GOST SEMINAR NOTES

THE BASICS OF SKOLEM HULLS, AND TRANSITIVE/MOSTOWSKI COLLAPSES

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Section 1. *Transitivity*

§ 1 A. Well-foundedness

Recall that a relation R is called *well-founded* iff there is no infinite R-descending sequence. This is a property that looks like it could be written easily in first order logic:

$$\text{“}\neg\exists x_0\ \exists x_1\ \cdots\ \left(\bigwedge_{i\in\mathbb{N}} x_{i+1}\ R\ x_i\right)\text{”.}$$

The only issue is that this isn't a formula since it is infinitely long. Instead, we can use the concepts of set theory to say that any infinite sequence S, like $\langle x_n : n \in \mathbb{N}\rangle$, is not an infinite R-descending sequence:

$$\text{“}\neg\forall x \in S\ \exists y \in S\ (y\ R\ x)\text{”.}$$

Indeed if there were such a collection, we'd have that R isn't well-founded. As a result, the two are equivalent for real world sets V.

1 A • 1. Definition

A relation $R \subseteq X^2$ is well-founded iff every non-empty subset of X has an R-least element.

As indicated above, there are a few equivalent ways to state this.

1 A • 2. Result

(DC) Let $R \subseteq X^2$ be a relation. Therefore the following are equivalent.

1. R is well-founded as in Definition 1 A • 1.
2. There are no infinite, R-decreasing sequences.
3. There is an R-minimal element, and for each $x \in \text{dom}(R)$, there is an R-minimal y with $x\ R\ y$.

The usefulness of these relations comes from the ordinals, which are just the canonical well-founded, linear orders—i.e. the well-orders. Most mathematicians only encounter the word “well-order” in the context of the natural numbers, which are already of great importance. The general idea is that the ordinals can be used to talk about well-founded relations, since any chain will be order isomorphic to an ordinal (it's *order type*). This allows us to use the mechanisms of transfinite induction and consequently arrive at the Mowstowski collapse. The importance of the Mostowski or transitive collapse[i] is that it allows us to identify well-founded orders (satisfying minor conditions) with transitive sets.

[i] There is no difference between the Mostowski collapse and the transitive collapse, they are just different names for the same concept: a transitive set with membership isomorphic to the relation

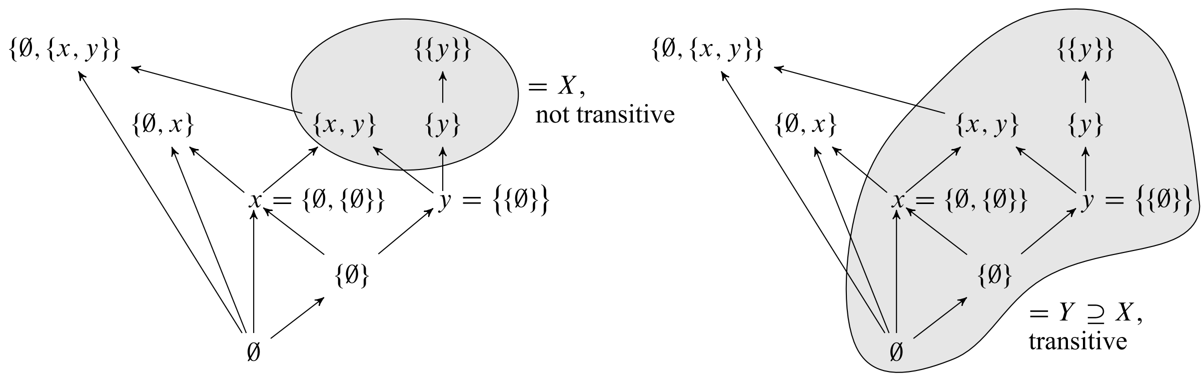
§ 1 B. Transitive sets

In some sense, "transitive set" is a misnomer, since it is not the set that is transitive, but the membership relation *restricted to membership in x*.

1 B • 1. Definition

A set x is *transitive* iff $\forall a, b\ (a \in b \in x \to a \in x)$.

Equivalently, x is transitive iff $b \in x$ implies $b \subseteq x$[ii]. In some sense, this means that transitive xs not only contain various a with $a \in b \in x$, but that we go *all the way down* to the basis of the universe: $\emptyset$.



1 B • 2. Figure: The membership relation compared to a transitive set

In another sense, x being transitive means that the structure $\mathbf{X} = \langle X, \in \rangle$ is a *submodel* of $\mathbf{V}$, and a good one at that: for any $y \in X$, we have

$$\{z : \mathbf{X} \vDash \text{“}z \in y\text{”}\} = \{z : z \in y\} = y.$$

As a result of this, we get some nice model-theoretic results about *absoluteness*.

Finding examples of transitive sets and examples of non-transitive sets is easy. In particular,

1. $\emptyset$ is transitive. $\{\emptyset\}$ is transitive.
2. If x is transitive, then $x \cup \{x\}$ is transitive (any element $b \in x \cup \{x\}$ is still a subset since $b \subseteq x \subseteq x \cup \{x\}$).
3. Writing $0 = \emptyset$, $1 = \{0\}$, and $2 = \{0, 1\}$, then from the above, 0, 1, 2, and $\{0, 1, 2\}$ are transitive, but $\{1\}$, $\{0, 2\}$, and $\{2\}$ are not.
4. If x is transitive and $y \subseteq x$, then $x \cup \{y\}$ is transitive.

The axiom of foundation can be motivated though the iterative conception of what a collection is: namely, collections are built up of smaller things that have come before in a certain sense. This will turn out to be equivalent to the axiom. Explicitly, foundation merely states that membership itself is well-founded.

1 B • 3. Corollary

Assume the axiom of foundation. Therefore:

1. We never have $x \in x$.
2. In fact, there are no finite loops $x_0 \in x_1 \in \cdots \in x_n \in x_0$.
3. If $x \neq \emptyset$ is transitive, $\emptyset \in x$ is the $\in$-minimal element of x.
4. x is transitive iff $x \cup \{x\}$ is transitive.
5. Every x is contained in a transitive $y \supseteq x$.

Proof .:.

1. Suppose $x \in x$. By foundation, there is a $\in$-minimal element of $\{x\}$, which must be x. So any $y \in x$ has $y \notin \{x\}$ by minimality. But $x \in x$ has $x \in \{x\}$, so we have a contradiction.

[ii] Of course, we cannot have a set where $\forall b\ (b \subseteq x \to b \in x)$ by the same reasoning as in Russell's paradox.

2. Consider the set $\{x_0, \cdots, x_n\}$, which exists by finite applications of union and pairing. This has no $\in$-minimal element, since any x_i has $x_{i-1} \in x_i$ for $i > 0$ or else $x_n \in x_i$ for $i = 0$.
3. If x is transitive, then every element $y \in x$ is a subset of x. Hence if $y \neq \emptyset$ is $\in$-minimal, then there is some $z \in y \in x$, which yields $z \in x$ and $z \in y$, contradicting the minimality of y. Hence any $\in$-minimal element must be $\emptyset$.
4. We know that x being transitive implies $x \cup \{x\}$ is transitive. For the other direction, if $x \cup \{x\}$ is transitive, then any $a \in b \in x \cup \{x\}$ must have either $a \in x$ or $a = x$. But a cannot equal x without us having a finite loop: either $x \in b \in x$ or $x \in b = x$. Hence $a \in b \in x \cup \{x\}$ requires $a \in x$. This clearly implies that x is transitive since $a \in b \in x \subseteq x \cup \{x\}$ implies $a \in x$.
5. Take $x_0 = x$ and define $x_{n+1} = \bigcup x_n$. With $x_\omega = \bigcup_{n\in\omega} x_n$, we arrive at a transitive set with $x \subseteq x_\omega$. $\dashv$

The proof of this last fact suggests an idea of how to talk about well-founded sets via *rank*: how many levels down we must go to reach the bottom.[iii] If $\emptyset$ is the base of the universe, then $\{\emptyset\}$ is just above it, and so has a rank one higher. Similarly, collections built from these like $\{\emptyset, \{\emptyset\}\}$ and $\{\{\emptyset\}\}$ are a rank higher than that.

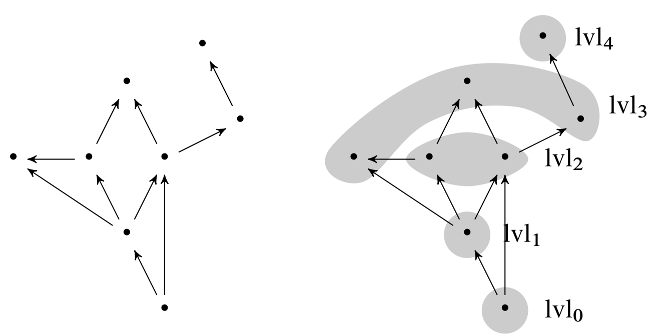
§ 1 C. The transitive collapse

The introduction of rank allows us to think of any well-founded, extensional graph just as a transitive set. We will show the following result, which holds even in the absence of foundation. In some sense, the result says that all well-founded models of a little set theory in **V** are isomorphic to transitive sets.[iv] This (unique) transitive set is referred to as the *transitive collapse* or the *Mostowski collapse*. There is no difference except in name.

1 C • 1. Theorem (The Mostowski Collapse)

Let $\mathbf{A} = \langle A, <_A\rangle$ (in V) be well-founded such that $\mathbf{A}$ satisfies the axiom of extensionality. Therefore $\mathbf{A} \cong \langle T, \in\rangle$ for a unique transitive set T.

Although we can prove the theorem outright at this point, to get a better perspective on what is going on with the proof, we will go through the idea of rank. Although all well-orders are isomorphic to ordinals, well-founded, extensional structures are not in general. But they can still make use of ordinals according to *chains*, which are then well-ordered. Really, this just means indexing the *levels* of the structure like with a tree. This is the idea of rank: we give an inductive definition. Note that this doesn't require that the structure be extensional[v]



1 C • 2. Figure: An example of a well-founded structure and its levels

1 C • 3. Lemma

Let $\mathbf{A} = \langle A, <_A\rangle$ be well-founded. Therefore there is a unique function $f : A \to \mathrm{Ord}$ such that

$$f(a) = \sup\{f(b) + 1 : b <_A a\}.$$

[iii] Or really, how many levels *up* from the bottom we need to reach the element.

[iv] So the axiom of foundation in some sense takes the converse to be true: all transitive sets are well-founded.

[v] In the sense that it satisfies the axiom of extensionality: "$\forall x\ \forall y\ (x = y \leftrightarrow \forall z\ (z < x \leftrightarrow z < y))$", equivalently, $x = y$ iff $\mathrm{pred}_<(x) = \mathrm{pred}_<(y)$.

Proof .:.

Uniqueness is immediate: for f, g two such functions and a $<_A$-minimal where $f(a) \neq g(a)$, we have that $f(a) = \sup\{f(b) + 1 : b <_A a\}$. By minimality of a, $f(b) = g(b)$ for $b <_A a$ so this supremum is $f(a) = \sup\{g(b) + 1 : b <_A a\} = g(a)$, a contradiction.

We construct such an f by transfinite recursion. Firstly, as **A** is well-founded, define by transfinite recursion

$$\mathrm{lvl}_0 = \emptyset$$
$$\mathrm{lvl}_{\alpha+1} = \left\{a \in A : a \text{ is } <_A\text{-minimal in } A \setminus \bigcup_{\beta \leq \alpha} \mathrm{lvl}_\beta\right\}$$
$$\mathrm{lvl}_\gamma = \emptyset, \text{ for } \gamma \text{ a limit.}$$

If $\mathrm{lvl}_{\alpha+1}$ is ever empty, then we stop, and so $\mathrm{lvl}_\alpha = A$. Then we define $f : A \to \mathrm{Ord}$ by taking $f(x)$ to be the least (and only) α such that $x \in \mathrm{lvl}_{\alpha+1}$. Now assuming A is a set, this process stops at some point so that $f \in \mathrm{V}$.

Note that $x, y \in \mathrm{lvl}_\alpha$ implies x and y are $<_A$-incomparable: $x \not<_A y$ and $y \not<_A x$ (otherwise, they wouldn't be minimal). Hence $f(x) = f(y)$ implies x and y are $<_A$-incomparable.

Claim 1

This f works, meaning $f(a) = \sup\{f(b) + 1 : b <_A a\}$ for each $a \in A$.

Proof .:.

First we show that $f(a) \geq \sup\{f(b) + 1 : b <_A a\}$. To see this, the above shows that if $b <_A a$ then $f(b) \neq f(a)$ and in fact $f(b) < f(a)$ ($f(a) < f(b)$ with $b <_A a$ contradicts that a is minimal in $A \setminus \bigcup_{\beta < f(a)} \mathrm{lvl}_\beta$). Hence $f(a) \geq f(b) + 1$ for each $b <_A a$ and therefore $f(a) \geq \sup\{f(b) + 1 : b <_A a\}$.

Now if $f(a) > \beta = \sup\{f(b) + 1 : b <_A a\}$, then by the definition of the lvl_αs, a wasn't minimal in $A \setminus \bigcup_{\gamma < \beta+1} \mathrm{lvl}_\gamma$, meaning that there is some $b \in A \setminus \bigcup_{\gamma < \beta+1} \mathrm{lvl}_\gamma$ with $b <_A a$. Taking a $<_A$-minimal such b yields that $f(b) = \beta + 1$, contradicting the definition of β. ⊣

The point of having a rank function is to proceed by induction on the levels. Indeed, the proof above just defines the function f by induction on the levels of A. So if we can prove something for the elements inductively by level, then we can prove it for the whole set. So we have the following definition. By uniqueness, we are justified in using "the" rank function, and defining the following as aspects of the structure alone, independent of any choice of rank function.

1 C • 4. Definition

For well-founded $\mathbf{A} = \langle A, <_A \rangle$;

- The *rank function* on **A** is the function $\mathrm{rank} : A \to \mathrm{Ord}$ such that $\mathrm{rank}(a) = \sup\{\mathrm{rank}(b) + 1 : b <_A a\}$.
- the *levels* of **A** are the sets $\mathrm{lvl}_\alpha(\mathbf{A}) = \{a \in A : \mathrm{rank}(a) = \alpha\}$ for all $\alpha \in \mathrm{Ord}$.
- the *height* or *length* of **A** is $\mathrm{ht}(\mathbf{A}) = \sup\{\mathrm{rank}(a) + 1 : a \in A\}$.

A structure $\mathbf{A} = \langle A, <_A \rangle$ is *extensional* iff it satisfies the axiom of extensionality:

$$\{z \in A : z \; R \; x\} = \{z \in A : z \; R \; y\} \quad \text{implies} \quad x = y.$$

We include the "+1" in the definition of height (and rank) to ensure that every element has a smaller rank than the height (or rank of the element we're considering). So the empty relation has height 0, and the set with one element has height 1 while the single element has rank 0. Note that for **A** a set, the height of **A** is an ordinal, and not just Ord itself. Note some other immediate facts.

1 C • 5. Result

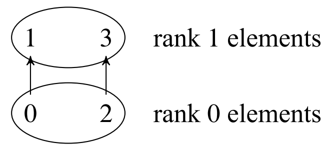
Let $\mathbf{A} = \langle A, <_A \rangle$ be well-founded with rank function, rank. Therefore, the following hold.

1. If $a <_A b$, then $\mathrm{rank}(a) < \mathrm{rank}(b)$.
2. If $a, b \in A$ are *comparable*—i.e. $a <_A b$ or $b <_A a$—then $\mathrm{rank}(a) < \mathrm{rank}(b)$ iff $a <_A b$.

Proof .:.

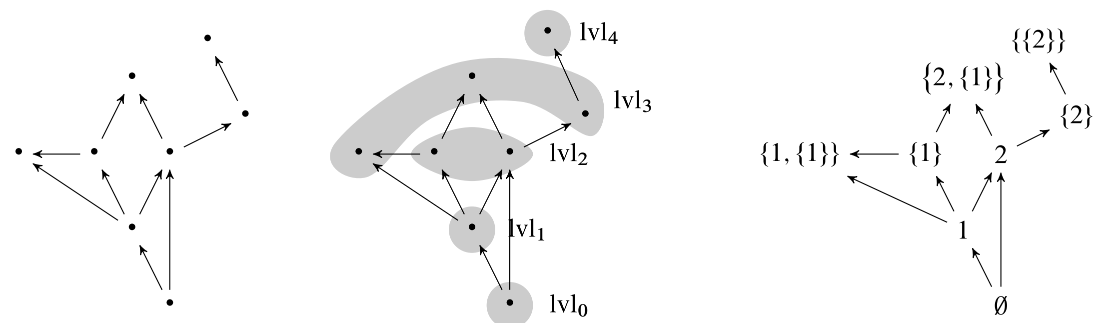
1. Clearly $a <_A b$ implies $\mathrm{rank}(b) > \sup\{\mathrm{rank}(x) : x <_A b\} \geq \mathrm{rank}(a)$ by definition of rank.
2. If a and b are comparable, then either $a <_A b$ (in which case $\mathrm{rank}(a) < \mathrm{rank}(b)$ implies $a <_A b$ by (1)), or $b <_A a$ (in which case $\mathrm{rank}(a) < \mathrm{rank}(b)$ implies $b <_A a$ vacuously by (1)).

Note that we cannot ensure in general that $\text{rank}(a) < \text{rank}(b)$ implies $a <_A b$, by the figure below:



1 C • 6. Figure: A non-extensional, well-founded relation

Taking $<_A = \{\langle 0, 1\rangle, \langle 2, 3\rangle\}$ yields a well-founded relation with $\text{rank}(2) = 0$, $\text{rank}(1) = 1$, but $2 \not<_A 1$. But this concept of rank is what allows us to *collapse* a well-founded, extensional set to a transitive set. We cannot do with with the above example, because it does not satisfy extensionality. It is extensionality that ensures we can *uniquely* describe elements by talking about their predecessors.



1 C • 7. Figure: An example of a well-founded, extensional structure and its transitive collapse

Proof of The Mostowski Collapse (1 C • 1) .:.

We would like to just state the isomorphism outright: define $f(a) = \{f(b) : b <_A a\}$. To show that this is actually a definition that makes sense requires us to proceed by recursion on the levels of $\mathbf{A}$. So this is how we proceed. Note that we have a starting point: as $\mathbf{A}$ satisfies extensionality, there is only one $<_A$-minimal element, $a_\emptyset$. This is because any $<_A$-minimal element a has $\text{pred}_{<_A}(a) = \emptyset = \text{pred}_{<_A}(a_\emptyset)$.

Proceed by recursion on the levels of $\mathbf{A}$ to define an isomorphism. In particular, we will define f_α on the elements of rank $\leq \alpha$. Since there is only one $<_A$-minimal element $a_\emptyset$, define $f_0(a_\emptyset) = \emptyset$. At limit stage γ define $f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha$. At successor stage $\alpha + 1$, define

$$f_{\alpha+1}(x) = \begin{cases} f_\alpha(x) & \text{if } x \in \text{dom}(f_\alpha) \\ \{f_\alpha(y) : y <_A x\} & \text{if } x \in \text{lvl}_{\alpha+1}(\mathbf{A}). \end{cases}$$

Note that this process is well-defined: inductively, $\text{dom}(f_\alpha) = \bigcup_{\beta\leq\alpha} \text{lvl}_\beta(\mathbf{A})$, and if $y <_A x \in \text{lvl}_{\alpha+1}(\mathbf{A})$, then $\text{rank}(y) < \text{rank}(x) = \alpha$ so that y is in the domain of f_α. Taking $f = \bigcup_{\alpha<\text{ht}(\mathbf{A})} f_\alpha$, it follows that $f(x) = \{f(y) : y <_A x\}$ for all $x \in A$.

Note that $T = \text{im } f$ is transitive: if $x \in f(a) = \{f(b) : b <_A a\} \in T$, then $x = f(b)$ for some $b <_A a$, and thus $x = f(b) \in T$. So it suffices to show that f is an isomorphism between $\mathbf{A}$ and $\langle T, \in\rangle$.

Claim 1

f is an isomorphism between $\langle A, <_A\rangle$ and $\langle T, \in\rangle$.

Proof .:.

Surjectivity of $f : A \to T$ follows just by definition of $T = \operatorname{im} f$. For injectivity, suppose not and let $a \in A$ be $<_A$-minimal where $f(a) = f(b)$ for some b. Let $f(x) \in f(b)$ for some $x <_A b$ so that $f(x) \in f(a)$ and thus $f(x) = f(y)$ for some $y <_A a$. By minimality of a, $y = x$ and therefore $x <_A a$. The same idea shows that if $x <_A a$ then $x <_A b$, and thus $a = b$ by extensionality.

Now if $a <_A b$ then $f(a) \in \{f(x) : x <_A b\} = f(b)$. Similarly, suppose $f(a) \in f(b)$. Thus $f(a) = f(x)$ for some $x <_A b$. By injectivity, $a = x$ and thus $a <_A b$. ⊣

To see that T is unique, suppose $g : A \to D$ is an isomorphism with D transitive. Let $a \in A$ be of least rank such that $f(a) \neq g(a)$. Note that by extensionality and the inductive hypothesis, $f(a) = \{f(x) : x <_A a\} = \{g(x) : x <_A a\} = g(a)$, a contradiction. ⊣

So again, The Mostowski Collapse (1 C • 1) should highlight the importance of transitive sets, as they allow us to consider any sort of well-founded, extensional relation.

1 C • 8. Definition

Let $\mathbf{A} = \langle A, <_A \rangle$ be well-founded and extensional. The *mostowski collapsing map* of $\mathbf{A}$ is an isomorphism $\pi : A \to T \subseteq \mathrm{V}$ defined by recursion on rank: for every $a \in A$, $\pi(a) = \{\pi(b) : b <_A a\}$. The *transitive collapse* of $\mathbf{A}$ is then $\langle \operatorname{im} \pi, \in \rangle$.

The proof of The Mostowski Collapse (1 C • 1) shows that π is well-defined, unique, and is in fact an isomorphism.

Note that there is a slightly more general version of The Mostowski Collapse (1 C • 1): we don't require that $\mathbf{A} \in \mathrm{V}$, but instead that $\operatorname{pred}_{<_A}(a) \in \mathrm{V}$ for each $a \in A$. This property is sometimes called being *set-like*. For example, $\mathbf{V}$ satisfies this, as $\operatorname{pred}_{\in}(x) = x \in \mathrm{V}$ for each $x \in \mathrm{V}$. The proof remains the same, as we never needed $\operatorname{ht}(\mathbf{A})$ to be an ordinal: it could be Ord itself, as with V. The result is that the collapsing map is a *class* rather than a set.

The point of this generalization is just in case we have a well-founded, partially ordered structure that is not a set. Then we can collapse it down to a transitive class (not necessarily a set) under membership. For now, we will have no use of this generality, but it will be incredibly important later, as we will collapse down various collections (especially ultrapowers) into "inner models".

To be slightly more precise than the previous paragraph, for A and R classes, if $\operatorname{pred}_{\mathrm{R}}(x)$ is a set for each $x \in \mathrm{A}$, then we can define the mostowski collapse as in Definition 1 C • 8 as a class, and so yield the image T as a transitive class, which is still isomorphic under membership to A under R.

Section 2. Skolem Hulls

The first idea we will consider is the idea of a model *generated* by a set and theory. There are two or three versions of this theorem. The first two versions are certainly useful for logic, and have the most applications outside of logic, especially algebra, in detailing what is first-order expressible. The third version is the most useful for our purposes, and implies the other two. First we introduce a definition.

2 • 1. Definition

Let A, and B be $\mathcal{L}$-models.
A is an elementary submodel of B, written $\mathsf{A} \preccurlyeq \mathsf{B}$, iff $A \subseteq B$, and for every $\mathcal{L}$-formula φ with parameters in $A \cap B = A$, we have $\mathsf{A} \vDash \varphi$ iff $\mathsf{B} \vDash \varphi$.

It should be clear that being an elementary submodel implies being a submodel just by looking at the atomic $\mathcal{L}$-formulas (with parameters). But being a submodel does not entail being elementary. For example, the order of the real numbers on the unit interval $\langle (0,1), \leq \rangle$ is the same as for the closed unit interval $\langle [0,1], \leq \rangle$ so that they are submodels: $\langle (0,1), \leq \rangle \subseteq \langle [0,1], \leq \rangle$. But $\langle [0,1], < \rangle \vDash$ "$\exists x \forall y (y \leq x)$" while $\langle (0,1), \leq \rangle \nvDash$ "$\exists x \forall y (y \leq x)$": $\langle [0,1], \leq \rangle$ has a maximal element whereas $\langle (0,1), \leq \rangle$ does not. In essence, being an elementary submodel is the strongest amount of agreement two models can have on first-order formulas. So note the following properties of elementary submodels: for all $\mathcal{L}$-models A, B, and C;

- $\mathsf{A} \preccurlyeq \mathsf{A}$.
- $\mathsf{A} \preccurlyeq \mathsf{B} \preccurlyeq \mathsf{A}$ iff $\mathsf{A} = \mathsf{B}$ (since $A \subseteq B \subseteq A$, and they interpret the signature the same way).
- $\mathsf{A} \preccurlyeq \mathsf{B} \preccurlyeq \mathsf{C}$ implies $\mathsf{A} \preccurlyeq \mathsf{C}$.
- $\mathsf{A} \preccurlyeq \mathsf{C}$ and $\mathsf{B} \preccurlyeq \mathsf{C}$ implies $\mathsf{A} \preccurlyeq \mathsf{B} \leftrightarrow A \subseteq B$.
- (the Tarski–Vaught theorem) For γ a limit ordinal and $\mathsf{A}_\alpha \preccurlyeq \mathsf{A}_\beta$ for $\alpha < \beta < \gamma$; $\mathsf{A}_\alpha \preccurlyeq \bigcup_{\alpha<\gamma} \mathsf{A}_\alpha$ (the direct limit) for each $\alpha < \gamma$.

§ 2 A. Versions of the Löwenheim–Skolem theorem

The next theorem, one of the versions of the Löwenheim–Skolem theorem, then tells us that we can *generate* elementary submodels using arbitrary subsets of the original model we start with.

2 A • 1. Theorem (Taking a Skolem Hull)

Let A be an infinite $\mathcal{L}$-model, and $X \subseteq A$. Therefore there's a model $\mathsf{Hull}^{\mathsf{A}}(X)$ called the *Skolem hull* of X where

1. $X \subseteq \mathrm{Hull}^{\mathsf{A}}(X) \subseteq A$;
2. $|X| \leq |\mathrm{Hull}^{\mathsf{A}}(X)| \leq |X| \cdot |\mathcal{L}| \cdot \aleph_0$;
3. $\mathsf{Hull}^{\mathsf{A}}(X) \preccurlyeq \mathsf{A}$.

To prove this result, we essentially do a careful proof of the completeness theorem of first-order logic, building up a model from X by closing under the functions of $\mathcal{L}$ and whatever witnesses existential statements need from A. So the following combinatorial result will be useful in showing that we do not add too many elements in building up the Skolem hull.

2 A • 2. Lemma

Let X be a set. Let f be a function with $X \subseteq \operatorname{dom} f$. Therefore the *closure* of X under f—meaning the $\subseteq$-least set Y with $X \subseteq Y$ and $f``Y \subseteq Y$—has size at most $|X| \cdot \aleph_0$.

Proof .:.

Write $X_0 = X$, and define $X_{n+1} = X_n \cup f``X_n$. Let $Y = \bigcup_{n\in\omega} X_n$. Note that for each $x \in Y$, $f(x) \in X_{n+1}$ where $x \in X_n$. Hence $f(x) \in Y$. Thus $Y$ is closed under $f$. Moreover, for each $n \in \omega$, $|X_{n+1}| \leq |X_n| + |X_n| = 2|X_n|$ because $|f``X_n| \leq |X_n|$. Therefore, inductively, $|X_n| \leq \aleph_0 \cdot |X|$ for each $n \in \omega$. Therefore the union Y has $|X| \leq |Y| \leq \aleph_0 \cdot \aleph_0 \cdot |X| = \aleph_0 \cdot |X|$. Regardless of whether Y is the $\subseteq$-least set containing X, any $Z \subseteq Y$ which is the real closure of X has $|Z| \leq |X| \cdot \aleph_0$. ⊣

As a result, we can close under entire sets of functions as well, and still we can bound the size of the resulting set.

2 A • 3. Corollary

Let X be a set. Let σ be a set of functions with $X \subseteq \operatorname{dom} f$ for each $f \in \sigma$. Therefore the *closure* of X under σ—meaning the $\subseteq$-least set Y with $X \subseteq y$ and $f"Y \subseteq Y$ for each $f \in \sigma$—has size at most $|X| \cdot |\sigma| \cdot \aleph_0$.

Proof .:.

As before, write $X_0 = X$, and define

$$X_{n+1} = X_n \cup \bigcup_{f \in \sigma} (\text{the closure of } X_n \text{ under } f).$$

Thus by Lemma 2 A • 2, $|X_{n+1}| \leq |X_n| + |X_n| \cdot |\sigma| \cdot \aleph_0 = |X_n| \cdot |\sigma| \cdot \aleph_0$ For each $n \in \omega$. So inductively, it follows that $|X_n| \leq |X| \cdot |\sigma|^n \cdot \aleph_0 = |X| \cdot |\sigma| \cdot \aleph_0$. Taking the union $Y = \bigcup_{n \in \omega} X_n$ yields that Y is closed under each $f \in \sigma$ as in Lemma 2 A • 2, and moreover, $|Y| \leq |X| \cdot |\sigma| \cdot \aleph_0^2$. ⊣

Therefore, when we build up the Skolem hull, we aren't adding too many elements to X. Note that in the following proof of Taking a Skolem Hull (2 A • 1), we have an elementary submodel by the idea of Skolem functions: functions which map existential statements to elements that witness them. This allows us to see that the agreement between **A** and $\mathbf{Hull}^{\mathbf{A}}(X)$ includes existential statements. The propositional connectives are practically free, and so by induction on formulas, this implies the hull is an elementary submodel. Note that the existence of such functions relies on (some form of) AC: generally the Skolem hull will not be unique unless we fix some well-ordering of the original model.

Proof of Taking a Skolem Hull (2 A • 1) .:.

For each existential $\mathcal{L}$-formula $\psi(\vec{x})$ being "$\exists v \varphi(v, \vec{x})$", add the function symbol f_ψ (with arity being the length of $\vec{x}$) to the signature. Thus we now consider the language

$$\mathcal{L}' = \mathcal{L} \cup \{f_\psi : \psi \text{ is an existential } \mathcal{L}\text{-formula}\}.$$

We interpret the functions f_ψ in the model **A** by the axiom of choice: for $\psi(\vec{x})$ being $\exists v\ \varphi(v, \vec{x})$, if $\mathbf{A} \vDash$ "$\exists v\ \varphi(v, \vec{x})$", choose $f_\psi^{\mathbf{A}}(\vec{x}) \in A$ such that $\mathbf{A} \vDash \varphi(f_\psi^{\mathbf{A}}(\vec{x}), \vec{x})$. Obviously, if $\mathbf{A} \nvDash \exists v\ \varphi(v, \vec{x})$, then we can set $f_\psi^{\mathbf{A}}(\vec{x})$ to be any particular, fixed element of A that we want (this is only done to ensure that f'_ψ is indeed a function defined over all of A). Hence we can consider the $\mathcal{L}'$-model $\mathbf{A}'$ with these new interpretations, noting that we have only added interpretations: we still have $X \subseteq A' = A$, for instance.

With this, by Corollary 2 A • 3, we can consider the closure of X under the functions of $\mathcal{L}'$, yielding $\text{Hull}^{\mathbf{A}}(X)$. This clearly has $X \subseteq \text{Hull}^{\mathbf{A}}(X) \subseteq A$, meaning (1) holds. Moreover, by Corollary 2 A • 3, $|\text{Hull}^{\mathbf{A}}(X)| \leq |X| \cdot |\mathcal{L}| \cdot \aleph_0$, meaning (2) holds.

Now we take the model $\mathbf{Hull}^{\mathbf{A}}(X)$ to have the same function and relation interpretations as **A**, but restricted to $\text{Hull}^{\mathbf{A}}(X)$. To show (3), suppose $w_0, \cdots, w_n \in \text{Hull}^{\mathbf{A}}(X)$. We proceed by induction on the $\mathcal{L}$-formula $\varphi(\vec{x})$ to show that $\mathbf{Hull}^{\mathbf{A}}(X) \vDash$ "$\varphi(\vec{w})$" iff $\mathbf{A} \vDash$ "$\varphi(\vec{w})$".

- The atomic is immediate by definition; and '$\neg$' & '$\wedge$' are immediate by the inductive hypothesis.
- For $\varphi(\vec{x})$ being $\exists v\ \psi(v, \vec{x})$, $\mathbf{A} \vDash$ "$\exists v\ \psi(v, \vec{w})$" iff $\mathbf{A} \vDash$ "$\psi(f_\varphi^{\mathbf{A}}(\vec{w}), \vec{w})$". Since $\mathbf{Hull}^{\mathbf{A}}(X)$ is closed under these Skolem functions, by the inductive hypothesis, this is equivalent to $\mathbf{Hull}^{\mathbf{A}}(X) \vDash$ "$\psi(f_\varphi^{\mathbf{A}}(\vec{w}), \vec{w})$", iff $\mathbf{Hull}^{\mathbf{A}}(X) \vDash$ "$\exists v\ \psi(v, \vec{w})$".

Hence by induction on $\mathcal{L}$-formulas, it follows that $\mathbf{Hull}^{\mathbf{A}}(X) \preccurlyeq \mathbf{A}$, and thus (1)–(3) hold. ⊣

Some immediate consequences of this are that if ZFC is consistent, then there is a *countable* model in addition to a model of size $\aleph_1$, and models of every cardinality.

§ 2 B. Common applications to set theory: hulls

Often we don't want to consider an elementary submodel directly, but instead a model which *maps* to an elementary submodel by way of an embedding.

2 B • 1. Definition

Let **A** and **B** be $\mathcal{L}$-models. For $f : A \to B$ an injective map, the structure $f``\mathbf{A}$ is the structure with universe $f``A$ and with interpretations of $\mathcal{L}$ given by f applied to the interpretations in **A**.
$f : A \to B$ is an *embedding* (**A** is *embedded* in **B**) iff $f``\mathbf{A} \subseteq \mathbf{B}$.
$f : A \to B$ is an *elementary embedding* (**A** is *elementarily embedded* in **B**) iff $f``\mathbf{A} \preccurlyeq \mathbf{B}$.

An alternative characterization of being an elementary embedding would be that for every $\mathcal{L}$-formula $\varphi(\vec{x})$ and $\vec{a}$ members of A, $\mathbf{A} \vDash$ "$\varphi(\vec{a})$" iff $\mathbf{B} \vDash$ "$\varphi(f(\vec{a}))$". This characterization is arguably a better way of thinking about it. Similarly, f is an embedding iff $\mathbf{A} \vDash$ "$R(\vec{a})$" iff $\mathbf{B} \vDash$ "$R(f(\vec{a}))$" for every relation R and $\vec{a}$ in A, and similarly for functions: $\mathbf{A} \vDash$ "$F(\vec{a}) = a_0$" iff $\mathbf{B} \vDash$ "$F(f(\vec{a})) = f(a_0)$".

For now, our main application will be with respect to Taking a Skolem Hull (2 A • 1) and elementarity. The great thing about taking Skolem hulls of transitive sets is that we end up with well-founded sets, and thus can collapse them.

2 B • 2. Result

Let $\mathbf{A} = \langle A, R\rangle$ be well-founded. Therefore any $\mathbf{B} = \langle B, R'\rangle$ embedded in **A** is also well-founded.

Proof .:.

Let $f : B \to A$ be an embedding and let $X \subseteq B$ be arbitrary. Since **A** is well-founded, $f``X \subseteq A$ has an $R$-minimal element $a \in f``X$. Thus for every $y \in f``X$, $\neg y \ R \ a$. As an embedding, $\neg(f^{-1}(y) \ R' \ f^{-1}(a))$ for each $y \in f``X$, meaning $\neg(x \ R' \ f^{-1}(a))$ for each $x \in X$. Therefore $f^{-1}(a)$ is R'-minimal. Thus **B** is also well-founded. ⊣

2 B • 3. Corollary

Let T be a transitive set and $X \subseteq T$. Therefore $\mathbf{Hull}^{\langle T,\in\rangle}(X)$ is well-founded, and is isomorphic to the transitive collapse $\mathbf{cHull}^{\langle T,\in\rangle}(X)$, which is then elementarily embedded in $\langle T, \in\rangle$. Moreover, if X is transitive, X is left uncollapsed: the collapsing map $\pi : \mathrm{Hull}^{\langle T,\in\rangle}(X) \to \mathrm{cHull}^{\langle T,\in\rangle}(X)$ has $\pi \restriction X = \mathrm{id} \restriction X$.

Proof .:.

Write **T** for $\langle T, \in\rangle$ and $\mathbf{T}'$ for $\mathbf{cHull}^{\mathbf{T}}(X)$. By Taking a Skolem Hull (2 A • 1), $\mathbf{Hull}^{\mathbf{T}}(X) \preccurlyeq \mathbf{T}$ so that the hull is well-founded. By elementarity, the hull satisfies the axiom of extensionality. By The Mostowski Collapse (1 C • 1), the hull is isomorphic to the transitive $\mathbf{T}'$ by the map inductively defined by $\pi(x) = \{\pi(a) : \mathbf{Hull}^{\mathbf{T}}(X) \vDash \text{"}a \in x\text{"}\}$. Note that as a substructure of **V**, for $a, x \in H$, $\mathbf{Hull}^{\mathbf{T}}(X) \vDash$ "$a \in x$" iff $a \in x$. Moreover, $\mathbf{Hull}^{\mathbf{T}}(X) \vDash$ "$a \in x$" implies $a \in H$ just by virtue of the semantics. Therefore $\mathbf{Hull}^{\mathbf{T}}(X) \vDash$ "$a \in x$" iff $a \in x \cap H$ and thus $\pi(x)$ is equal to $\{\pi(a) : a \in x \cap H\}$. In particular, if X is transitive, the inductive hypothesis tells us that $\pi(x)$ for $x \in X$ is equal to $\{\pi(a) : a \in x \cap H\} = \{a : a \in x \cap H\} = x \cap H$. Since X is transitive, $x \subseteq X \subseteq H$ so that $x \cap H = x$. Therefore $\pi(x) = x$ and so $\pi \restriction X = \mathrm{id} \restriction X$ by induction on rank. ⊣

Note that the use of "collapse" especially makes sense here, because every $\pi(x) \in \mathrm{cHull}^{\mathbf{T}}(X)$ has $\mathrm{rank}(\pi(x)) \leq \mathrm{rank}(x)$. Of course, strict inequality requires that $x \not\subseteq \mathrm{Hull}^{\mathbf{T}}(X)$. Using direct limits, we can build up Skolem hulls to have less and less collapsed while still being relatively small.

In particular, if we take the hull that includes all of an ordinal, we get a model that contains all of the ordinals below it. Using the elementary chains, this allows us to conclude the following, showing we can get ordinals in our uncollapsed model before collapsing.

2 B • 4. Corollary

Let T be a transitive set with $\kappa \in T$ an uncountable, regular cardinal and $X \subseteq T$ of size $< \kappa$. Therefore, there is an elementary $\mathbf{H} \preccurlyeq \langle T, \in\rangle$ with $H \cap \mathrm{Ord}$ an ordinal, $|H| < \kappa$, and $X \subseteq H$.

Proof .:.

Take the Skolem hull $\mathbf{H}_0 = \mathbf{Hull}^{\langle T,\in\rangle}(X)$. This may not have $H_0 \cap \mathrm{Ord}$ as an ordinal although it will satisfy that $\mathbf{H}_0 \preccurlyeq \langle T, \in\rangle$ and $|H_0| \leq \aleph_0 \cdot 1 \cdot |X| < \kappa$. For $\mathbf{H}_n$ already defined, if $H_n \cap \mathrm{Ord}$ is an ordinal, then stop the process, and take $\mathbf{H} = \mathbf{H}_n$. Otherwise let $\mathbf{H}_{n+1} = \mathbf{Hull}^{\langle T,\in\rangle}(H_n \cup \sup(H_n \cap \mathrm{Ord}))$. As a regular cardinal, $\sup(H_n \cap \mathrm{Ord}) < \kappa$ because inductively $|H_n| < \kappa$, which also tells us that $|H_{n+1}| < \kappa$. Define $\mathbf{H}_\omega$ to be the direct limit of the $\mathbf{H}_n$s (i.e. the union).

Note that $\mathsf{H}_\omega \preccurlyeq \langle T, \in\rangle$ with $X \subseteq H_\omega$ and $|H_\omega| \leq \aleph_0 \cdot \sup_{n\in\omega} |H_n|$. As each $|H_n| < \kappa$ and κ has cofinality $\kappa > \omega$, it follows that $\sup_{n\in\omega} |H_n| < \kappa$ and thus $|H_\omega| < \kappa$. To see that $H_\omega \cap \mathrm{Ord}$ is an ordinal, it suffices to show that $H_\omega \cap \mathrm{Ord}$ is transitive. For $\beta \in H_\omega \cap \mathrm{Ord}$, it follows that $\beta \in H_n \cap \mathrm{Ord}$ for some $n < \omega$. Thus $\beta \subseteq H_{n+1}$ and so $\beta \subseteq H_\omega \cap \mathrm{Ord}$. $\dashv$

Section 3. *The constructible hierarchy* L

Recall that the levels of V were defined by iteratively taking the powerset operation. Gödel's definition of the constructible universe, L, does the same, but restricts to subsets which are definable over the previous levels.

3 • 1. Definition

Let $\mathrm{L} = \bigcup_{\alpha<\mathrm{Ord}} \mathrm{L}_\alpha$ where $\alpha \mapsto \mathrm{L}_\alpha$ is defined by transfinite recursion: $\mathrm{L}_0 = \emptyset$, $\mathrm{L}_\gamma = \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha$ for γ a limit, and

$$\mathrm{L}_{\alpha+1} = \{x \in \mathcal{P}(\mathrm{L}_\alpha) : x \text{ is definable over } \langle \mathrm{L}_\alpha, \in \rangle\}.$$

Here, x being definable over L_α means that there is a formula with parameters $\varphi(y)$ where $\mathsf{L}_\alpha \vDash$ "$\varphi(y)$" iff $y \in x$.

The importance of L to set theory is hard to overstate. There are three main ideas why. Firstly, every transitive model of enough set theory has an interpretation of L, and this interpretation is the same across all transitive models of $\mathsf{ZF} - \mathsf{P}$. Secondly, it's the *only* model with this property, demonstrating a strong minimality condition. In fact, it's defining formula is so rigid that any transitive model elementarily equivalent to one of the L_α levels is actually one of the L_α levels. Thirdly, as a result of all of this, L is always the smallest inner model. And thus it can be seen as the transitive model "generated" by the theory of ZFC in that it is the smallest such model. This is analogous to the situation with arithmetic, where $\mathbb{N}$ is the smallest model of the peano axioms, and so can be thought of as being generated by them.

As stated before, L has many "canonicity" properties. In particular, it has a strong minimality condition, being contained (up to a given height) in any transitive model of $\mathsf{ZF} - \mathsf{P}$. As a result, it's the smallest inner model, and is determined by its theory. We state these three facts as follows. Firstly, we have the absoluteness of L, leading to L being the smallest inner model.

3 • 2. Theorem (Absoluteness of L)

For any transitive model $\mathsf{M} \vDash \mathsf{ZF} - \mathsf{P}$, writing $\mathrm{L}_{\mathrm{Ord}}$ for L;

1. For each $\alpha \in \mathrm{Ord} \cap \mathrm{M}$, $\mathrm{L}_\alpha \subseteq \mathrm{M}$.
2. $\mathrm{L}^{\mathsf{M}} = \mathrm{L}_{\mathrm{Ord}\cap\mathrm{M}}$.

Proof .:.

Firstly, note that y being a first-order definable subset with parameters in x is absolute between these types of models, because y just needs to be in the closure of a set under the operations of intersection, complementation, and projection (corresponding to the operations of first-order logic) using membership in x and equality on x. This closure is given by recursion (repeatedly applying the operations). Given that each of the operations is clearly absolute, it follows that the output of this is absolute.

Proceed by induction on α to show $\mathrm{L}_\alpha^{\mathsf{M}} = \mathrm{L}_\alpha$ and thus $\mathrm{L}_\alpha \subseteq \mathrm{M}$ for $\alpha \in \mathrm{Ord} \cap \mathrm{M}$. Clearly, for $\alpha = 0$, $\mathrm{L}_\alpha = \emptyset \in \mathrm{M}$. Similarly, by the absoluteness of unions and the inductive hypothesis, for limit $\gamma \in \mathrm{Ord} \cap \mathrm{M}$, $\mathrm{L}_\gamma^{\mathsf{M}} = \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha^{\mathsf{M}} = \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha = \mathrm{L}_\gamma$. For the successor stage $\alpha + 1$, the argument above tells us that $\mathrm{L}_{\alpha+1}^{\mathsf{M}}$ is the first-order definable subsets (as interpretted in M) with parameters in $\mathrm{L}_\alpha^{\mathsf{M}}$ which is—inductively and by absoluteness—the same as the first-order definable subsets with parameters in L_α: $\mathrm{L}_{\alpha+1}$. Hence $\mathrm{L}_\alpha \in \mathrm{M}$ for each $\alpha \in \mathrm{Ord} \cap \mathrm{M}$.

For (2), we have $\mathrm{L}^{\mathsf{M}} = \bigcup_{\alpha\in\mathrm{Ord}\cap\mathrm{M}} \mathrm{L}_\alpha^{\mathsf{M}} = \bigcup_{\alpha\in\mathrm{Ord}\cap\mathrm{M}} \mathrm{L}_\alpha = \mathrm{L}_{\mathrm{Ord}\cap\mathrm{M}} \subseteq \mathrm{M}$. In particular, for M an inner model, $\mathrm{L}^{\mathsf{M}} = \mathrm{L} \subseteq \mathrm{M}$. ⊣

In particular, $\mathrm{L}^{\mathsf{L}} = \mathrm{L}$. In fact, if $\mathrm{Ord} \subseteq \mathrm{M}$, then all of L is contained in M.

3 • 3. Corollary (Smallest Inner Model)

$\mathrm{L} \subseteq \mathrm{M}$ for any inner model M of $\mathsf{ZF} - \mathsf{P}$.

Next, since we can write "V = L" as a $\{\in\}$-sentence, considering it as an axiom yields the following.

3 • 4. Corollary (Condensation)

Suppose $\mathsf{M} \vDash \mathsf{ZF} - \mathsf{P} +$ "V = L" where M is transitive. Therefore $\mathrm{M} = \mathrm{L}_{\mathrm{Ord}\cap\mathrm{M}}$.

Proof .:.

Since $\mathsf{M} \vDash$ "$\mathrm{V} = \mathrm{L}$", $\mathrm{M} = \mathrm{V}^{\mathsf{M}} = \mathrm{L}^{\mathsf{M}} = \mathrm{L}_{\mathrm{M}\cap\mathrm{Ord}}$ by Absoluteness of L (3 • 2). ⊣

There are other versions of Condensation (3 • 4) that don't matter for our purposes.

§ 3 A. CH in L

The importance of Condensation (3 • 4) comes from its use with Taking a Skolem Hull (2 A • 1) in the form of Corollary 2 B • 3. In particular, since any Skolem hull is elementarily equivalent to a level of L, when we collapse it, it becomes a level of L.

3 A • 1. Result

Suppose $\alpha \in \mathrm{Ord}$ has $\mathsf{L}_\alpha \vDash \mathsf{ZF} - \mathsf{P}$. Therefore, for any $X \subseteq \mathrm{L}_\alpha$, $\mathsf{cHull}^{\mathsf{L}_\alpha}(X) = \mathsf{L}_\beta$ for some $\beta \leq \alpha$.

Proof .:.

Note that such an α also has $\mathsf{L}_\alpha \vDash$ "$\mathrm{V} = \mathrm{L}$". Taking a Skolem hull yields $\mathsf{Hull}^{\mathsf{L}_\alpha}(X) \preccurlyeq \mathsf{L}_\alpha$ so that $\mathsf{Hull}^{\mathsf{L}_\alpha}(X) \vDash \mathsf{ZF} - \mathsf{P} +$ "$\mathrm{V} = \mathrm{L}$". Taking the transitive collapse yields an isomorphic, transitive $\mathsf{cHull}^{\mathsf{L}_\alpha}(X) \cong \mathsf{Hull}^{\mathsf{L}_\alpha}(X)$ and thus the two are elementarily equivalent: $\mathsf{cHull}^{\mathsf{L}_\alpha}(X) \vDash \mathsf{ZF} - \mathsf{P} +$ "$\mathrm{V} = \mathrm{L}$". By Condensation (3 • 4), it follows that $\mathrm{cHull}^{\mathsf{L}_\alpha}(X) = \mathrm{L}_\beta$ for some β, and it's not hard to see that $\beta \leq \alpha$. ⊣

If we investigate further the levels of L, we get some quick examples of models of "$\mathrm{V} = \mathrm{L}$". Note that the levels of L, although defined similarly, develop differently to the levels of V. In particular, $\mathrm{V}_\alpha \neq \mathrm{L}_\alpha$ in general, even if we assume $\mathrm{V} = \mathrm{L}$. An easy example of this is that in ZFC, $\wp(\omega) \subseteq \mathrm{V}_{\omega+1}$, meaning $|\mathrm{V}_{\omega+1}| \geq 2^{\aleph_0} > \aleph_0$. But $\mathrm{L}_{\omega+1}$ has only countably many new elements, corresponding to the defining formulas and parameters[vi]. Hence $|\mathrm{L}_{\omega+1}| = \aleph_0$. So the point is that subsets of ω don't appear in $\mathrm{L}_{\omega+1}$. In particular, one should not make the mistake of thinking $\mathrm{V}^{\mathrm{L}}_\alpha = \mathrm{L}_\alpha$. This is (almost always) false.

3 A • 2. Lemma

Let $\alpha \geq \omega$. Therefore $|\mathrm{L}_\alpha| = |\alpha|$.

Proof .:.

Proceed by induction on α. For $\alpha = \omega$, this is clear as L_ω is the countable union of sets, each of which is countable by induction: $\mathrm{L}_0 = \emptyset$ is clearly countable, and $\mathrm{L}_{n+1} \subseteq \wp(\mathrm{L}_n)$ which is also finite for $n < \omega$. For $\alpha + 1$, $\mathrm{L}_{\alpha+1}$ is the closure of L_α under countably many operations and is thus $|\mathrm{L}_{\alpha+1}| \leq |\mathrm{L}_\alpha| \cdot \aleph_0$. Since clearly $\omega \subseteq \mathrm{L}_\alpha$ for $\alpha \geq \omega$ and $\mathrm{L}_\alpha \subseteq \mathrm{L}_{\alpha+1}$, it follows that the reverse inequality holds and in fact $|\mathrm{L}_{\alpha+1}| = |\mathrm{L}_\alpha| = |\alpha| = |\alpha + 1|$.

For limit γ, $|\mathrm{L}_\gamma| = \left|\bigcup_{\alpha<\gamma} \mathrm{L}_\alpha\right| \leq |\gamma| \cdot \sup_{\alpha<\gamma} |\mathrm{L}_\gamma| = |\gamma| \cdot \sup_{\alpha<\gamma} |\alpha| = |\gamma|$. ⊣

This allows us to more precisely understand what the levels of L look like. The proof of the following isn't particularly enlightening, mostly just being combinatorial problems for the harder axioms, or just straight up definitions for the easier ones.

3 A • 3. Result

Let $\kappa > \aleph_0$ be a regular cardinal. Therefore $\mathsf{L}_\kappa \vDash \mathsf{ZFC} - \mathsf{P} +$ "$\mathrm{V} = \mathrm{L}$".

One of the more important corollaries of Result 3 A • 3 and Condensation (3 • 4) is what happens when we take Skolem hulls.

3 A • 4. Corollary

Let $\kappa > \aleph_0$ be a regular cardinal. Let $X \subseteq \mathrm{L}_\kappa$. Therefore the collapsed Skolem hull $\mathrm{cHull}^{\mathsf{L}_\kappa}(X) = \mathrm{L}_\alpha$ for some $\alpha < \kappa$. Moreover, if X is transitive, then $X \subseteq \mathrm{cHull}^{\mathsf{L}_\kappa}(X)$.

[vi] More formally, it's the closure of L_ω under countably many operations, and hence adds only countably many elements.

Proof .:.

By The Mostowski Collapse (1 C • 1), the collapsed hull models "V = L":

$$\mathsf{cHull}^{\mathsf{L}_\kappa}(X) \cong \mathsf{Hull}^{\mathsf{L}_\kappa}(X) \preccurlyeq \mathsf{L}_\kappa \vDash \mathsf{ZFC} - \mathsf{P} + \text{“V = L”}.$$

Hence by Condensation (3 • 4), $\mathrm{cHull}^{\mathsf{L}_\kappa}(X) = \mathrm{L}_\alpha$ for some α. As $\mathrm{Hull}^{\mathsf{L}_\kappa}(X) \subseteq \mathrm{L}_\kappa$, α can be calculated as $\alpha = \mathrm{Ord} \cap \mathrm{cHull}^{\mathsf{L}_\kappa}(X) \leq \kappa$. Note that Corollary 2 B • 3 implies $X \subseteq \mathrm{cHull}^{\mathsf{L}_\kappa}(X)$ if X is transitive. ⊣

3 A • 5. Theorem

$\mathsf{L} \vDash \mathsf{GCH}$, meaning $\mathsf{L} \vDash$ "$\forall\kappa\ (|\kappa| = \kappa \geq \aleph_0 \to 2^\kappa = \kappa^+)$". We only prove CH here, but the proof generalizes.

Proof .:.

Argue in a model of "V = L" to suppress so many superscripts of **L**. Let $x \in \mathcal{P}(\aleph_0)$ be arbitrary so that $x \in \mathrm{L}_\alpha$ for some $\alpha \in \mathrm{Ord}$. Let θ be a regular cardinal larger than $\max(\aleph_0, \alpha)$ (for example, $\theta = \max(\aleph_1, |\alpha|^+)$ works, but we just need it to be regular and sufficiently large). Therefore $\mathsf{L}_\theta \vDash \mathsf{ZF} - \mathsf{P} +$ "V = L".

Let $H = \mathrm{cHull}^{\mathsf{L}_\theta}(\{x\} \cup \aleph_0)$ so that $H = \mathrm{L}_\alpha$ for some $\alpha < \theta$ by Result 3 A • 1. As $|H| \leq \aleph_0 \cdot |\aleph_0 \cup \{x\}| = \aleph_0$, it follows by Lemma 3 A • 2 that $\alpha \leq \aleph_1$. Note also that $\aleph_0 \cup \{x\}$ is transitive, so that $\aleph_0 \cup \{x\} \subseteq H$ by Corollary 3 A • 4. In particular, $x \in \mathrm{L}_\alpha \subseteq \mathrm{L}_{\aleph_0^+}$. As $x \in \mathcal{P}(\aleph_0)$ was arbitrary, $\mathcal{P}(\aleph_0) \subseteq \mathrm{L}_{\aleph_1}$ and therefore $2^{\aleph}_0 \leq |\mathrm{L}_{\aleph_1}| = \aleph_1$. By Cantor's theorem, $\aleph_1 \leq 2^{\aleph}_0$ and thus we have equality. ⊣

Note that this shows there is no hope of proving the consistency of $\neg\mathsf{GCH}$ from ZFC with our current methods: trying to define an inner model with this true in it. Any attempts to define a class C by a formula φ to show $\mathsf{ZFC} \vdash \mathsf{ZFC}^{\mathrm{C}} + \neg\mathsf{GCH}^{\mathrm{C}}$ would also need to have $\mathsf{ZFC} +$ "V = L" $\vdash \mathsf{ZFC}^{\mathrm{C}} + \neg\mathsf{GCH}^{\mathrm{C}}$. But as the smallest inner model, any model $\mathbf{M} \vDash \mathsf{ZFC} +$ "V = L" has by absoluteness of L,

$$\mathrm{L}^{\mathsf{M}} = \mathrm{L}^{\mathrm{C}^{\mathsf{M}}} \subseteq \mathrm{C}^{\mathsf{M}} \subseteq \mathrm{M} = \mathrm{L}^{\mathsf{M}}$$

and thus would have $\mathbf{C}^{\mathsf{M}} = \mathbf{L}^{\mathsf{M}} \vDash \neg\mathsf{GCH}$, a contradiction.