerate in $\mathrm{V}[G]$

the dense open sets of $\mathrm{M}[G]$ by $\{D_\alpha : \alpha < \lambda^+\}$. We then form a generic by iteratively considering $p_0 \leqslant p^*$ in D_0 and $p_\alpha \in D_\alpha$ below each previous p_β, $\beta < \alpha$, which exists by $< \lambda^+$-directed closure. The resulting $\{p_\alpha : \alpha < \lambda^+\}$ generates a $\mathbb{R}^{\mathrm{M}[G]}_{[\lambda,j(\lambda))}$-generic filter H over $\mathrm{M}[G]$. It's not hard to see that because $p^* \in H$, $j``G \subseteq H$.

This allows us to lift $j : \mathrm{V} \to \mathrm{M}$ to $j^+ : \mathrm{V}[G] \to \mathrm{M}[G * H]$. To see that $\mathrm{M}[G * H]$ is still closed under λ-sequences, witnessing the λ-supercompactness of κ in $\mathrm{V}[G]$, use Lemma 5 • 17. We know that $\mathbb{P}_\lambda$ is λ-cc and thus $\mathrm{M}[G]$ is closed under λ-sequences. $\mathbb{R}^{\mathrm{M}[G]}_{[\lambda,j(\lambda))}$, being $\leq \lambda$-distributive closed, adds no λ-length sequences and so $\mathrm{M}[G * H]$ is also closed under these sequences: $\mathrm{V}[G] \vDash$ "${}^\lambda\mathrm{M}[G * H] \subseteq \mathrm{M}[G * H]$" as desired. ⊣

Hence while full indestructibility for all degrees of supercompactness is incompatible with multiple supercompacts, weak indestructibility is compatible with a proper class of supercompacts.

5 • 19. Corollary. *Assume there is a proper class of supercompacts reflecting supercompacts and* GCH *holds. Therefore the modified preparation* $\mathbb{P} \Vdash$ UWISSc + *"there is a proper class of supercompacts".*

Proof. For the sake of space, write *scrsc* for "supercompact reflecting supercompacts". Note that any scrsc is an inaccessible limit of supercompacts, so having a proper class of these allows us to assume any scrsc is λ-scrsc for unboundedly many inaccessible limits of supercompacts λ. Using the modified preparation $\mathbb{P}$ from Definition 5 • 9, it follows that any scrsc cardinal in V is λ-supercompact in $\mathrm{V}^{\mathbb{P}_\lambda}$ by Result 5 • 18. Since the tail forcing after this point adds no λ-length sequences, the cardinal remains λ-supercompact in $\mathrm{V}^{\mathbb{P}}$ and therefore supercompact there. Thus $\mathrm{V}^{\mathbb{P}}$ has a proper class of supercompacts, and Corollary 5 • 13 tells us UWISSc holds there too. ⊣

This can be generalized to smaller amounts of supercompacts in a similar way to Theorem 4 • 1, although (using this method) one must assume a proper class of inaccessible limits of supercompact cardinals to make use of Result 5 • 18. One can also modify the arguments from Theorem 4.A • 1, Result 4.A • 4, and Lemma 4.A • 10 about Woodin cardinals to get analogous results for cardinals that are Woodin for supercompactness.

One open problem that remains is to ensure universal weak indestructibility for every κ's $< \kappa^{+\sharp\sharp}$-strength or $< \kappa^{+\$}$-supercompactness in the presence of multiple strong or supercompact cardinals. The results here are limited to weak indestructibility for any κ's $\kappa^{+\sharp}$-strength or supercompactness. Indeed, we get much more than this, but generally speaking, the methods in this document don't ensure any κ's $< \kappa^{+\sharp\sharp}$-strength or $< \kappa^{+\$}$-supercompactness is weakly indestructible, even if true for certain κ. The main obstacle to this is control over the next stage of forcing: the next stage after κ is $\kappa^{+\sharp\sharp}$ in Definition 3.A • 3 or $\kappa^{+\$}$ in Definition 5 • 9, and any non-trivial forcing there could reduce $\kappa^{+\sharp\sharp}$ or $\kappa^{+\$}$ to be merely measurable. This not only bumps up the calculation of $\kappa^{+\sharp\sharp}$ or $\kappa^{+\$}$, but also has the potential to resurrect degrees of strength of κ. This is also an issue if one wants to generalize weak indestructibility to full indestructibility: the calculation of $\kappa^{+\sharp}$ might change and also resurrect degrees of κ below the new $\kappa^{+\sharp}$. This issue is of course bypassed by cutting off the universe at, say, $\kappa^{+\sharp}$ whenever κ has the desired form of indestructibility up to $\kappa^{+\sharp}$. But obviously, this isn't desirable when we want lots of larger cardinals above.

Some of the open problems stated above are collected here for convenience.

5 • 20. Questions. *All of the questions below about strength can also be rephrased in terms of supercompactness.*

1. *Is it possible to have every κ's $< \kappa^{+\sharp\sharp}$-strength weakly indestructible in the presence of multiple strong cardinals?*
2. *To what extent can the reflection properties in the embeddings of a measurable cardinal be (weakly) indestructible?*
3. *Is it possible to have a strong reflecting strongs cardinal (with a strong above it) such that this strength and reflection of (ground model) strongs is weakly indestructible?*
4. *To what extent can we control the resurrection of degrees after destroying degrees of strength in a preparation like Definition 3.A • 3?*
5. *If a poset is $\leq \kappa$-strategically closed, is it $< \kappa^{+}$-strategically closed?*
6. *Is Corollary 5 • 19 an equiconsistency result like for strength in Theorem 1 • 2?*

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