GOST SEMINAR NOTES

THE THEORY OF FORCING

James Holland
math.rutgers.edu/~jch258
jch258@math.rutgers.edu

Start Date: 2020-02-21
End Date: 2020-03-11
Updated: 2020-09-16

Contents

Part I. Forcing Terminology

Section 1. *The Purpose of Forcing*

§ 1 A. The Main Results

The main idea behind forcing is to expand a model of set theory by a new set. Moreover, we should do this in a *minimal* way, and we should hope to preserve the membership relation, meaning that the new model should be transitive.

1 A • 1. Theorem

Let $\mathrm{V} \vDash \mathsf{ZFC}$ be a transitive model. Let $\mathbb{P} \in \mathrm{V}$ be a poset.
A *generic extension* of V by $G \notin \mathrm{V}$, written $\mathrm{V}[G]$, has the following properties:

1. $G \subseteq \mathbb{P}$;
2. $\mathrm{V}[G] \vDash \mathsf{ZFC}$ is transitive;
3. $\mathrm{V}[G]$ is the $\subseteq$-least transitive model M of ZFC with $\mathrm{V} \subseteq M$ and $G \in M$.

(1) is a really where $\mathbb{P}$ comes into play: we attempt to find a set G not in V, but which still has some intelligible structure to it. (2) is just a nice result of $\mathbb{P}$ being a set. (3) is the most important and motivating idea for us. The idea is that, despite G not being in V, we carry out a bunch of potential constructions of $\mathrm{V}[G]$ *inside* V (so-called $\mathbb{P}$-names). It is only through using G as a kind of oracle that allows us to form $\mathrm{V}[G]$ by interpretting these constructions in V.

To figure out which $G \subseteq \mathbb{P}$ are appropriate, we have the following theorem relating truth in $\mathrm{V}[G]$ with $\mathbb{P}$ in V. Here $p \Vdash \varphi$ is a notion definable in V which we will introduce later: it's the *forcing* relation.

1 A • 2. Theorem

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let $G \subseteq \mathbb{P}$ be "generic". Therefore, $\mathrm{V}[G] \vDash \varphi$ iff there is some $p \in G$ with $p \Vdash \varphi$.

Interpretting the forcing relation requires a lot of work, and there are many perspectives to take on it. Regardless, one can always take the formal approach, using the definition of it in V from Appendix A.

§ 1 B. Philosophy behind the method

If we're working with the actual universe of sets, the existence of these generic sets $G \notin \mathrm{V}$ is called into question. This worry can be alleviated in several ways. Firstly, we can regard ourselves as working in a relatively small inner model, assuming that for each $\mathbb{P} \in \mathrm{V}$, there is a $G \notin \mathrm{V}$. L, for example, consistently has relatively few sets, so it should be a little more understandable for there to be these $G \notin \mathrm{L}$, although this still depends on what sets exist in the ambient universe.

Secondly, many authors work with *countable* transitive models as sort of toy models of set theory to play with. With such models, the existence of these Gs is provable from ZFC. Unfortunately, the existence of these models does not follow from ZFC by Gödel incompleteness. This sub-worry can be alleviated when we acknowledge what the purpose of forcing is: consistency results. In particular, we only need to worry about countable transitive models of finite fragments of ZFC, which do exist assuming ZFC is consistent.[i] So if we're trying to show the consistency of some

[i] The coded version of this statement doesn't follow from ZFC. More precisely, for every actually finite fragment of ZFC, we get these models. But

theory $\mathrm{Con}(\mathsf{ZFC}+T)$ assuming $\mathrm{Con}(\mathsf{ZFC})$, it suffices to show $\mathrm{Con}(\Delta+T)$ from $\mathrm{Con}(\Delta')$ where $\Delta,\Delta'\subseteq\mathsf{ZFC}$ are sufficiently large fragments of ZFC. In other words, we get sufficiently large finite fragments of $\mathsf{ZFC}+T$ for some theory T, demonstrating that $\mathsf{ZFC}+T$ is consistent by the compactness theorem.

There is a third interpretation of the generic extension by way of Boolean valued models, where truth value is not taken to be either 0 or 1, but instead an element of a Boolean algebra. Under this interpretation, a suitable ultrafilter G tells us how to interpret the non-strictly-true and non-strictly false statements as either true or false. Alternatively, if we forgo the existence of such a G, the resulting Boolean algebra can still allow us to see whether a certain theory T is consistent relative to ZFC: in the Boolean valued model, each formula of ZFC has truth value 1, and perhaps so do all the elements of T. One can show using logic that this shows the consistency of $\mathsf{ZFC}+T$.

We will mostly just consider V to be a transitive inner model in the ambient universe, and we just assume that every poset $\mathbb{P}\in\mathrm{V}$ and $p\in\mathbb{P}$ we consider will have a generic G.

some models of ZFC think there are finite fragments of ZFC that are inconsistent, because it misinterprets what "finite" means (and so it misinterprets both ZFC and first-order logic).

Section 2. Names, Possible Worlds, and the Forcing Relation

First we will consider what $\mathrm{V}[G]$ will actually *be*, and then we will consider *truth* in $\mathrm{V}[G]$. The idea behind what $\mathrm{V}[G]$ will be is a bunch of conditional constructions that, once we have access to G *is*, can be thinned out to yield what we were trying to construct dependent on G. This yields a kind of *forcing* to be true in that knowing just a bit about G can already determine the outcome of some constructions.

§ 2 A. Names

For $G \notin \mathrm{L}$, there is already the notation of $\mathrm{L}[G]$. Recall that L is defined recursively:

$$\mathrm{L}_0 = \emptyset \qquad \mathrm{L}_{\alpha+1} = \{x \subseteq \mathrm{L}_\alpha : x \text{ is definable over } \langle \mathrm{L}_\alpha, \in \rangle\} \qquad \mathrm{L}_\gamma = \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha, \text{ for } \gamma \text{ a limit.}$$

So at each stage, we're taking definable subsets. What is the natural model of L that includes G (given that $G \subseteq \mathbb{P} \in \mathrm{L}$)? Well, we just make G a definable subset of $\mathbb{P}$: rather than consider definable subsets of $\langle \mathrm{L}_\alpha, \in \rangle$, we allow membership in G as a predicate:

$$\mathrm{L}_0[G] = \emptyset \qquad \mathrm{L}_{\alpha+1}[G] = \{x \subseteq \mathrm{L}_\alpha[G] : x \text{ is definable over } \langle \mathrm{L}_\alpha[G], \in, G \rangle\} \qquad \mathrm{L}_\gamma[G] = \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha[G], \text{ for } \gamma \text{ a limit.}$$

Clearly if G were in L already, this wouldn't make a difference: $\mathrm{L}[G] = \mathrm{L}$. Moreover, any inner model M with $G \in M$ can do this construction: $\mathrm{L} \subseteq M$ already, and $G \in M$ allows one to consider all of these stages. Hence $\mathrm{L}[G]$ is the least inner model M of ZFC with $G \in M$ (assuming $G \subseteq \mathbb{P} \in \mathrm{L}$). This is kind of the gold standard we want to emulate when forming the generic extension, and it motivates the idea of a *name*.

Suppose V has access to G. What sets can V form from G? V doesn't know what G is, but it *can* at consider constructions from $\mathbb{P}$, and then thin these out if it had access to G. The idea is to tag elements at each stage of the construction with elements of $\mathbb{P}$: look at things of the form $\langle x, p \rangle$ for $p \in \mathbb{P}$. Once we have access to G, if $p \notin G$, we throw out x, and if $p \in G$, we include it. For example, $\{\langle p, p \rangle : p \in \mathbb{P}\}$ will be thinned out to G, as we only include the first coordinate of $\langle p, p \rangle$ where $p \in G$.

The idea is that each set is tagged with an element of $\mathbb{P}$, and we just consider the elements tagged with an element of G. We can also iterate this concept: the set $\{\langle 0, p \rangle, \langle 1, q \rangle\}$ will be thinned out to

$$\begin{cases} \{0,1\} & \text{if } p, q \in G \\ \{0\} & \text{if } p \in G \wedge q \notin G \\ \{1\} & \text{if } p \notin G \wedge q \in G \\ \emptyset & \text{if } p, q \in G. \end{cases}$$

2 A • 1. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a set. A $\mathbb{P}$-name is defined by recursion on rank: define

- $\mathrm{V}^{\mathbb{P}}_0 = \emptyset$;
- $\mathrm{V}^{\mathbb{P}}_\gamma = \bigcup_{\alpha<\gamma} \mathrm{V}^{\mathbb{P}}_\alpha$ for γ a limit;
- $\mathrm{V}^{\mathbb{P}}_{\alpha+1} = (\wp(\mathrm{V}^{\mathbb{P}}_\alpha \times \mathbb{P}))^{\mathrm{V}}$.

Say that τ is a $\mathbb{P}$-*name* iff τ is in $\mathrm{V}^{\mathbb{P}} = \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{V}^{\mathbb{P}}$.

Each $x \in \mathrm{V}^{\mathbb{P}}$ can be "thinned out" once we're given access to G. In particular, we have the following interpretations.

2 A • 2. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a set, and $G \subseteq \mathbb{P}$, possibly not in V. For a $\mathbb{P}$-name τ, define by recursion on $\mathbb{P}$-name

$$\tau_G = \{\sigma_G : \exists p \in G\ (\langle \sigma, p\rangle \in \tau)\}.$$

So, for example, $\emptyset$ is a $\mathbb{P}$-name, as is $\tau = \{\langle 0, p\rangle : p \in \mathbb{P}\}$. As $G \neq \emptyset$, this is a $\mathbb{P}$-name for $\tau_G = \{0\} = 1$. Note that there can be multiple $\mathbb{P}$-names for a single set. For example, for $p, q \in G$ and $p', q' \notin G$, $\{\langle x, p\rangle, \langle x, q\rangle\}_G = \{x_G\}$. We also have a $\mathbb{P}$-name for G itself.

2 A • 3. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a set. For $x \in \mathrm{V}$ define $\check{x} \in \mathrm{V}^{\mathbb{P}}$ recursively to be $\{\check{y} : y \in x\} \times \mathbb{P}$.

For example,

- $\check{\emptyset} = \emptyset \times \mathbb{P}$ is a $\mathbb{P}$-name for $\emptyset$;
- $\check{1} = \{\check{0}\} \times \mathbb{P}$ is a $\mathbb{P}$-name for 1;
- $\check{2} = \{\check{0}, \check{1}\} \times \mathbb{P}$ is a $\mathbb{P}$-name for 2;
- $\widecheck{(x \cup y)} = \check{x} \cup \check{y}$ is a $\mathbb{P}$-name for $x \cup y$;
- $\widecheck{\{x, y\}} = \{\langle \check{x}, p\rangle, \langle \check{y}, p\rangle : p \in \mathbb{P}\}$ is a $\mathbb{P}$-name for $\{x, y\}$; etc.

One can easily see that $\check{x}$ is inductively a $\mathbb{P}$-name, and that its interpretation is just x.

2 A • 4. Result

Let $\mathbb{P}$ be a set and $\emptyset \neq G \subseteq \mathbb{P}$. Let $x \in \mathrm{V}$. Therefore $(\check{x})_G = x$.

Proof .:.

Proceed by induction on the rank of x. Inductively, $(\check{y})_G = y$ for each y of lower rank, in particular, for $y \in x$. Hence we can calculate

$$\begin{aligned}(\check{x})_G &= \{\langle \check{y}, p\rangle : y \in x \wedge p \in \mathbb{P}\}_G \\ &= \{(\check{y})_G : \exists p\ (\langle \check{y}, p\rangle \in \check{x})\} \\ &= \{(\check{y})_G : y \in x\} = \{y : y \in x\} = x \dashv\end{aligned}$$

This also gives a $\mathbb{P}$-name for G itself: $\{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$. Hence, by the following definition, $\mathrm{V} \subseteq \mathrm{V}[G]$ and $G \in \mathrm{V}[G]$.

2 A • 5. Theorem

Let $\mathbb{P} \in \mathrm{V}$ be a set and $\emptyset \neq G \subseteq \mathbb{P}$. Define $\mathrm{V}[G] = \{\tau_G : \tau \in \mathrm{V}^{\mathbb{P}}\}$. Therefore,

1. $\mathrm{V} \subseteq \mathrm{V}[G]$ and $G \in \mathrm{V}[G]$;
2. $\mathrm{V}[G]$ is transitive; and
3. Any transitive $M \vDash \mathsf{ZFC}$ with $\mathrm{V} \subseteq M$, $G \in M$ has $\mathrm{V}[G] \subseteq M$.

Proof .:.

1. $\check{x}$ has $(\check{x})_G = x \in \mathrm{V}[G]$ so that $\mathrm{V} \subseteq \mathrm{V}[G]$. We have the name $\dot{G} = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$ for G so that $G = (\dot{G})_G \in \mathrm{V}[G]$.
2. To see that $\mathrm{V}[G]$ is transitive, let $x \in \tau_G \in \mathrm{V}[G]$ where $\tau \in \mathrm{V}^{\mathbb{P}}$. Therefore, $x = \sigma_G$ for some $\sigma \in \mathrm{V}^{\mathbb{P}}$ so that $x = \sigma_G \in \mathrm{V}[G]$.
3. Any such M with $\mathrm{V} \subseteq M$ can construct each $\mathbb{P}$-name: $\mathrm{V}^{\mathbb{P}} \subseteq M$. Since $G \in M$, for each $\mathbb{P}$-name τ of V, we can construct τ_G (M knows enough set theory to carry out these constructions). Hence $\mathrm{V}[G] \subseteq M$. $\dashv$

So already we have a kind of "minimal" model by expanding V to $\mathrm{V}[G]$. But it's not obvious how we can know whether $\mathrm{V}[G] \vDash \mathsf{ZFC}$ or not. Indeed, it's not at all obvious how to calculate truth in $\mathrm{V}[G]$. Just from it being transitive, finding the right $\mathbb{P}$-names allows us to argue by absoluteness that $\mathrm{V}[G]$ models pairing, union, and some of the other simple axioms. But going further than this isn't easy, especially when $\mathrm{V}[G]$ differs from V on some statement.

§ 2 B. Posets, information, and Forcing

Recall the definition of a poset (a partially ordered set).

2 B • 1. Definition

A *poset* is a structure $\langle \mathbb{P}, \leqslant \rangle$ where $\leqslant \subseteq \mathbb{P} \times \mathbb{P}$ is reflexive, transitive, and anti-symmetric, meaning
- (reflexive) for all $p \in \mathbb{P}$, $p \leqslant p$;
- (anti-symmetric) for all $p, q \in \mathbb{P}$, $p \leqslant q \leqslant p$ implies $p = q$; and
- (transitive) for all $p, q, r \in \mathbb{P}$, $p \leqslant q \leqslant r$ implies $p \leqslant r$.

In some sense, we don't really need reflexivity, as any non-reflexive $\langle \mathbb{P}, < \rangle$ has a corresponding reflexive version $\langle \mathbb{P}, \leqslant \rangle$ where we just consider $\leqslant \, = \, < \cup \{\langle p, p \rangle : p \in \mathbb{P}\}$. The concept of a poset should be fairly familiar, as it corresponds to directed graphs (that are transitive) with no loops.[ii]

We will be viewing posets as coding information. Consider the following analogy with knowledge and discovery. Currently, we have a fair amount of knowledge p. At a later point in time, we could make discoveries such that we know q or r. In this way, we have an ordering on our knowledge. Given that more precise, specific information is less likely to be true in general, we say that $p^* \leqslant p$ to represent that p^* has more information than p. This gives a kind of poset, and motivates the terminology of $p \in \mathbb{P}$ as a "condition". Moreover, this analogy allows us to introduce the forcing relation already: p forces something to be true if p has enough information to determine it.

2 B • 2. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a poset with $p \in \mathbb{P}$. We say p *forces* a formula φ, written $p \Vdash \varphi$, iff $p \in G$ implies $\mathrm{V}[G] \vDash \varphi$ for all appropriate $G \subseteq \mathbb{P}$ (to be made precise later).

2 B • 3. Motivation

Let $\mathbb{P}$ be a poset. For $p \in \mathbb{P}$, write $p^* \in \mathbb{P}$ for an arbitrary $p^* \leqslant p$ (an arbitrary point in time after p). Therefore,
1. if $p \Vdash \varphi$ then every $p^* \Vdash \varphi$;
2. $p \Vdash$ "$\neg\varphi$" iff every $p^* \nVdash \varphi$, i.e. you can conclude it's false iff you will never discover that it's true;
3. $p \Vdash$ "$\varphi \wedge \psi$" iff $p \Vdash \varphi$ and $p \Vdash \psi$;
4. if $p \Vdash$ "$\exists x\ \varphi(x)$" then there is some $p^* \leqslant p$ and τ where $p^* \Vdash$ "$\varphi(\tau)$"; and
5. if $p \Vdash \varphi$, and φ is logically equivalent to ψ, then $p \Vdash \psi$;

Note that this is very intuitionistic. There is actually a fairly close connection between forcing and intuitionistic logic [1]. Note, for example, $p \nVdash \varphi$ is not equivalent to $p \Vdash$ "$\neg\varphi$". The motivation behind (4) is that if we know something is true, we should be able to discover an example.

2 B • 4. Corollary

Let $\mathbb{P}$ be a poset, $p \in \mathbb{P}$, and φ a formula. Therefore, $p \Vdash \varphi$ iff every $p^* \Vdash \varphi$.

Proof .:.

The "$\rightarrow$" direction is clear. For the "$\leftarrow$" direction, suppose $p \nVdash \varphi$, i.e. $p \nVdash$ "$\neg\neg\varphi$". By (2) of Motivation 2 B • 3, there is then an extension $p^* \Vdash$ "$\neg\varphi$", contradicting that every $p^* \Vdash \varphi$.

The idea given by the proof also shows the following: if it's currently unclear whether something is true, it will be decided later.

2 B • 5. Corollary

Let $\mathbb{P}$ be a poset, $p \in \mathbb{P}$, and φ a formula. Therefore, $p \nVdash \varphi$ and $p \nVdash$ "$\neg\varphi$" implies there are $q, r \leqslant p$ where $q \Vdash \varphi$ and $r \Vdash$ "$\neg\varphi$".

In the end, our goal will be the following.

2 B • 6. Theorem

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let $G \subseteq \mathbb{P}$ be "generic". Therefore, $\mathrm{V}[G] \vDash \varphi$ iff there is some $p \in G$ with $p \Vdash \varphi$.

[ii] We don't even need anti-symmetry really: the result are *preorders* where we can mod out by the equivalence relation $\approx$—defined by $p \approx q$ iff $p \leqslant q \leqslant p$—to get a poset. Indeed that poset is *forcing equivalent* (gives the same generic extensions) as the original preorder, so there's no harm in using only preorders, but posets are a more familiar object to most mathematicians.

Proof .:.

Clearly if $p \in G$ has $p \Vdash \varphi$ then $\mathrm{V}[G] \vDash \varphi$ by Definition 2 B • 2. So suppose $\mathrm{V}[G] \vDash \varphi$, but every $p \in G$ has $p \not\Vdash \varphi$. So every $p^* \not\Vdash \varphi$ too for $p \in G$ because G is "generic". Thus $p \Vdash$ "$\neg\varphi$" for $p \in G$ by the unproven Motivation 2 B • 3. Therefore $\mathrm{V}[G] \vDash \neg\varphi$, a contradiction. ⊣

So really we need a notion that ensure the forcing relation as defined in Definition 2 B • 2 obeys Motivation 2 B • 3. So this partially motivates what G should look like: we obviously need all $p \in G$ to be compatible with each other in a precise sense. We also need G to interact nicely with extensions: we need to be able to extend elements of G with certain properties as needed. We will see later that this amounts to being a filter and intersecting dense sets.

Section 3. *Poset Topology*

Properties of the generic G will be induced by the topology of the corresponding poset $\mathbb{P}$. The topology on posets is given by their ordering relation. In particular, basic open sets are just sets that are closed downward.

3 • 1. Definition

Let $\mathbb{P}$ be a poset. For $p \in \mathbb{P}$, the *basic open neighborhood* around p is $\mathbb{P}_{\leq p} = \{q \in \mathbb{P} : p \leqslant q\}$, i.e. everything below p. The *poset topology* on $\mathbb{P}$ is the topology generated by these: $\{\mathbb{P}_{\leq p} : p \in \mathbb{P}\}$.

3 • 2. Corollary

Let $\mathbb{P}$ be a poset. Therefore $U \subseteq \mathbb{P}$ is open iff U is closed downward: $p \in U$ and $q \leqslant p$ implies $q \in U$.

Proof .:.

If U is closed downward, then for every $p \in U$, $\mathbb{P}_{\leq p} \subseteq U$ and hence U is open. If U is open, note that U has been generated by the basic open neighborhoods, which are closed downward. To be generated by these basic open sets, U must be an arbitrary union of finite intersections of these $\mathbb{P}_{\leq p}$s. These finite intersections are easily seen to be closed downward, and unions of sets closed downward are also closed downward. Hence U is closed downward. ⊣

Without appealling to topology, one can make the following result a definition.

3 • 3. Corollary

A set $D \subseteq \mathbb{P}$ is *dense* iff D intersects every open set. Thus $D \subseteq \mathbb{P}$ is dense iff for every $p \in \mathbb{P}$, there is some $p^* \leqslant p$ where $p^* \in D$. In other words, D is dense iff we can always extend a p to a $p^* \in D$.

Proof .:.

Suppose D is dense with $p \in \mathbb{P}$. Since $D \cap \mathbb{P}_{\leq p} \neq \emptyset$, there is some $p^* \leqslant p$ with $p^* \in D$.

§ 3 A. Density

For the most part, the above ideas are not used: we do not care about topological definitions in general. We really only care about sets closed downward, and sets where we can always go downward into the set. In the analogy with knowledge and discovery, these correspond to things that always remain true (closed downward), and things that always have the potential to be true (density) in that for any point of time p, it's always possible to discover at a later time p^* that it's true.

The notion of being able to extend an element is incredibly important for us. We thus have two additional notions for posets.

3 A • 1. Definition

Let $\mathbb{P}$ be a poset. Let $p, q \in \mathbb{P}$. We say that p and q are *compatible* iff there is a common extension $r \leqslant p, q$.
We say that p and q are *incompatible*, written $p \perp q$, otherwise: there is no common extension.

Easy examples of compatible elements include any two comparable elements: $p^* \leqslant p$ implies p and p^* are compatible. The basic pictures of compatibility and incompatibility are below.

There need not be a common predecessor to p and q if $p \perp q$, as the figure above suggests. But we will only consider posets where this occurs, sometimes artificially adding a maximal element to $\mathbb{P}$ to ensure that this happens.

In the context of forcing, if p and q are compatible, then there are no conflicts with what they force: there is no φ with

$$\begin{tikzcd} & r \arrow[dl] \arrow[dr] & \\ p & & q \end{tikzcd} \qquad \begin{tikzcd} p \arrow[dr] & & q \arrow[dl] \\ & r & \end{tikzcd}$$

Incompatible elements — Compatible elements

3 A • 2. Figure: Compatibility of p and q in example posets

$p \Vdash \varphi$ and $q \Vdash$ "$\neg\varphi$". This follows from (1) of Motivation 2 B • 3: a common extension $r \leqslant p, q$ would need to have $r \Vdash$ "$\varphi \wedge \neg\varphi$", which would mean any G with $r \in G$ has $\mathrm{V}[G] \vDash$ "$\varphi \wedge \neg\varphi$", which is impossible. This still, of course, depends on some knowledge about what G can be, but it provides some motivation on *what* we want G to be.

So we now have the fundamental concepts with posets: extending individual elements, and extending perhaps incomparable (but still compatible) elements. The notion of density is closely connected with the idea that "most" elements have the property or at least are compatible with the property in a loose sense.

3 A • 3. Lemma

Let $\mathbb{P}$ be a poset. Let D_0 and D_1 be open, dense sets in $\mathbb{P}$. Therefore $D_0 \cap D_1$ is open, dense (and in particular, non-empty).

Proof .:.

Since both D_0 and D_1 are closed downwards, so is $D_0 \cap D_1$, meaning it's open. To show density, suppose $p \in \mathbb{P}$. We can extend this to some $p^* \in D_0$ and then to some $p^{**} \in D_1$. Since D_0 is closed downward, $p^{**} \in D_0$. By transitivity, $p^{**} \leqslant p$ so that $D_0 \cap D_1$ is dense. $\dashv$

Section 4. *Generic Filters*

§ 4 A. A new set

We are now in a position to say what properties G needs to have. Firstly, consider the following theorem of ZFC.

4 A • 1. Theorem

Let $\mathbb{P}$ be a poset with $p \in \mathbb{P}$. Let $\mathcal{D}$ be a countable collection of dense sets. Therefore, there is a filter $G \subseteq \mathbb{P}$ where $p \in G$ and $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Proof .:.

Enumerate $\mathcal{D} = \{D_n : n < \omega\}$. As D_0 is dense, let $p_0 \leqslant p$ be in D_0. As in Lemma 3 A • 3, just by continually expanding, we get a sequence $\langle p_n \in \mathbb{P} : n \in \omega \rangle$, where $p_{n+1} \leqslant p_n \leqslant p$ and $p_n \in D_n$. Taking the upward closure of this chain $G = \{q \in \mathbb{P} : \exists n \in \omega\ (p_n \leqslant q)\}$ yields a filter (a set closed upward where all elements are compatible) where $p \in G$, and $p_n \in G \cap D_n$ for each $n < \omega$. $\dashv$

The question then becomes: how many dense sets can we intersect? The generic G is one that intersects all dense sets of V. Of course, $\mathbb{P}$ itself also has this property, but we require in addition that all the elements of G are compatible to ensure Motivation 2 B • 3 holds.

4 A • 2. Definition

Let $\mathbb{P}$ be a poset. Let $\mathcal{D}$ be a collection of dense sents. A set $G \subseteq \mathbb{P}$ is said to be $\mathbb{P}$-*generic over* $\mathcal{D}$ iff

- $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$; and
- G is a filter (i.e. closed upward under $\leqslant$, and any two elements of G are compatible).

We say that G is *generic over* V iff G is $\mathbb{P}$-generic over $\{D \in \mathrm{V} : D \text{ is dense}\}$.

It will turn out that $G \notin \mathrm{V}$ if $\mathbb{P}$ satisfies some weak requirements. So what posets are appropriate to use in forcing? The following terminology is non-standard, and is really just short-hand for the concept.

4 A • 3. Definition

A poset $\mathbb{P}$ is *appropriate for forcing* iff

- there is a $\leqslant$-maximal element $\mathbb{1}_{\mathbb{P}}$; and
- for every $p \in \mathbb{P}$, there are $q, r \leqslant p$ where $q \perp r$.

If $\mathbb{P}$ has no maximal element, we can artificially consider $\mathbb{P}' = \mathbb{P} \cup \{\mathbb{1}\}$ and say $\mathbb{1} \geqslant p$ for each $p \in \mathbb{P}$. $\mathbb{P}'$ then has a maximal element. The reason we want these properties is the following.

4 A • 4. Theorem

Let $\mathbb{P} \in \mathrm{V}$ be a poset appropriate for forcing. Therefore, there is no $G \in \mathrm{V}$ that is $\mathbb{P}$-generic over V.

Proof .:.

Suppose $G \cap D \neq \emptyset$ for every dense $D \in \mathrm{V}$. Clearly $G \neq \mathbb{P}$, as there are incompatible elements in $\mathbb{P}$, but all elements of G must be compatible. So consider $\mathbb{P} \setminus G$. This set will be dense.

To see that $\mathbb{P} \setminus G$ is dense, let $p \in \mathbb{P}$ be arbitrary. There are then two incompatible conditions $q \perp r$ below p. Since any two elements of G are compatible, we cannot have both $q, r \in G$. So one of these is in $\mathbb{P} \setminus G$, meaning we have an extension of p in $\mathbb{P} \setminus G$. Hence $\mathbb{P} \setminus G$ is dense, and thus $G \notin \mathrm{V}$ as otherwise genericity over V implies $G \cap (\mathbb{P} \setminus \mathrm{G}) \neq \emptyset$, a contradiction. $\dashv$

This is in contrast to Theorem 4 A • 1. In particular, if V is countable in our universe, then there are only countably many dense sets in V and hence a generic exists in the real world. It's just that V just doesn't see this subset of $\mathbb{P}$.

We will never actually confirm that a poset is appropriate for forcing, because we rarely care whether there is a generic already in V: sometimes $V = V[G]$. This is the case with trivial posets, for example: a poset $\mathbb{P} = \{\mathbb{1}\}$ has $G = \mathbb{P} \in V$ as generic over V. So whether $G \in V$ or not is practically irrelevant: we care more what properties $V[G]$ has.

§ 4 B. The generic extension

We've already defined the generic extension in Subsection 2 A, but we repeat it here for ease of reference. Recall that a $\mathbb{P}$-name is just a potential construction hinging on G in the sense that it is (inductively) some potential constructions marked with elements of $\mathbb{P}$. When interpretting a $\mathbb{P}$-name τ, we just take those elements tagged with an element in G: $\tau_G = \{\sigma_G : \exists p \in G\ (\langle \sigma, p\rangle \in \tau)\}$. Note that the collection of $\mathbb{P}$-names depends on the sets V can construct, hence the notation "$V^{\mathbb{P}}$" for the colleciton of $\mathbb{P}$-names in V.

4 B • 1. Definition
Let $\mathbb{P} \in V$ be a poset and G $\mathbb{P}$-generic over V. The *generic extension* $V[G] = \{\tau_G : \tau \text{ is a } \mathbb{P}\text{-name}\}$.

We now can formally define the forcing relation as in Definition 2 B • 2

4 B • 2. Definition
Let $\mathbb{P}$ be a poset and $p \in \mathbb{P}$. Write $p \Vdash \varphi$, iff $V[G] \vDash \varphi$ for every G that is $\mathbb{P}$-generic over V with $p \in G$.

This allows us to confirm the results of Motivation 2 B • 3. The proof of this fact is quite long, however, and is only given at the end of Appendix A. A very thorough treatment of the forcing relation in general can be found in Appendix A as well as in Chapter VII of [2] (which the appendix is based on).

4 B • 3. Lemma
Let $\mathbb{P} \in V$ be a poset and φ a formula. Therefore the relation $\{\langle p, \vec{\tau}\rangle : p \Vdash \text{“}\varphi(\vec{\tau})\text{”}\}$ is definable over V.

4 B • 4. Corollary
Let $\mathbb{P} \in V$ be a poset appropriate for forcing and $p \in \mathbb{P}$. Write $p^* \in \mathbb{P}$ for an arbitrary $p^* < p$. Therefore,
1. $p \Vdash \varphi$ iff every $p^* \Vdash \varphi$;
2. $p \Vdash$ "$\neg\varphi$" iff every $p^* \nVdash \varphi$;
3. $p \Vdash$ "$\varphi \wedge \psi$" iff $p \Vdash \varphi$ and $p \Vdash \psi$;
4. $p \Vdash$ "$\exists x\ \varphi(x)$" iff there is some $\mathbb{P}$-name τ and extension $p^* \leqslant p$ where $p^* \Vdash$ "$\varphi(\tau)$"; and
5. For φ and ψ logically equivalent, $p \Vdash \varphi$ iff $p \Vdash \psi$.

As before with Theorem 2 B • 6, this allows us to characterize truth in $V[G]$.

4 B • 5. Corollary
Let $\mathbb{P} \in V$ be a poset. Let G be $\mathbb{P}$-generic over V. Therefore $V[G] \vDash \varphi$ iff $\exists p \in G\ (p \Vdash \varphi)$.

Moreover, through tedious checking, we can confirm each individual axiom of ZFC in $V[G]$.

4 B • 6. Theorem
Let $\mathbb{P} \in V$ be a poset. Let G be $\mathbb{P}$-generic over V. Therefore $V[G] \vDash$ ZFC.

The proof of this is quite tedious, but can also be found later in Appendix B and in Chapter VII of [2].

Part II. Examples of Forcing, and Further Ideas

If forcing is all about adding new objects into the universe, we should think about what sorts of objects we want to add. There are all sorts of posets that generically add in all sorts of objects. There are, of course, limits to what we can add with forcing,[i] but commonly we add in functions and subsets. So this will be the purpose of the first few posets: add in subsets and functions.

One immediate question that pops up is how do we choose what poset to force with? Commonly, the idea begins with the goal in mind: we want to add in some $G \subseteq \mathrm{V}$. Our poset will often consist of V's approximations to G where $p \leqslant q$ iff p approximates more than q. In the context of functions and subsets, this ordering is usually containment: $p \leqslant q$ iff $p \supseteq q$.

Section 5. *Collapsing Cardinals*

Recall that cardinals are really just special ordinals. They are determined by what functions[ii] the model V has. For example, to calculate ω_1, the general idea is that V looks at each ordinal α, determines whether there's a bijection with ω. Then, the first place it has no bijection, it stops and says "this is ω_1".

But because this is all based on what functions V has, if we add in a bijection with ω and, say, $\alpha = \omega_1^{\mathrm{V}}$, we can show this ordinal α is countable in $\mathrm{V}[G]$, meaning $\omega_1^{\mathrm{V}} = \alpha < \omega_1^{\mathrm{V}[G]}$. We can also generalize this, but let's stick with "collapsing" a cardinal to ω for now.

5 • 1. Definition

Let κ be an infinite ordinal. The poset $\mathrm{Col}(\aleph_0, \alpha) \in \mathrm{V}$ consists of functions $f \in \mathrm{V}$ where (all interpreted in V)

- $|f| < \aleph_0$; and
- $\mathrm{dom}(f) \subseteq \omega$, $\mathrm{im}(f) \subseteq \kappa$.

We write $f \leqslant g$ iff $f \supseteq g$.

We say f is a *partial function* from ω to κ in the sense that $f : A \to \kappa$ for some subset $A \subseteq \omega$. This shorthand is quite useful as each f is an approximation to a full-fledged function from ω to κ.

The first thing to confirm is that this gives us what we want: G codes a surjection from $\aleph_0$ onto κ. We will show this slowly with all the detail. The main idea is that $\bigcup G = g$ inherets properties from the approximations in $\mathrm{Col}(\aleph_0, \alpha)$. So since we can always add in an $n < \omega$ into the domain of these approximations, and always add in an $\alpha < \kappa$ into the range, it follows that these form dense sets. Hence each $n < \omega$ is in the domain of g, and each $\alpha < \kappa$ is in the range of g.

5 • 2. Theorem

Let κ be an infinite ordinal. Let G be $\mathrm{Col}(\aleph_0, \alpha) = \mathbb{P}$-generic over V. Therefore $\bigcup G = g$ is a surjection from ω to κ. In particular, $\mathrm{V}[G] \vDash$ "$|\kappa| = \aleph_0$".

Proof .:.

We need to confirm several things: that g is in fact a function. That $\mathrm{dom}(g) = \omega$, and that $\mathrm{im}(g) = \kappa$ so that g is a surjection.

[i] For example, we can't add ordinals with forcing.

[ii] In particular, what *bijections*

Claim 1

g is a function

Proof .:.

This is a simple application of compatibility of G. In particular, if $\langle n, \alpha\rangle, \langle n, \beta\rangle \in g$ for some $n < \omega$ and $\alpha, \beta < \kappa$, there are some $f, g \in G$ with $f(n) = \alpha$ and $g(n) = \beta$. By compatibility, there is some $h \in G$ with $h \leqslant f, g$, meaning a finite, partial function $h \supseteq f, g$. But then $\langle n, \alpha\rangle, \langle n, \beta\rangle \in f \cup g \subseteq h$ requires that $\alpha = \beta$ for h to be a function at all. Hence $\langle n, \alpha\rangle, \langle n, \beta\rangle \in g$ implies $\alpha = \beta$ and thus g is a function. ⊣

Just by definition of $g = \bigcup G \subseteq \omega \times \kappa$, we have that g has domain $\mathrm{dom}(g) \subseteq \omega$ and $\mathrm{im}(g) \subseteq \kappa$. The issue, however, is whether we have equality. This is where dense sets come into play.

Claim 2

$g : \omega \to \kappa$, i.e. $\mathrm{dom}(g) = \omega$.

Proof .:.

We need to show that for each $n < \omega$, there is some α with $\langle n, \alpha\rangle \in \bigcup G$. The only real way we have to ensure something is in G is to find an appropriate dense set. Then G intersects it, and we have a witness. So for our case, we need some $f \in G$ where $\langle n, \alpha\rangle \in f$ for some α. For each $n < \omega$, consider the set of these f:

$$D_n = \{f \in \mathbb{P} : n \in \mathrm{dom}(f)\}.$$

Note that this is dense in $\mathbb{P}$, since for any $p \in \mathbb{P}$, if $n \in \mathrm{dom}(p)$, we're done. If $n \notin \mathrm{dom}(p)$, then we just choose some $\alpha < \kappa$ not already in $\mathrm{im}(p)$ (p is finite while κ is infinite, so this is possible), and then consider $q = p \cup \{\langle n, \alpha\rangle\} \leqslant p$. This $q \in D_n$ and extends our arbitrary $p \in \mathbb{P}$, so each D_n is dense.

In particular, $G \cap D_n \neq \emptyset$ for each n, and thus $n \in \mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ for some $f \in G$, implying each $n \in \mathrm{dom}(g)$. Therefore $\omega \subseteq \mathrm{dom}(g)$. Since clearly $\mathrm{dom}(g) \subseteq \omega$, we have equality. ⊣

So all that remains to be shown is that g is surjective. To see this, we proceed exactly like in Claim 2 for the range. Let $\alpha < \kappa$ be arbitrary. Consider the set

$$E_\alpha = \{f \in \mathbb{P} : \alpha \in \mathrm{im}(f)\}.$$

This set is dense by the same reason as above: since $p \in \mathbb{P}$ is finite, take $n \in \omega \setminus \mathrm{dom}(p)$ and add in $\langle n, \alpha\rangle$: $q = p \cup \{\langle n, \alpha\rangle\} \leqslant p$ has $q \in E_\alpha$ and thus E_α is dense. Therefore there is some $f \in G \cap E_\alpha$ and so $\alpha \in \mathrm{im}(f) \subseteq \mathrm{im}(g)$. As $\alpha < \kappa$ was arbitrary, $\kappa \subseteq \mathrm{im}(g)$. We obviously have $\mathrm{im}(g) \subseteq \kappa$, and thus equality. This means $g : \omega \to \kappa$ is a surjection. By AC in $\mathrm{V}[G]$, it follows that $\mathrm{V}[G] \vDash$ "$|\kappa| = \aleph_0$". ⊣

Where exactly did the forcing relation come into play here? The idea is that $f \in \mathbb{P}$ has $f \Vdash$ "$\check{f} \subseteq \dot{g}$", where $\dot{g}$ is a name for g[iii] and it's this sense that our $f \in \mathbb{P}$ is an approximation to g.

The above forcing notion gives us the idea of "collapsing" a cardinal in the following sense. This allows us to consider other forcing notions that *do not* collapse cardinals.

5 • 3. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a poset. We say that $\mathbb{P}$ *preserves cardinals* iff every $\alpha \in \mathrm{V}$ such that $\mathrm{V} \vDash$ "$\alpha = |\alpha|$" has $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\alpha} = |\check{\alpha}|$". Equivalently, $\mathbb{P}$ preserves cardinals iff for every $\mathbb{P}$-generic G over V, $\mathrm{V} \vDash$ "$\alpha = |\alpha|$" iff $\mathrm{V}[G] \vDash$ "$\alpha = |\alpha|$".

The above definition also makes sense for other properties. For example, one can say that $\mathbb{P}$ *preserves cofinalities* whenever V and $\mathrm{V}[G]$ agree on the function $\alpha \mapsto \mathrm{cof}(\alpha)$. Similarly, $\mathbb{P}$ *preserves stationary sets* whenever $S \in \mathrm{V}$ is stationary implies $\mathrm{V}[G] \vDash$ "S is stationary". So it should be obvious that $\mathrm{Col}(\omega, \kappa)$ does not preserve cardinality when $\kappa > \aleph_0$. But $\mathrm{Col}(\omega, \kappa)$ *does* preserve cardinals $\leq \aleph_0$.[iv]

We can define more generally $\mathrm{Col}(\lambda, \kappa)$, which forces that $|\kappa| = \lambda$ in the generic extension (so long as $\kappa > \lambda$). While

[iii] for example, $\{\langle\langle n, \alpha\rangle\check{}, p\rangle \in (\omega \times \kappa)\check{} \times \mathbb{P} : p(n) = \alpha\}$.

[iv] All forcings do this as ω and $n < \omega$ are absolute between transitive models of set theory.

this poset also collapses cardinals, it *doesn't* collapse all of them. In particular, it leaves cardinals $\leq \lambda$ alone. To prove this, we need some more concepts related to posets.

5 • 4. Definition

Let $\lambda < \kappa$ be infinite ordinals. The poset $\mathrm{Col}(\lambda, \kappa) \in \mathrm{V}$ consists of functions $f \in \mathrm{V}$ where (interpretted in V)

- $|f| < \lambda$; and
- $\mathrm{dom}(f) \subseteq \lambda$, $\mathrm{im}(f) \subseteq \kappa$.

We write $f \leqslant g$ iff $f \supseteq g$.

We of course have a similar property as before:

5 • 5. Theorem

Let $\lambda < \kappa$ be an infinite cardinals in V. Let G be $\mathrm{Col}(\lambda, \kappa) = \mathbb{P}$-generic over V. Therefore $\bigcup G = g$ is a surjection from λ to κ. In particular, $\mathrm{V}[G] \vDash$ "$|\kappa| = \lambda$".

Proof .:.

By the same reasoning as before, we already know g is a function with domain $\mathrm{dom}(g) \subseteq \lambda$ and image $\mathrm{im}(g) \subseteq \kappa$. So we want to show equality of each of these. For each $\alpha < \lambda$ and each $\beta < \kappa$, consider the sets

$$D_\alpha = \{f \in \mathbb{P} : \alpha \in \mathrm{dom}(f)\}$$
$$E_\beta = \{f \in \mathbb{P} : \beta \in \mathrm{im}(f)\}.$$

Since $\kappa > \lambda$, any $f \in \mathbb{P}$ (which then has size $|f| < \lambda$) is not a bijection: $\mathrm{dom}(f) \subsetneq \lambda$ and $\mathrm{im}(f) \subsetneq \kappa$. If $\alpha \notin \mathrm{dom}(f)$ and $\beta \notin \mathrm{im}(f)$, $f^* = f \cup \{\langle \alpha, \beta \rangle\} \leqslant f$ has $f^* \in D_\alpha \cap E_\beta$ and thus each is dense. Hence $G \cap D_\alpha \neq \emptyset$ and $G \cap E_\beta \neq \emptyset$ for each $\alpha < \lambda$ and $\beta < \kappa$. In particular, this yields that $\alpha \in \mathrm{dom}(g)$ and $\beta \in \mathrm{im}(g)$ for each $\alpha < \lambda$ and $\beta < \kappa$, meaning $g : \lambda \to \kappa$ is a surjection. $\dashv$

This shows $\mathrm{Col}(\lambda, \kappa)$ collapses κ to λ. $\mathrm{Col}(\lambda, \kappa)$ still *preserves* some cardinals by the following fact.

5 • 6. Lemma

$\mathrm{Col}(\lambda, \kappa)$ is $< \mathrm{cof}(\lambda)$-closed, meaning if $\langle p_\alpha \in \mathbb{P} : \alpha < \gamma \rangle$ is a $\leqslant$-decreasing sequence (in V) of length $\gamma < \mathrm{cof}(\lambda)$, there is some condition $p \in \mathbb{P}$ below all of them: $p \leqslant p_\alpha$ for each $\alpha < \gamma$.

Proof .:.

Let $\langle p_\alpha : \alpha < \gamma \rangle$ be as in the statement. Thus $p = \bigcup_{\alpha < \gamma} p_\alpha$ is a partial function from λ to κ. Moreover, as $\gamma < \mathrm{cof}(\lambda)$ and each p_α has size $|p_\alpha| < \lambda$, it follows that this union has size $|p| \leq |\gamma| \cdot \sup_{\alpha < \gamma} |p_\alpha| < \lambda$. Hence p is a condition in $\mathbb{P}$, and clearly lies below each p_α. $\dashv$

This gives the following corollary. It's also a nice exercise to see how the result should change if λ is not regular.

5 • 7. Corollary

Let λ be a regular, infinite cardinal. Let $\lambda < \kappa$. Therefore $\mathrm{Col}(\lambda, \kappa)$ preserves cardinals $\leq \lambda$. In other words, for G $\mathrm{Col}(\lambda, \kappa) = \mathbb{P}$-generic over V and $\theta < \lambda$, $\mathrm{V} \vDash$ "$\theta = |\theta|$" iff $\mathrm{V}[G] \vDash$ "$\theta = |\theta|$".

Proof .:.

Just by downward absoluteness, if $\mathrm{V}[G] \vDash$ "$\theta = |\theta|$", then clearly $\mathrm{V} \vDash$ "$\theta = |\theta|$", because if $\mathrm{V}[G]$ has no bijections from smaller ordinals to θ then neither does V. So suppose $\mathrm{V} \vDash$ "$\theta = |\theta|$", but $\mathrm{V}[G] \vDash$ "$|\theta| = \rho < \theta$" as witnessed by a bijection $f : \rho \to \theta$ in $\mathrm{V}[G]$.

$f \in \mathrm{V}[G]$ has a name $\dot{f} \in \mathrm{V}^{\mathbb{P}}$ and by Corollary 4 B • 5, there is some $p \in G$ where $p \Vdash$ "$\dot{f}$ is a function from $\check{\rho}$ to $\check{\theta}$". We'd like to do the following that doesn't actually work. It *does* provide motivation, however, of continually deciding more and more of f. The $< \lambda$-closure of $\mathbb{P}$ allows us to decide all of f with a single condition.

Construct a $\leqslant$-decreasing sequence $\langle p_\alpha \in G : \alpha < \rho \rangle$ where $\exists \beta < \theta$ ($p_\alpha \Vdash$ "$\dot{f}(\check{\alpha}) = \check{\beta}$"). We would do this recursively: $p_0 = p$ and for $f(\alpha) = \beta$, let $q_\alpha \in G$ be such that $q_\alpha \Vdash$ "$\dot{f}(\check{\alpha}) = \check{\beta}$". Then we find a common extension $p_{\alpha+1} \leqslant q_\alpha, p_\alpha$. At limit stages $\gamma < \rho$, we would appeal to $< \lambda$-closure to find a $p_\gamma \leqslant p_\alpha$ for all

$\alpha < \gamma$. Then a $p^* \leqslant p_\alpha$ for all $\alpha < \rho$ (which is supposed to exist by $< \lambda$-closure) has

$$f = \{\langle \alpha, \beta \rangle : p^* \Vdash \text{“}\dot{f}(\check{\alpha}) = \check{\beta}\text{”}\} \in \mathrm{V}.$$

This implies $f \in \mathrm{V}$ is a bijection from ρ to θ, contradicting that $\mathrm{V} \vDash$ “$\theta = |\theta|$”.

The issue with this approach is that this construction of $\langle p_\alpha : \alpha < \rho \rangle$ takes place outside of V. In other words, because V doesn't have access to G, it cannot form this sequence. This is especially obvious when V is some countable transitive model where it's clear V doesn't contain all countable sequences (V thinks $\mathbb{P}$ is $< \lambda$-closed just in case all sequences *in* V have lower bounds, but many sequences outside V may not). So we must use a slightly different approach with the same motivating idea just translated into terms of dense sets. But if the reader understands the idea above, this is enough, as the actual approach below doesn't add much understanding.[v]

For each $\alpha < \rho$, consider

$$D_\alpha = \{q \leqslant p : \exists \beta < \theta\ (q \Vdash \text{“}\dot{f}(\check{\alpha}) = \check{\beta}\text{”})\}.$$

This will be open by Corollary 4 B • 4 (1), and also dense below p. To see density, let $q \leqslant p$ be arbitrary. Let H be generic with $q \in H$. Thus $\mathrm{V}[H] \vDash$ “$f(\alpha) = \beta$” for some β. There is then some $r \in H$ forcing this: $r \Vdash$ “$\dot{f}(\check{\alpha}) = \check{\beta}$”. A common extension $q^* \leqslant r, q$ then yields a condition $q^* \in D_\alpha$. So D_α is dense below p.

Claim 1

For each $\gamma \leq \rho$, $\bigcap_{\alpha<\gamma} D_\alpha$ is open (by Corollary 4 B • 4 (1)) and dense below p.

Proof .:.

Suppose the result holds for all ordinals below γ. Let $q \leqslant p$ be arbitrary. Choose $p_\beta \in \bigcap_{\alpha<\beta} D_\alpha$ for $\beta < \gamma$. Without loss of generality, choose the p_βs so that they are $\leqslant$-decreasing (density allows this) with $p_0 \leqslant q$. As $\gamma < \rho < \lambda$, by $< \lambda$-closure, there is some $q^* \leqslant p_\beta$ for every $\beta < \gamma$. By density of D_γ, there is some $p_\gamma \leqslant q^* \leqslant q$ in D_γ and in fact in $\bigcap_{\alpha<\gamma} D_\alpha$. Thus $\bigcap_{\alpha<\gamma} D_\alpha$ is dense below p. ⊣

Note that each $p^* \in D_\rho$ decides all of $\dot{f}$. In particular, for $p^* \in G \cap D_\rho$,

$$f = \{\langle \alpha, \beta \rangle : p^* \Vdash \text{“}\dot{f}(\check{\alpha}) = \check{\beta}\text{”}\} \in \mathrm{V},$$

tells us that $f \in \mathrm{V}$ and we get the contradiction as before. ⊣

This same proof generalizes to show that if a poset $\mathbb{P}$ is $< \lambda$-closed, then $\mathbb{P}$ preserves cardinals $\leq \lambda$.

[v] it does, however, motivate the definition of the *distibutivity* of a poset.

Section 6. Forcing ¬CH

The above forcing shows that cardinals are not absolute between transitive models of set theory. Now we will show both that CH is independent of ZFC,[vi] and that we can both preserve cardinals and change *cardinality*. To argue this, we will need a little more technology. The argument given in Corollary 5 • 7 from Lemma 5 • 6 tells us that $<\lambda$-closure of a poset implies preserving cardinals $\leq \lambda$. To argue that we preserve cardinals $> \lambda$, we need to talk more about names in the next subsection.

But first, let's introduce the poset, and then show this gives what we want. Recall that $\mathcal{P}(\omega)$ can be identified with characterisitic functions $f : \omega \to 2$: $X \subseteq \omega$ is just $\{n < \omega : \chi_X(n) = 1\}$ where $\chi_X(n) = 1$ if $n \in X$ and 0 otherwise. If we want to add a subset of ω, we could then consider the poset which looks at finite approximations of characteristic functions: finite partial functions from ω to 2. If we want to add a *lot* of partial functions to bump up the size of $\mathcal{P}(\omega)$, we can instead index them: adding in $\{f_\alpha \in {}^{\omega}2 : \alpha < \kappa\}$ for κ some cardinal. This is equivalent to adding in a single function $f : \kappa \times \omega \to 2$ where each f_α is just the slice $n \mapsto f(\alpha, n)$. So this is what we are approximating.

6 • 1. Definition

Let κ be a cardinal of V. Define $\mathrm{Add}(\aleph_0, \kappa) \in \mathrm{V}$ to be the poset consisting of functions p where (interpreted in V)

- $|p| < \aleph_0$;
- $\mathrm{dom}(p) \subseteq \kappa \times \omega$, and $\mathrm{im}(p) \subseteq \{0, 1\} = 2$.

We say $p \leqslant q$ iff $p \supseteq q$.

So if we consider the forcing relation, for $g = \bigcup G$, $p \Vdash$ "$\check{p} \subseteq \dot{g}$". As with $\mathrm{Col}(\lambda, \kappa)$, we can show this does what we want. Note that we freely identify $g : \kappa \times \omega \to 2$ as a function $g : \kappa \to {}^{\omega}2$ just by taking α to the map g_α defined by $g_\alpha(n) = g(\alpha, n)$.

6 • 2. Theorem

Let $\kappa > |\mathcal{P}(\omega)|^{\mathrm{V}}$ be an infinite, regular cardinal. Let G be $\mathrm{Add}(\aleph_0, \kappa) = \mathbb{P}$ generic over V. Therefore $g = \bigcup G \in \mathrm{V}[G]$ yields an injection from κ to ${}^{\omega}2$. In particular, $\mathrm{V}[G] \vDash$ "$|\mathcal{P}(\omega)| \geq |\kappa|$".

Proof .:.

It should be clear that g is a function by compatibility of G. By considering for each $\alpha < \kappa$ and $n < \omega$

$$D_{\alpha,n} = \{p \in \mathbb{P} : \langle \alpha, n\rangle \in \mathrm{dom}(p)\},$$

which is clearly dense (recall each $p \in \mathbb{P}$ is finite, so we can just add $\langle\langle \alpha, n\rangle, 1\rangle$ to p and get an extension in $D_{\alpha,n}$), it should be clear that $g : \kappa \times \omega \to 2$: $f \in D_{\alpha,n} \cap G$ has $\langle \alpha, n\rangle \in f \subseteq g$.

It then suffices to show that g is injective, or rather the map $\alpha \mapsto g_\alpha$ is injective, where $g_\alpha(n) = g(n, \alpha)$. To do this, for each $\alpha \neq \beta < \kappa$, consider the set

$$E_{\alpha,\beta} = \{p \in \mathbb{P} : \exists n < \omega(\langle \alpha, n\rangle, \langle \beta, n\rangle \in \mathrm{dom}(p) \wedge p(\alpha, n) \neq p(\beta, n))\}.$$

So if $p \in E_{\alpha,\beta} \cap G$, then then g_α and g_β disagree somewhere. Note that each $E_{\alpha,\beta}$ is dense, since each $p \in \mathbb{P}$ is finite: there are only finitely many n where $\langle \alpha, n\rangle, \langle \beta, n\rangle \in \mathrm{dom}(p)$. Hence some n beyond all of these yields an extension $p^* = p \cup \{\langle\langle \alpha, n\rangle, 1\rangle, \langle\langle \beta, n\rangle, 0\rangle\} \leqslant p$ with $p^* \in E_{\alpha,\beta}$. Therefore, $G \cap E_{\alpha,\beta} \neq \emptyset$ for each α, β, implying $g_\alpha \neq g_\beta$ for each $\alpha \neq \beta < \kappa$. So $\mathrm{V}[G]$ has an injection from κ to $({}^{\omega}2)^{\mathrm{V}[G]}$, and therefore to $\mathcal{P}(\omega)^{\mathrm{V}[G]}$. So $\mathrm{V}[G] \vDash$ "$|\mathcal{P}(\omega)| \geq |\kappa|$". ⊣

This doesn't tell us, however, that κ is preserved. A priori, we could have $\mathrm{V}[G] \vDash$ "$|\kappa| = \aleph_1$" so that the above theorem says $\mathrm{V}[G] \vDash$ "$|\mathcal{P}(\omega)| \geq \aleph_1$", which we already know is true since $\mathrm{V}[G] \vDash$ ZFC. To show that κ is preserved, we basically need to show that there aren't any bijections from smaller cardinals. This, in essence, amounts to showing

[vi] Since L ⊨ ZFC + CH, this shows CH is relatively consistent with ZFC, so we will merely show ZFC + ¬CH is relatively consistent with ZFC.

that there aren't too many "choices" $\mathbb{P}$ can allow, and this is related to the concept of antichains.

§ 6 A. Antichains

Density has a close connection to *antichains*. Again, this is a general topological concept that one can prove is equivalent to the definition below in the context of posets. But we have no need for the general definition.

6 A • 1. Definition

Let $\mathbb{P}$ be a poset. A set $A \subseteq \mathbb{P}$ is an *antichain* iff any two distinct $p, q \in A$ have $p \perp q$.

Maximal antichains also have some nice properties with forcing. One thing that will show this is the following result.

6 A • 2. Lemma

Let $\mathbb{P}$ be a poset. Let $A \subseteq \mathbb{P}$ be an antichain. Therefore A is $\subseteq$-maximal iff every $p \in \mathbb{P}$ has some $q \in A$ where p, q are compatible.

Proof .:.

Suppose A is maximal. Suppose $p \in \mathbb{P}$ is incompatible with every element of A. Thus $A \cup \{p\}$ is an antichain extending A, contradicting maximality.

So suppose A is not maximal. Hence there is some antichain $\mathcal{A} \subseteq \mathbb{P}$ with $A \subsetneq \mathcal{A}$. Any $p \in \mathcal{A} \setminus A$ has, since $\mathcal{A}$ is an antichain, $p \perp q$ for each $q \in A$. Thus there is an element of $\mathbb{P}$ with no $q \in A$ where p and q are compatible. This is the contrapositive of the desired direction. ⊣

Note that this has a similar flavor to density: a dense set D allows you to always extend to enter D. Similarly, a maximal antichain A allows you to always find an incompatible element to enter A.

6 A • 3. Result

Let $\mathbb{P}$ be a poset. Suppose every $p \in \mathbb{P}$ has an extension $p^* \in \mathbb{P}$ (meaning there are no bottom nodes).

- Let $A \subseteq \mathbb{P}$ be an antichain. Therefore $\mathbb{P} \setminus A$ is dense.
- In fact, every dense set contains a $\subseteq$-*maximal* antichain.

Proof .:.

To show that $\mathbb{P} \setminus A$ is dense, let $p \in \mathbb{P}$ be arbitrary. If $p \in A$, then an extension $p^* \leqslant p$ cannot be in A (as p^* and p are compatible with the obvious common extension p^*). If $p \notin A$, then $p \leqslant p$ has $p \in \mathbb{P} \setminus A$. Hence $\mathbb{P} \setminus A$ is dense.

Now suppose $D \subseteq \mathbb{P}$ is open and dense. We will show that there is a $\subseteq$-maximal antichain $A \subseteq D$. We know by Zorn's lemma that there is a $\subseteq$-maximal element in the set of antichains $\{A \subseteq D : A \text{ is an antichain of } \mathbb{P}\}$. So it suffices to show that this maximal element $\mathcal{A}$ is a maximal antichain in the context of the rest of $\mathbb{P}$. So let $p \in \mathbb{P}$ be arbitrary. As D is dense, there is some $p^* \in D$ extending p. Now working in D the same reasoning in Lemma 6 A • 2 tells us that there is some element $q \in \mathcal{A}$ compatible with p^*. But then q is compatible with p: there is an $r \leqslant q$ and $r \leqslant p^* \leqslant p$. ⊣

So there is a nice interplay between dense sets, and maximal antichains. How does this help us? Well, antichains represent choices: if G is $\mathbb{P}$-generic over V with $A \in$ V a maximal antichain, then $G \cap A$ is a singleton. Moreover, the above result tells us that for G to be generic, G must intersect all maximal antichains.

6 A • 4. Corollary

Let $\mathbb{P} \in$ V be a poset. Let A be a maximal antichain in V. Let G be $\mathbb{P}$-generic over V. Therefore, $|G \cap A| = 1$.

Proof .:.

Clearly as any two elements of G are compatible while any two elements of A are incompatible, $|G \cap A| \leq 1$. Consider the downward closure of A, $A\downarrow$, which is dense by Lemma 6 A • 2: any $p \in \mathbb{P}$ has some $q \in A$ where p and q are compatible, and therefore there is some common extension $q^* \in A\downarrow$. By density, $G \cap A\downarrow \neq \emptyset$ and so there is some $q^* \leqslant q \in A$ where $q^* \in G \cap A\downarrow$. Since G is closed upward, $q \in G$, showing $|G \cap A| \geq 1$ and so we have equality. ⊣

This is useful, because we can now talk about what kinds of antichains $\mathbb{P}$ has.

6 A • 5. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let κ be a cardinal of V. $\mathbb{P}$ is *κ-cc* (has the *κ-chain condition*) iff every antichain $A \in \mathrm{V}$ of $\mathbb{P}$ has size $|A|^{\mathrm{V}} < \kappa$. We say $\mathbb{P}$ is *ccc* if it is $\aleph_1$-cc.

We introduce this, because κ-cc posets preserve cardinals and cofinalities $\geq \kappa$ for κ regular. We show this only for ccc posets, but the proof generalizes. Again, it's a good exercise to see how this changes for κ-cc posets when κ is singular.

6 A • 6. Theorem

Let $\mathbb{P}$ be ccc. Therefore, $\mathbb{P}$ preserves all cardinals and cofinalities, meaning if $\mathrm{V} \vDash$ "$\alpha = \mathrm{cof}(\beta)$", then $\mathrm{V}[G] \vDash$ "$\alpha = \mathrm{cof}(\beta)$", for any G that is $\mathbb{P}$-generic over V.

Proof .:.

Suppose not. We have two possibilities.

- A cofinality $\mathrm{cof}^{\mathrm{V}}(\beta)$ is not preserved. Thus $\mathrm{cof}^{\mathrm{V}}(\beta)$—a regular cardinal in V—is not regular in $\mathrm{V}[G]$.
- A cardinal κ is not preserved. If κ was a limit cardinal in V, then for $\mathrm{V}[G] \vDash$ "$|\kappa| = \lambda < \kappa$" has $(\lambda^+)^{\mathrm{V}}$ a regular cardinal of V no longer regular in $\mathrm{V}[G] \vDash$ "$|(\lambda^+)^{\mathrm{V}}| \leq |\kappa| = \lambda$". Similarly, if κ is a successor cardinal in V, then it's no longer regular in $\mathrm{V}[G]$.

So it suffices to show that every regular $\kappa \in \mathrm{V}$ is regular in $\mathrm{V}[G]$. Let $\vec{\alpha} \in \mathrm{V}[G]$ be a λ-length sequence in κ with $\lambda < \kappa$. We will show this is bounded in $\mathrm{V}[G]$. Let $\dot{\vec{\alpha}}$ be a $\mathbb{P}$-name for $\vec{\alpha}$, and let $p \in \mathbb{P}$ force that $\dot{\vec{\alpha}}$ is a function from $\check{\lambda}$ to $\check{\kappa}$. In V, we can consider the possible values of $\dot{\alpha}_\xi$ for each $\xi < \lambda$:

$$A_\xi = \{\beta < \kappa : \exists p \in \mathbb{P}\ (p \Vdash \text{“}\dot{\vec{\alpha}} \text{ is a function from } \check{\lambda} \text{ to } \check{\kappa} \text{ and } \dot{\alpha}_\xi = \check{\beta}\text{”})\}.$$

Note that this is the result of an antichain of $p \in \mathbb{P}$: no two compatible p, q can force different values of $\dot{\alpha}_\xi$. In other words, for each $\beta \in A_\xi$, let $p_\beta \leq p$ have $p_\beta \Vdash$ "$\dot{(\alpha)}_\xi = \check{\beta}$". Therefore, $\mathcal{A}_\xi = \{p_\beta \in \mathbb{P} : \beta < \kappa\}$ is an antichain. Since $\mathbb{P}$ is ccc, $\mathcal{A}_\xi$ is countable, and thus so is A_ξ for each $\xi < \lambda$. But then $\sup_{\xi<\lambda} A_\xi$ is bounded in κ since κ is regular, $\lambda < \kappa$, and each $|A_\xi| \leq \aleph_0$. Therefore for $\kappa > \beta > \sup_{\xi<\lambda} A_\xi$, each $q \in \mathbb{P}$ forces "$\dot{\alpha}_\xi < \check{\beta}$", meaning $\dot{\vec{\alpha}}_G = \vec{\alpha}$ can't be unbounded in κ. ⊣

Note that being ccc or κ-cc is a statement about V: V may have fewer antichains than $\mathrm{V}[G]$ does.

§ 6 B. Showing we actually did force ¬CH

Let's return to $\mathrm{Add}(\omega, \kappa)$ where κ is regular. We can pretty easily show that $\mathrm{Add}(\omega, \kappa)$ is ccc.

6 B • 1. Lemma

For every ordinal κ, $\mathbb{P} = \mathrm{Add}(\omega, \kappa) \in \mathrm{V}$ is ccc.

Proof .:.

Clearly if $\mathbb{P}$ is countable (e.g. if κ is countable), then every antichain is countable. So let $A \subseteq \mathbb{P}$ be an uncountable subset of $\mathbb{P}$. Consider the set of domains of $p \in A$: $D = \{\mathrm{dom}(p) : p \in A\}$. D must also be uncountable, since each $d \in D$ has only countably many (in fact, finitely many) functions from d to 2, so if D were countable, then would A be too.

Claim 1 (The Δ-System Lemma)

There is an uncountable $D' \subseteq D$ and $r \in \mathbb{P}$ such that any two distinct $p, q \in B$ have $p \cap q = r$. In other words, D' forms a Δ-system.

Now we can lift this into A: $A' = \{p \in A : \mathrm{dom}(p) \in D'\}$, another uncountable set, but this time, for any two distinct $p, q \in A$, $\mathrm{dom}(p) \cap \mathrm{dom}(q)$ is some fixed, finite $X \subseteq \kappa \times \omega$. Since there are only countably many functions from X to 2, there must be some $p : X \to 2$ with uncountably many $q, q' \in A'$ with $q \restriction X = q' \restriction x = p$.

Note that this implies any two $p, q \in A'$ are compatible: $p \cup q$ is a function since p and q never disagree. $p \cup q$

is still a finite partial function from $\kappa \times \omega$ to 2, and thus A cannot be an antichain. So the only antichains of $\mathbb{P}$ are countable. ⊣

Proving the Δ-system lemma isn't particularly interesting. It's a common idea that can be found in most any standard, introductory reference for set theory, e.g. [2].

6 B • 2. Corollary

Let κ be an uncountable cardinal in V. Let G be $\mathrm{Add}(\omega, \kappa) = \mathbb{P}$-generic over V. Therefore κ is not collapsed. In particular, $\kappa = \omega_2$ has $\mathrm{V}[G] \vDash$ "$|\wp(\omega)| \geq \aleph_2$".

Proof .:.

By Lemma 6 B • 1 $\mathrm{Add}(\omega, \kappa)$ is ccc. So by Theorem 6 A • 6, κ is still a cardinal in $\mathrm{V}[G]$. For $\kappa = \omega_2^{\mathrm{V}}$, we then have $\kappa > \omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[G]}$ and thus $\mathrm{V}[G] \vDash$ "$|\wp(\omega)| \geq \kappa = \omega_2$" by Theorem 6 • 2. So $\mathrm{V}[G] \vDash \mathsf{ZFC} + \neg\mathsf{CH}$. ⊣

This just yields a lower bound on $2^{\aleph_0}$ in $\mathrm{V}[G]$. But how do we actually calculate $(2^{\aleph_0})^{\mathrm{V}[G]}$? The answer lies in counting names for subsets of ω. Note that for any particular $x \subseteq \omega$ in $\mathrm{V}[G]$, there are a proper class of names for x just by considering $\dot{x} \cup \{\langle \check{\alpha}, p\rangle\}$ for $\alpha \in \mathrm{Ord}$ and $p \notin G$ where $\dot{x} \in \mathrm{V}^{\mathbb{P}}$ is any $\mathbb{P}$-name for x.

§ 6 C. Nice names

A "nice name" is just a name that has the sort of properties you would want it to have as a subset of another name. There are a variety of different kinds of names one can consider. Firstly, consider the following.

6 C • 1. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Let $x \in \mathrm{V}[G]$ be arbitrary with name $\dot{x}$. Let $y \subseteq x$ be in $\mathrm{V}[G]$. A *kinda nice name* for y is a $\tau \in \mathrm{V}^{\mathbb{P}}$ such that $\mathrm{dom}(\tau) \subseteq \mathrm{dom}(\dot{x})$.

6 C • 2. Result

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Let $y \subseteq x \in \mathrm{V}[G]$ be arbitrary with $y \in \mathrm{V}[G]$. Therefore there is a kinda nice name for y.

Proof .:.

We know y has some name $\dot{y} \in \mathrm{V}^{\mathbb{P}}$. Consider
$$\pi = \{\langle \sigma, p\rangle \in \mathrm{dom}(\dot{x}) \times \mathbb{P} : p \Vdash \text{“}\sigma \in \dot{y} \wedge \sigma \in \dot{x}\text{”}\}.$$
Clearly $\pi_G \subseteq y$, since any $p \in G$ with $\langle \sigma, p\rangle \in \pi$ has $\sigma_G \in \dot{y}_G = y$. Similarly, any $\sigma_G \in y$ has some $\sigma' \in \mathrm{dom}(\dot{x})$ with $\mathrm{V}[G] \vDash$ "$\sigma_G = \sigma'_G$". This is forced by some $p \in \mathbb{P}$ where then $\langle \sigma', p\rangle \in \pi$ and so $\sigma'_G \in \pi$. Thus $y \subseteq \pi_G$, and so we have equality. ⊣

The benefit of nice names is that they allow us to consider just names of a certain form rather than all names, which again form a proper class. In particular, we have the following result.

6 C • 3. Result

Let $\mathbb{P} \in \mathrm{V}$ be a poset. For any $\mathbb{P}$-name $\dot{x}$, there are at most $2^{|\mathrm{dom}(\dot{x}) \times \mathbb{P}|}$ kinda nice names for subsets of $\dot{x}$.

Proof .:.

Every nice name for a subset of $\dot{x}_G$ (where G is generic) is in $\wp(\mathrm{dom}(\dot{x}) \times \mathbb{P})^{\mathrm{V}}$. ⊣

In particular, if $\mathbb{P}$ is countable, then $2^{\aleph_0} = (2^{\aleph_0})^{\mathrm{V}}$ in $\mathrm{V}[G]$. To see this, in V, there are at most $2^{|\mathrm{dom}(\check{\omega}) \times \mathbb{P}|} = 2^{\aleph_0}$ kinda nice names for subsets of ω. In particular, since $\mathbb{P}$ is ccc and so preserves cardinals, if $\mathrm{V} \vDash \mathsf{CH}$, then $\mathrm{V}[G] \vDash$ "$2^{\aleph_0} = (2^{\aleph_0})^{\mathrm{V}} = \aleph_1^{\mathrm{V}} = \aleph_1$" and so $\mathrm{V}[G] \vDash \mathsf{CH}$.

Another kind of nice name uses antichains in conjunction with chain conditions to do a better job at counting.

6 C • 4. Definition

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Let $x \in \mathrm{V}[G]$ be arbitrary with name $\dot{x}$. Let $y \subseteq x$ be in $\mathrm{V}[G]$. A *nice name* for y is a $\dot{y} \in \mathrm{V}^{\mathbb{P}}$ such that

- $\mathrm{dom}(\dot{y}) \subseteq \mathrm{dom}(\dot{x})$, i.e. $\dot{y}$ is a kinda nice name; and
- for each $\tau \in \mathrm{dom}(\dot{y})$, $\{p \in \mathbb{P} : \langle \tau, p\rangle \in \dot{y}\}$ is an antichain.

Equivalently, $\dot{y}$ is of the form $\bigcup_{\sigma \in \mathrm{dom}(\dot{x})} \{\sigma\} \times A_\sigma$ where each A_σ is an antichain of $\mathbb{P}$ or else $\emptyset$.

6 C • 5. Result

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Let $y \subseteq x \in \mathrm{V}[G]$ be arbitrary with $y \in \mathrm{V}[G]$. Therefore there is a nice name for y.

Proof .:.

We know there is a kinda nice name $\dot{y} \in \mathrm{V}^{\mathbb{P}}$ for y. For each $\sigma \in \mathrm{dom}(\dot{y}) \subseteq \mathrm{dom}(\dot{x})$, let $A_\sigma = \{p \in \mathbb{P} : p \Vdash$ "$\sigma \in \dot{y}$"$\}$. A_σ is non-empty, of course, since $\langle \sigma, p\rangle \in y$ implies $p \in A_\sigma$. Of the antichains contained in this set, let $\mathcal{A}_\sigma$ be maximal among the subsets of A_σ. Therefore, every $p \in \mathbb{P}$ that forces "$\sigma \in \dot{y}$" is compatible with an element of $\mathcal{A}_\sigma$. So consider the name

$$\pi = \bigcup_{\sigma \in \mathrm{dom}(\dot{y})} \{\sigma\} \times \mathcal{A}_\sigma.$$

This is clearly a nice name, so it suffices to show $\pi_G = y$.

To show $y \subseteq \pi_G$, let $\sigma_G \in y$ have a $p \in G$ forcing $\sigma \in \dot{y}$, meaning $p \in A_\sigma$. There must then be some $q \in \mathcal{A}_\sigma \cap G$ compatible with p and thus $\langle \sigma, q\rangle \in \pi$, meaning $q \Vdash$ "$\sigma \in \dot{y}$".

Similarly, for $\sigma_G \in \pi_G$, we have $\langle \sigma, p\rangle \in \pi$ for some $p \in G$, meaning $p \in \mathcal{A}_\sigma$ and thus $p \Vdash$ "$\sigma \in \dot{y}$" so that $\sigma_G \in y$. Hence $\pi_G \subseteq y$, and so we have equality: $\pi_G = y$. This means π is a nice name for y. ⊣

The above has been stated in a somewhat concrete way, but alternatively, we can say that for any two names $\dot{x}, \dot{y} \in \mathrm{V}^{\mathbb{P}}$, there is a nice name $\tau \in \mathrm{V}^{\mathbb{P}}$ for a subset of $\dot{x}$ such that $p \Vdash$ "$\dot{y} \subseteq \dot{x} \to \tau = \dot{y}$" for every $p \in \mathbb{P}$.

6 C • 6. Corollary

Let $\mathbb{P} \in \mathrm{V}$ be κ^+-cc, and let $\dot{x}$ be a $\mathbb{P}$-name. Therefore, there are at most $|\mathbb{P}|^{\kappa \cdot |\mathrm{dom}(\dot{x})|}$ nice names for subsets of $\dot{x}$.

Proof .:.

There are at most $|\mathbb{P}|^\kappa \le \kappa$-sized subsets of $\mathbb{P}$. Hence there are at most that many antichains. Since each nice name is given by a function from $\mathrm{dom}(\dot{x})$ to antichains of $\mathbb{P}$, there are at most $|\mathbb{P}|^{\kappa \cdot |\mathrm{dom}(\dot{x})|}$ many nice names for subsets of $\dot{x}$. ⊣

In particular, in V, there is a bijection between this ordinal $|\mathbb{P}|^{\kappa \cdot |\mathrm{dom}(\dot{x})|}$ and the nice names for subsets of $\dot{x}$. So in $\mathrm{V}[G]$, there is still this bijection that—with the help of G to interpret the $\mathbb{P}$-names—yields a surjection from this ordinal to $\wp(x)$. Hence we can say $\mathrm{V}[G] \vDash$ "$|\wp(x)| \le \left|\left(|\mathbb{P}|^{\kappa \cdot |\mathrm{dom}(\dot{x})|}\right)^{\mathrm{V}}\right|$". If $\mathbb{P}$ is ccc and thus preserves cardinals, this simplies to $\mathrm{V}[G] \vDash$ "$|\wp(x)| \le \left(|\mathbb{P}|^{|\mathrm{dom}(\dot{x})|}\right)^{\mathrm{V}}$".

6 C • 7. Corollary

Let κ be a regular, uncountable cardinal of V such that $\mathrm{V} \vDash$ "$\kappa^{\aleph_0} = \kappa$". Let G be $\mathrm{Add}(\aleph_0, \kappa)$-generic over V. Therefore $\mathrm{V}[G] \vDash$ "$2^{\aleph_0} = \kappa$".

Proof .:.

By Lemma 6 B • 1, $\mathrm{Add}(\aleph_0, \kappa)$ is ccc. We know from Theorem 6 • 2 that $\mathrm{V}[G] \vDash$ "$2^{\aleph_0} \ge |\kappa|$", and $\mathrm{V}[G] \vDash$ "$|\kappa| = \kappa$" by preservation of cardinals: Theorem 6 A • 6. By counting nice names for subsets of $\check{\omega}$—which has $|\mathrm{dom}(\check{\omega})| = \aleph_0$—it follows by Corollary 6 C • 6 that $\mathrm{V}[G]$ has at most $(\kappa^{\aleph_0})^{\mathrm{V}} = \kappa$ subsets of ω, meaning $\mathrm{V}[G] \vDash$ "$2^{\aleph_0} \le \kappa$", and thus we have equality. ⊣

Section 7. *Forcing* CH

We've seen that we can force ¬CH pretty easily, but it took some work to confirm that CH fails in the generic extension. Similarly, we can pretty easily force that CH holds in the generic extension, but it will take some work to show this. We will take the expected approach: add in some surjection from $\aleph_1$ and $\wp(\omega)$. A worry one might have is that both ω_1 and $\wp(\omega)$ might change in the generic extension: perhaps one of the following holds:

1. $\omega_1^{\mathrm{V}[G]} \neq \omega_1^{\mathrm{V}}$; or
2. $\wp(\omega)^{\mathrm{V}} \neq \wp(\omega)^{\mathrm{V}[G]}$.

We will need to confirm that this doesn't happen: ω_1 isn't collapsed, and we don't add too many subsets of ω. Note that the failure of (2) implies the failure of (1) for us, since if we have a bijection between $\aleph_1^{\mathrm{V}}$ and $\wp(\omega)^{\mathrm{V}} = \wp(\omega)^{\mathrm{V}[G]}$, then $\mathrm{V}[G] \vDash$ "$\aleph_1^{\mathrm{V}} = |\wp(\omega)| \geq \aleph_1$" and we already know $\mathrm{V}[G] \vDash$ "$\aleph_1 \geq \aleph_1^{\mathrm{V}}$". We know already that $\leq \omega$-closed posets preserve $\aleph_1$, but they also preserve $\wp(\omega)$.

7 • 1. Lemma

Suppose $\mathbb{P} \in \mathrm{V}$ is a $\leq \kappa$-closed poset. Suppose G is $\mathbb{P}$-generic over V. Therefore, $\wp(\kappa)^{\mathrm{V}[G]} = \wp(\kappa)^{\mathrm{V}}$.

Proof .:.

The basic idea is that $\mathbb{P}$ being $\leq \kappa$-closed means that we can collect together κ-much information in V already. The motivating idea is as follows, although the real argument is in the next paragraph. In particular, for $y \subseteq \kappa$ with $y \in \mathrm{V}[G]$, we have a kinda nice name $\dot{y} \in \mathrm{V}^{\mathbb{P}}$ for y. For each $\alpha < \kappa$, we either have $\mathrm{V}[G] \vDash$ "$\alpha \in y$" or $\mathrm{V}[G] \vDash$ "$\alpha \notin y$" and thus we have some element of the poset p_α that either forces "$\check{\alpha} \in \dot{y}$" for "$\check{\alpha} \notin \dot{y}$". By continually expanding, we get a $\leqslant$-decreasing sequence of elements in the poset which continually decide more and more of $\dot{y}$. Hence there is some $p \in \mathbb{P}$ with $p \leqslant p_\alpha$ for each $\alpha < \kappa$. This p then decides whether any α is in y: $y = \{\alpha : p \Vdash \text{“}\check{\alpha} \in \dot{y}\text{”}\} \in \mathrm{V}$, implying $\wp(\kappa)^{\mathrm{V}[G]} \subseteq \wp(\kappa)^{\mathrm{V}}$. The other containment is obvious since $\mathrm{V}[G] \subseteq \mathrm{V}$.

As with Corollary 5 • 7, the above argument actually needs to be translated in terms of dense sets. A terse argument in that style is given below: let $\dot{y}$ be a kinda nice name for a subset of κ. For each α, consider $D_\alpha = \{p \in \mathbb{P} : p \Vdash \text{“}\check{\alpha} \in \dot{y}\text{” or } p \Vdash \text{“}\check{\alpha} \notin \dot{y}\text{”}\}$. Each D_α is dense and open. By $\leq \kappa$-closure, $\bigcap_{\alpha<\kappa} D_\alpha = D_\kappa$ is also dense and open. Any $p \in D_\kappa \cap G$ yields that $\dot{y}_G = y = \{\alpha : p \Vdash \text{“}\check{\alpha} \in \dot{y}\text{”}\} \in \mathrm{V}$, meaning $\wp(\kappa)^{\mathrm{V}[G]} \subseteq \wp(\kappa)^{\mathrm{V}}$. ⊣

7 • 2. Corollary

Let $\mathbb{P} \in \mathrm{V}$ be a $\leq \omega$-closed poset. Suppose G is $\mathbb{P}$-generic over V. Therefore $\omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[G]}$ and $\wp(\omega)^{\mathrm{V}} = \wp(\omega)^{\mathrm{V}[G]}$.

So we will consider the following poset, adding a bijection between $\aleph_1$ and $\wp(\omega)$. To make this countably closed, we can't work with finite functions as we have been doing before: the countable union of finitely many functions isn't necessarily finite. So we use the next best idea: *countable* partial functions. Since these will still have relatively small domain compared to $\aleph_1$, we have enough flexibility when using them as approximations.

7 • 3. Definition

Let $\mathrm{Fn}_{<\aleph_1}(\omega_1, \wp(\omega))$ be the poset of *partial* functions of size $< \aleph_1$ (i.e. countable) from ω_1 to $\wp(\omega)$ ordered by inclusion: $p \leqslant q$ iff $p \supseteq q$.

It should be clear that $\mathrm{Fn}_{<\aleph_1}(\omega_1, \wp(\omega))$ is countably closed, since the union of any countable chain is still countable, and is obviously still a partial function from ω_1 to $\wp(\omega)$.

7 • 4. Theorem

Let G be $\mathbb{P} = \mathrm{Fn}_{<\aleph_1}(\omega_1, \wp(\omega))$-generic over V. Therefore $\mathrm{V}[G] \vDash \mathrm{CH}$.

Proof .:.

Really this just amounts to showing that $\bigcup G = g$ is a surjection from ω_1^{V} to $\wp(\omega)^{\mathrm{V}}$. By countable closure, Lemma 7 • 1 tells us that $\wp(\omega)^{\mathrm{V}[G]} = \wp(\omega)^{\mathrm{V}}$ and $\omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[G]}$, meaning g would be a surjection from $\omega_1^{\mathrm{V}[G]}$ to $\wp(\omega)^{\mathrm{V}[G]}$ and so $\mathrm{V}[G] \vDash \mathsf{CH}$. Since the two interpretations are equal, we just write "$\wp(\omega)$" and "$\aleph_1$".

But that $g : \omega_1 \to \wp(\omega)$ is a surjection is clear: for each $\alpha < \omega_1$ and each $x \in \wp(\omega)$, the following are dense

$$D_\alpha = \{p \in \mathbb{P} : \alpha \in \mathrm{dom}(p)\} \qquad \text{and} \qquad E_x = \{p \in \mathbb{P} : x \in \mathrm{im}(p)\}.$$

To see that D_α is dense, just extend any $p \in \mathbb{P}$ with $\langle \alpha, \emptyset \rangle$. To see that E_x is dense, just note that $p \in \mathbb{P}$ being countable implies $\mathrm{dom}(p) \neq \omega_1$ and thus we can choose some $\alpha \in \omega_1 \setminus \mathrm{dom}(p)$ and extend p with $\langle \alpha, x \rangle$. This new (partial) function remains countable and so is in E_x.

But this means each $x \in \wp(\omega)$ has a $p \in G \cap E_x$ where then $x \in \mathrm{im}(p) \subseteq \mathrm{im}(g)$ so that $\wp(\omega) \subseteq \mathrm{im}(g)$ and g is surjective. Given that $\mathrm{dom}(g) = \omega_1$ (by the density of the D_αs), we get the result: $\mathrm{V}[G] \vDash$ "$|\wp(\omega)| = \aleph_1$". ⊣

There are actually a great number of posets that force CH. For example, $\mathrm{Add}(\omega, 1)$ does this. In fact, $\mathrm{Add}(\omega, 1)$ forces $\diamondsuit$. It's not a bad exercise (although moderately difficult) to show that this holds.

In general, we have many different options when it comes to adding a generic G with certain properties. This is in part due to the vagueness of "approximation" when using a poset of sets supposed to approximate G. Many of these posets turn out to be equivalent in the sense that a generic $G \subseteq \mathbb{P}$ yields a generic $H \subseteq \mathbb{Q}$ where $\mathrm{V}[G] = \mathrm{V}[H]$. For example, the forcing we used with $\mathrm{Add}(\omega, \kappa)$—Cohen forcing—is equivalent to the subposet where all conditions have domains that not only are finite subsets of ω, but are actual natural numbers: $\mathrm{dom}(p) = n$ for some $n < \omega$.

There are also many posets that are not equivalent, but that *can* give similar generics. For example, forcing with $\mathrm{Col}(\lambda, \kappa)$ collapses $|\kappa|$ to λ, but leaves $|\lambda| = \lambda$ in the generic extension. If instead of conditions of size $< \lambda$ we consider *finite* conditions, we still end up with a generic G with $\bigcup G = g$ as a surjection from λ to κ, but we also end up with a surjection from ω to κ. This means κ is (and so subsequently all cardinals $\leq \kappa$, including λ, are) collapsed down to ω.[vii]

This is all just to say that it's generally not difficult to come up with a poset that adds some object serving whatever purpose you want *in the ground model*. But it's far more difficult to show it doesn't muck things up in the generic extension. This is the purpose of the discussion of antichains, nice names, and further ideas not covered here.

[vii] To see that we collapse κ to ω if we consider the poset $\mathbb{P}$ of *finite* partial functions from λ to κ (ordered by inclusion), just consider $g \restriction \omega = \bigcup G \restriction \omega$. Each $p \in \mathbb{P}$ is finite, and so $\mathrm{dom}(p) \subsetneq \omega$. For each $\alpha < \kappa$, there is some $n \in \omega \setminus \mathrm{dom}(p)$ where then $p \cup \{\langle n, \alpha \rangle\} \in D_\alpha = \{p \in \mathbb{P} : \alpha \in \mathrm{im}(p \restriction \omega)\}$. This implies each D_α is dense and so $p \in G \cap D_\alpha$ implies $\alpha \in \mathrm{im}(p \restriction \omega) \subseteq \mathrm{im}(g \restriction \omega)$ implying $\kappa \subseteq \mathrm{im}(g \restriction \omega)$ and therefore $g \restriction \omega$ is a surjection onto κ, meaning $\mathrm{V}[G] \vDash$ "$|\kappa| = \aleph_0$". This argument doesn't work with $\mathrm{Col}(\lambda, \kappa)$, since conditions there can be infinite and so contain all of ω in their domains, preventing us from extending into D_α.

Bibliography

[1] Melvin Chris Fitting, *Intuitionistic Logic Model Theory and Forcing*, Studies in Logic and the Foundations of Mathematics, vol. 60, North-Holland, 1969.

[2] Kenneth Kunen, *Set Theory*: *An Introduction to Independence Proofs*, 6th ed., Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, 1995.

[3] Andrzej M. Zarach, *Replacement* $\nrightarrow$ *Collection*, Gödel '96: Logical foundations of mathematics, computer science and physics—Kurt Gödel's legacy, Brno, Czech Republic, August 1996, proceedings (Petr Hájek, ed.), Lecture Notes in Logic, vol. 6, Springer-Verlag, 1996.

Appendix A. Gory Detail of the Forcing Relation and Its Definability

It's not recommended to read any of this section. The results of it are useful, but the proofs are long, technical, and uninteresting. I will repeat for emphasis: *do not read this section if you do not have to*. This is mostly for the curious and the skeptical. First we show the definability of each relation $\{\langle p, \vec{\tau}\rangle \in \mathbb{P} \times V^{\mathbb{P}} : p \Vdash \text{“}\varphi(\vec{\tau})\text{”}\}$ for φ a formula. To do this, we just straight up define a relation $\Vdash^*$ with all the properties we'd like it to have, and then we show that it is equivalent to $\Vdash$. The motivation behind the definition is a result about $\Vdash$. First, we have the following useful fact.

A • 1. Definition

Let $\mathbb{P}$ be a poset with $p \in \mathbb{P}$. A set $D \subseteq \mathbb{P}$ is *dense below* p iff for every $q \leqslant p$, there is some $r \leqslant q$ with $r \in D$. Equivalently, D is dense below p iff $D \cup (\mathbb{P} \setminus \mathbb{P}_{\leq p})$ is dense in $\mathbb{P}$ (here $\mathbb{P}_{\leq p} = \{q \in \mathbb{P} : q \leqslant p\}$).

A • 2. Result

Let $\mathbb{P} \in V$ be a poset appropriate for forcing. Let G be $\mathbb{P}$-generic for V with $p \in G$. Therefore, $G \cap D \neq \emptyset$ for every $D \subseteq \mathbb{P}$ that is dense below p.

Proof .:.

As $\mathbb{P}$ is appropriate for forcing, we can always extend. In particular, a set D is dense iff $D \setminus \{q \in \mathbb{P} : p \leqslant q\}$ is dense (removing an initial segment doesn't change long-term behavior of being able to extend into the set). Hence $D \cup (\mathbb{P} \setminus \mathbb{P}_{\leq p})$ is dense implies $D \cup (\mathbb{P} \setminus \{q \in \mathbb{P} : q \leqslant p \vee p \leqslant q\})$ is dense. Hence G has a non-empty intersection with this. So there is some $q \in G$ in D or else not comparable to p.

A • 3. Motivation

Let $\mathbb{P} \in V$ be a poset appropriate for forcing. Let φ be a formula. Therefore, the following are equivalent.

- $p \Vdash \varphi$;
- $\forall p^* < p \ (p^* \Vdash \varphi)$;
- $D = \{p^* < p : p^* \Vdash \varphi\}$ is dense below p.

The proof of this result will follow from the rest of our work in this section.

A • 4. Definition

Let $\mathbb{P}$ be a poset. We define $p \Vdash^* \text{“}\varphi(\vec{\tau})\text{”}$, read as p $*$-forces “$\varphi(\vec{\tau})$”, by structural induction on φ and $\mathbb{P}$-name rank of $\vec{\tau}$.

- $p \Vdash^* \text{“}\tau_1 = \tau_2\text{”}$ iff for each $\langle \sigma_1, q_1\rangle \in \tau_1$, D_1 is dense below p; and D_2 is too for each $\langle \sigma_2, q_2\rangle \in \tau_2$; where
$$D_1 = \{p^* \leqslant p : p^* \leqslant q_1 \to \exists \langle \sigma, q\rangle \in \tau_2 \ (p^* \leqslant q \wedge p^* \Vdash^* \text{“}\sigma = \sigma_1\text{”})\}$$
$$D_2 = \{p^* \leqslant p : p^* \leqslant q_2 \to \exists \langle \sigma, q\rangle \in \tau_1 \ (p^* \leqslant q \wedge p^* \Vdash^* \text{“}\sigma = \sigma_2\text{”})\}.$$
- $p \Vdash^* \text{“}\tau_1 \in \tau_2\text{”}$ iff $\{p^* \leqslant p : \exists \langle \sigma, q\rangle \in \tau_2 \ (p^* \leqslant q \wedge q \Vdash^* \text{“}\sigma = \tau_1\text{”})\}$ is dense below p.
- $p \Vdash^* \text{“}\varphi(\vec{\tau}) \wedge \psi(\vec{\tau})\text{”}$ iff $p \Vdash^* \text{“}\varphi(\vec{\tau})\text{”}$ and $p \Vdash^* \text{“}\psi(\vec{\tau})\text{”}$.
- $p \Vdash^* \text{“}\neg\varphi(\vec{\tau})\text{”}$ iff every $p^* \leqslant p$ has $p \nVdash^* \text{“}\varphi(\vec{\tau})\text{”}$.
- $p \Vdash^* \text{“}\exists x \ \varphi(x, \vec{\tau})\text{”}$ iff $\{p^* \leqslant p : \exists \sigma \in V^{\mathbb{P}} \ (p^* \Vdash^* \ \varphi(\sigma, \vec{\tau}))\}$ is dense below p.

So we always have $p \Vdash^* \text{“}\emptyset = \emptyset\text{”}$ just vacuously. So to confirm whether $p \Vdash^* \text{“}\{\langle \emptyset, p\rangle\} = \{\langle \emptyset, q\rangle\}\text{”}$, we need to see whether (plugging $\sigma_1 = \emptyset = \sigma_2$ and $q_1 = p, q_2 = q$ into Definition A • 4)

$$\begin{aligned} D_1 &= \{p^* \leqslant p : p^* \leqslant p \to (p^* \leqslant q \wedge p^* \Vdash \text{“}\emptyset = \emptyset\text{”})\} \\ &= \{p^* \leqslant p : p^* \leqslant q \wedge p^* \Vdash \text{“}\emptyset = \emptyset\text{”}\} \\ &= \{p^* \leqslant p : p^* \leqslant q\} \quad \text{is dense below } p \text{, and} \\ D_2 &= \{p^* \leqslant p : p^* \leqslant q \to (p^* \leqslant p \wedge p^* \Vdash^* \text{“}\emptyset = \emptyset\text{”})\} \\ &= \{p^* \leqslant p : p^* \in \mathbb{P}\} \quad \text{is dense below } p. \end{aligned}$$

Here, D_1 represents where $\tau_1 = \{\langle\emptyset, p\rangle\}$ will be a subset of $\tau_2 = \{\langle\emptyset, q\rangle\}$ according to p. Similarly, D_2 represents when τ_2 will be a subset of τ_1 according to p: always. So $p \Vdash^*$ "$\tau_1 = \tau_2$" iff (because D_2 is clearly dense below p) $\{p^* \leqslant p : p^* \leqslant q\}$ is dense below p. Note that with a $\mathbb{P}$-generic filter G, $p \in G$ implies $G \cap D_1 \neq \emptyset$, which implies $q \in G$ and thus $(\tau_1)_G = \{\emptyset\} = (\tau_2)_G$.

This gives some motivation that $\Vdash^*$ is well-defined: because we're always decreasing $\mathbb{P}$-name rank in the atomic formulas, eventually we go down to the $\mathbb{P}$-name $\emptyset$ where equality and membership are easy to calculate. Of course, this doesn't mean $\Vdash^*$ is easy to calculate, as it can be unclear whether certain sets are dense or not.

Inductively, we have the following result about $\Vdash^*$.

A • 5. Lemma

Let $\mathbb{P} \in \mathrm{V}$ be a poset appropriate for forcing. Let φ be a formula and $\vec{\tau}$ $\mathbb{P}$-names. Therefore, the following are equivalent.

1. $p \Vdash^*$ "$\varphi(\vec{\tau})$";
2. $\forall p^* \leqslant p$ ($p^* \Vdash^*$ "$\varphi(\vec{\tau})$");
3. $D = \{p^* \leqslant p : p^* \Vdash^* \text{"}\varphi(\vec{\tau})\text{"}\}$ is dense below p, i.e. $D \cup (\mathbb{P} \setminus \mathbb{P}_{\leq p})$ is dense in $\mathbb{P}$.

Proof .:.

- That (2) implies (1) is immediate since $p \leqslant p$. That (2) implies (3) is also immediate since then $D = \mathbb{P}_{\leq p}$.

- Suppose (1) holds, working towards (2). We proceed by structural induction on φ and the $\mathbb{P}$-rank of $\vec{\tau}$. More precisely, let φ be the $<_{\text{lex}}$-least formula with some $\vec{\tau}$ where (1) holds but (2) fails. Then we set $\vec{\tau}$ to be witnesses of least $\mathbb{P}$-rank. Let $p^* \leqslant p$ be such that $p^* \nVdash$ "$\varphi(\vec{\tau})$".
  - If "$\varphi(\vec{\tau})$" is of the form "$\tau_1 = \tau_2$", "$\tau_1 \in \tau_2$", or "$\exists x\ \psi(x, \vec{\tau})$", then (2) follows easily, since those sets being dense below p implies they are dense below p^*.
  - If "$\varphi(\vec{\tau})$" is of the form "$\neg\psi(\vec{\tau})$", then every $p^* \leqslant p$ has $p^* \nVdash^*$ "$\psi(\vec{\tau})$". In particular, for any $p^{**} \leqslant p^* \leqslant p$, $p^{**} \nVdash^*$ "$\psi(\vec{\tau})$" and therefore $p^* \Vdash^*$ "$\neg\psi(\vec{\tau})$" by definition. So (2) holds.
  - If "$\varphi(\vec{\tau})$" is of the form "$\theta(\vec{\tau}) \wedge \psi(\vec{\tau})$", then $p \Vdash^*$ "$\theta(\vec{\tau})$" and $p \Vdash^*$ "$\psi(\vec{\tau})$" so inductively every $p^* \leqslant p$ has $p^* \Vdash^*$ "$\theta(\vec{\tau})$" and $p^* \Vdash^*$ "$\psi(\vec{\tau})$" and so the conjunction is $*$-forced: $p^* \Vdash^*$ "$\varphi(\vec{\tau})$" so (2) holds.

- Suppose (3) holds, working towards (1). We again proceed by induction on φ and $\vec{\tau}$.
  - If "$\varphi(\vec{\tau})$" is of the form "$\tau_1 = \tau_2$", "$\tau_1 \in \tau_2$", or "$\exists x\ \psi(x, \vec{\tau})$", then (2) follows easily just from properties of denseness. Explicitly, if $\{p^* \leqslant p : D' \text{ is dense below } p^*\}$ is dense below p, then D' is dense below p: we just extend twice. To show (1), we just need to notice that we don't need to restrict the dense set definitions of Definition A • 4 to extensions of p. Then we're working with the same dense sets for all elements and thus we can apply this observation.
  - If "$\varphi(\vec{\tau})$" is of the form "$\neg\psi(\vec{\tau})$", then suppose (1) fails: there is some $p^* \leqslant p$ with $p^* \Vdash^*$ "$\psi(\vec{\tau})$". By density of D, there is a $p^{**} \Vdash^*$ "$\neg\psi(\vec{\tau})$", meaning $p^{**} \nVdash$ "$\psi(\vec{\tau})$". But this contradicts that (1) implies (2) since p^* $*$-forces it and $p^{**} \leqslant p^*$. Hence (1) holds.
  - Suppose "$\varphi(\vec{\tau})$" is of the form "$\theta(\vec{\tau}) \wedge \psi(\vec{\tau})$". Since
  $$D = \{p^* \leqslant p : p^* \Vdash^* \text{"}\theta(\vec{\tau}) \wedge \psi(\vec{\tau})\text{"}\}$$
  is dense below p, then in particular,
  $$D \subseteq D_0 = \{p^* \leqslant p : p^* \Vdash^* \text{"}\theta(\vec{\tau})\text{"}\}$$
  $$D \subseteq D_1 = \{p^* \leqslant p : p^* \Vdash^* \text{"}\psi(\vec{\tau})\text{"}\}$$
  are both dense below p. Inductively, then $p \Vdash^*$ "$\theta(\vec{\tau})$" and $p \Vdash^*$ "$\psi(\vec{\tau})$", so p $*$-forces the conjunction: $p^* \Vdash^*$ "$\varphi(\vec{\tau})$", meaning so (1) holds. ⊣

This allows us to show the analogues of Definition 4 B • 2 and Theorem 2 B • 6 for $*$-forcing. We unfortunately need to prove these simultaneously rather than focusing on just one or the other.

A • 6. Lemma

Let $\mathbb{P} \in \mathrm{V}$ be a poset appropriate for forcing, $p \in \mathbb{P}$, $\vec{\tau} \in \mathrm{V}^{\mathbb{P}}$, and G $\mathbb{P}$-generic over V. Therefore

(a) $p \Vdash^* $ "$\varphi(\vec{\tau})$" implies $p \Vdash$ "$\varphi(\vec{\tau})$"; and

(b) if $\mathrm{V}[G] \vDash$ "$\varphi(\vec{\tau}_G)$", then there is some $p \in G$ with $p \Vdash^*$ "$\varphi(\vec{\tau})$".

Proof .:.

As before, proceed by structural induction on φ and $\mathbb{P}$-rank of $\vec{\tau}$ (by which I mean induction on the max of the $\mathbb{P}$-ranks of $\vec{\tau}$). Let G be $\mathbb{P}$generic over V with $p \in G$. First we show the result for atomic formulas, and then we induct on φ.

(a) Suppose $p \Vdash^*$ "$\varphi(\vec{\tau})$". We must show $\mathrm{V}[G] \vDash$ "$\varphi(\vec{\tau}_G)$".

- Suppose "$\varphi(\vec{\tau})$" is "$\tau_1 = \tau_2$". We shall show $(\tau_1)_G \subseteq (\tau_2)_G$ in $\mathrm{V}[G]$, because the other containment is similar. So let $(\sigma_1)_G \in (\tau_1)_G$ be arbitrary. Therefore, there is some $q_1 \in G$ with $\langle \sigma_1, q_1 \rangle \in \tau_1$. Since $p^* \Vdash$ "$\tau_1 = \tau_2$", by Definition A • 4, the set
$$D_1 = \{p^* \leqslant p : p^* \leqslant q_1 \to \exists \langle \sigma, q \rangle \in \tau_2 \, (p^* \leqslant q \wedge p^* \Vdash^* \text{"}\sigma = \sigma_1\text{"})\}$$
is dense below p. Hence, as G is generic, $G \cap D_1 \neq \emptyset$. Hence there is some p^* and $\langle \sigma, q \rangle \in \tau_2$ such that
  1. $p^* \in G$;
  2. $p^* \leqslant p, q_1, q$; and
  3. $p^* \Vdash^*$ "$\sigma = \sigma_1$".

  (1) and (2) imply $q \in G$ and thus $\sigma_G \in (\tau_2)_G$. (3) implies by the inductive hypothesis that $p \Vdash$ "$\sigma = \sigma_1$" (we're looking at the same formula φ but now with parameters σ and σ_1 which have maximum $\mathbb{P}$-rank less than the maximum $\mathbb{P}$-rank of τ_2 and τ_1) and so $\mathrm{V}[G] \vDash$ "$(\sigma_1)_G = \sigma_G \in (\tau_2)_G$". As $(\sigma_1)_G$ was arbitrary, it follows that $\mathrm{V}[G] \vDash$ "$(\tau_1)_G \subseteq (\tau_2)_G$". The other containment follows analogously.
- Suppose "$\varphi(\vec{\tau})$" is "$\tau_1 \in \tau_2$". Since $p \Vdash^*$ "$\tau_1 \in \tau_2$", the set
$$D = \{p^* \leqslant p : \exists \langle \sigma, q \rangle \in \tau_2 \, (p^* \leqslant q \wedge q \Vdash^* \text{"}\sigma = \tau_1\text{"})\}$$
is dense below p. In particular, $G \cap D \neq \emptyset$ and so there is a p^* and $\langle \sigma, q \rangle \in \tau_2$ such that
  1. $p^* \in G$;
  2. $p^* \leqslant p, q$; and
  3. $p^* \Vdash^*$ "$\sigma = \tau_1$".

  (1) and (2) imply that $q \in G$ so that $\mathrm{V}[G] \vDash$ "$\sigma_G \in (\tau_2)_G$". (3) implies by the previous case above that $\mathrm{V}[G] \vDash$ "$\sigma_G = (\tau_1)_G$" and thus $\mathrm{V}[G] \vDash$ "$(\tau_1)_G \in (\tau_2)_G$".

(b) Suppose $\mathrm{V}[G] \vDash$ "$\varphi(\vec{\tau})$". We must show there is some $p \in G$ with $p \Vdash^*$ "$\varphi(\vec{\tau})$".

- Suppose $\mathrm{V}[G] \vDash$ "$(\tau_1)_G = (\tau_2)_G$". To see that some $p \in G$ has $p \Vdash^*$ "$\tau_1 = \tau_2$", it suffices to consider the dense set of Lemma A • 5. In particular, consider the set of all $p^* \leqslant p$ such that $p^* \Vdash^*$ "$\tau_1 = \tau_2$". This isn't exactly easy to get a handle on, so instead consider the set D of p where $p \Vdash$ "$\tau_1 = \tau_2$" or we have a conflict with the dense sets of Definition A • 4: either
  (i) there is a $\langle \sigma_1, q_1 \rangle \in \tau_1$ where $p \leqslant q_1$, and for every $\langle \sigma_2, q_2 \rangle \in \tau_2$ and every $q_2^* \leqslant q_2$, if $q_2^* \Vdash^*$ "$\sigma_1 = \sigma_2$" then $q_2^* \perp p$; or
  (ii) there is a $\langle \sigma_2, q_2 \rangle \in \tau_2$ where $p \leqslant q_2$, and for every $\langle \sigma_1, q_2 \rangle \in \tau_1$ and every $q_1^* \leqslant q_1$, if $q_1^* \Vdash^*$ "$\sigma_1 = \sigma_2$", then $q_1^* \perp p$.

  The idea is that there can be no $p \in G$ that satisfies either (i) or (ii). The issue is that (the handling of these two are analogous) if p satisfies (i) along with $\langle \sigma_1, q_1 \rangle \in \tau_1$, we would have $q_1 \in G$ so that $\mathrm{V}[G] \vDash$ "$(\sigma_1)_G \in (\tau_1)_G = (\tau_2)_G$", meaning $\mathrm{V}[G] \vDash$ "$(\sigma_1)_G = (\sigma_2)_G$" for some $\langle \sigma_2, q_2 \rangle \in \tau_2$. By the inductive hypothesis on $\mathbb{P}$-name rank, there is then some $q \in G$ where $q \Vdash^*$ "$\sigma_1 = \sigma_2$". Without loss of generality (G is a filter) we can assume $q \leqslant q_2$ so that by (i) $q \perp p$, contradicting that $q, p \in G$ and G is a filter.

  Thus if there are no $p \Vdash^*$ "$\tau_1 = \tau_2$" in G, then $G \cap D = \emptyset$. So it suffices to show that D is dense, yielding that $G \cap D \neq \emptyset$ and thus there is a $p \in G$ that $*$-forces "$\tau_1 = \tau_2$". So let $p \in \mathbb{P}$ be arbitrary, working towards a $p^* \in D$. Assume without loss of generality that $p \nVdash^* $ "$\tau_1 = \tau_2$". Thus by Definition A • 4 (the other possibility being similar, yielding an extension witnessing to (ii)) there is a $\langle \sigma_1, q_1 \rangle \in \tau_1$ with

$$D_1 = \{p^* \leqslant p : p^* \leqslant q_1 \to \exists \langle \sigma_2, q_2 \rangle \in \tau_2\ (p^* \leqslant q \wedge p^* \Vdash^* \text{"}\sigma_1 = \sigma_2\text{"})\}$$

not dense below p. In particular, there is some $p^* \leqslant p$ with no $p^{**} \in D_1$, meaning for all $p^{**} \leqslant p^*$,

$$p^{**} \leqslant q_1 \wedge \forall \langle \sigma_2, q_2 \rangle \in \tau_2\ (p^{**} \nleqslant q_2 \vee p^{**} \nVdash^* \text{"}\sigma_1 = \sigma_2\text{"}).$$

We now show that p^* satisfies (i). The above shows that in particular, $p^* \leqslant p^*$ has $p^* \leqslant q_1$. Note that if $\langle \sigma_2, q_2 \rangle \in \tau_2$, $q_2^* \leqslant q_2$ and $q_2^* \Vdash^*$ "$\sigma_1 = \sigma_2$", then any common extension $r \leqslant q_2^*, p^*$ has $r = p^{**}$ contradict the above. Thus this would imply $q_2 \perp p^*$ and thus that p^* satisfies (i). Therefore D is dense.

- Suppose $\mathrm{V}[G] \vDash$ "$(\tau_1)_G \in (\tau_2)_G$". This means there is some $\langle \sigma_2, q_2 \rangle \in \tau_2$ with $q_2 \in G$ and $\mathrm{V}[G] \vDash$ "$(\tau_1)_G = (\sigma_2)_G$". By the argument above, there is then some $p \in G$ with $p \Vdash^*$ "$\tau_1 = \sigma_2$". So if $p^* \leqslant p, q_2$, by Lemma A • 5, every $p^{**} \leqslant p^*$ has $p^{**} \leqslant q_2$ and $p^{**} \Vdash^*$ "$\tau_1 = \sigma_2$". Thus by definition, $p^* \Vdash$ "$\tau_1 \in \tau_2$".

So far, we have (a) and (b) for atomic formulas. The inductive steps are much easier, and we prove (a) and (b) by structural induction on φ.

- Suppose φ is "$\theta \wedge \psi$".
  - (a) If $p \Vdash^*$ "$\varphi(\vec{\tau})$" then p $*$-forces each conjunct. By the inductive hypothesis, p forces each conjunct, and thus the conjunction "$\varphi(\vec{\tau})$".
  - (b) If $\mathrm{V}[G] \vDash$ "$\varphi(\vec{\tau}_G$" then inductively there are $p_1, p_2 \in G$ with $p_1 \Vdash^*$ "$\theta(\vec{\tau})$" and $p_2 \Vdash^*$ "$\psi(\vec{\tau})$". As G is a filter, there is some common extension $p \leqslant p_1, p_2$ where then p $*$-forces both (by Lemma A • 5) and thus the conjunction "$\varphi(\vec{\tau})$".
- Suppose φ is "$\neg\psi$". This case is the only reason why we needed to prove (a) and (b) together.
  - (a) Suppose $p \Vdash^*$ "$\neg\psi(\vec{\tau})$" but some generic G has $p \in G$ with $\mathrm{V}[G] \vDash$ "$\psi(\vec{\tau})$". By the inductive hypothesis on (b), there is some $q \in G$ with $q \Vdash^*$ "$\psi(\vec{\tau})$". But then a common extension $p^* \leqslant p, q$ has (by Lemma A • 5) $p^* \Vdash^*$ "$\psi(\vec{\tau})$", contradicting Definition A • 4.
  - (b) Suppose $\mathrm{V}[G] \vDash$ "$\neg\psi(\vec{\tau})$". Consider the set D of $p \in \mathbb{P}$ that decide ψ:

    $$D = \{p \in \mathbb{P} : p \Vdash^* \text{"}\psi(\vec{\tau})\text{"} \vee p \Vdash^* \text{"}\neg\psi(\vec{\tau})\text{"}\}.$$

    It should be clear that D is dense in V, since either we can extend an arbitrary p to a $p^* \Vdash^*$ "$\psi(\vec{\tau})$" or else every $p^* \leqslant p$ doesn't $*$-force "$\psi(\vec{\tau})$", in which case $p \leqslant p$ $*$-forces "$\neg\psi(\vec{\tau})$". Hence $G \cap D \neq \emptyset$ as witnessed by some $p \in G$. We obviously can't have $p \Vdash^*$ "$\psi(\vec{\tau})$" as this would imply inductively that $p \Vdash$ "$\psi(\vec{\tau})$" and thus $\mathrm{V}[G] \vDash$ "$\psi(\vec{\tau})$". Hence $p \Vdash^*$ "$\neg\psi(\vec{\tau})$" witnesses the result.
- Suppose φ is "$\exists x\ \psi$".
  - (a) Suppose $p \Vdash^*$ "$\exists x\ \psi(x, \vec{\tau})$" so that $\{p^* \leqslant p : \exists \sigma \in \mathrm{V}^{\mathbb{P}}\ (p^* \Vdash^* \text{"}\psi(\sigma, \vec{\tau})\text{"})\}$ is dense below p. Thus if G is generic and $p \in G$, then there is some $p^* \leqslant p$ with $p^* \Vdash^*$ "$\psi(\sigma, \vec{\tau})$" for some $\mathbb{P}$-name σ. By the inductive hypothesis, $\mathrm{V}[G] \vDash$ "$\psi(\sigma_G, \vec{\tau}_G)$" and thus $\mathrm{V}[G] \vDash$ "$\exists x\ \psi(x, \vec{\tau}_G)$". As G was arbitrary, it follows that $p \Vdash$ "$\exists x\ \psi(x, \vec{\tau})$".
  - (b) Suppose $\mathrm{V}[G] \vDash$ "$\exists x\ \psi(x, \vec{\tau}_G)$". Let σ_G be a witness to this. Thus $\mathrm{V}[G] \vDash$ "$\psi(\sigma_G, \vec{\tau}_G)$" and so inductively, there is some $p \in G$ with $p \Vdash^*$ "$\psi(\sigma, \vec{\tau})$" and in particular, since every $p^* \leqslant p$ $*$-forces this, by Definition A • 4 $p \Vdash^*$ "$\exists x\ \psi(x, \vec{\tau})$". ⊣

This allows us to prove the desired results about the actual forcing relation.

A • 7. Corollary

Let $\mathbb{P} \in \mathrm{V}$ be a poset appropriate for forcing. Let $p \in \mathbb{P}$, $\vec{\tau} \in \mathrm{V}^{\mathbb{P}}$, and φ a formula. Therefore $p \Vdash^*$ "$\varphi(\vec{\tau})$" iff $p \Vdash$ "$\varphi(\vec{\tau})$".

Proof .:.

The ($\rightarrow$) direction follows from Lemma A • 6 (a). For the ($\leftarrow$) direction, suppose $p \Vdash$ "$\varphi(\vec{\tau})$", but $p \nVdash^*$ "$\varphi(\vec{\tau})$". Hence by Lemma A • 5, $D = \{p^* \leqslant p : p^* \Vdash^*$ "$\varphi(\vec{\tau})$"$\}$ is not dense below p, meaning there is some $p^* \leqslant p$ that cannot be extended into D, i.e. every $p^{**} \leqslant p^*$ has $p^{**} \nVdash$ "$\varphi(\vec{\tau})$", i.e. $p^* \Vdash^*$ "$\neg\varphi(\vec{\tau})$". But then $p^* \Vdash$ "$\neg\varphi(\vec{\tau})$", contradicting that $p \geqslant p^* \Vdash$ "$\varphi(\vec{\tau})$". ⊣

The above proof requires some philosophical assumptions: namely if $\mathbb{P} \in \mathrm{V}$, then any $p \in \mathbb{P}$ has a $\mathbb{P}$-generic G over V with $p \in G$. This is clear when V is countable by Theorem 4 A • 1[viii], but otherwise, one could read the conclusion of the proof above that instead p^* has no generic G with $p^* \in G$.

One consequence of the equivalence between forcing and $*$-forcing is the following from Lemma A • 6 (b).

A • 8. Corollary

Let $\mathbb{P} \in \mathrm{V}$ be a poset appropriate for forcing. Let G be $\mathbb{P}$-generic over V. Let $\vec{\tau} \in \mathrm{V}^{\mathbb{P}}$. Therefore $\mathrm{V}[G] \vDash$ "$\varphi(\vec{\tau}_G)$" iff there is some $p \in G$ with $p \Vdash$ "$\varphi(\vec{\tau})$".

Moreover, we can finally confirm the results of Motivation 2 B • 3. The only interesting case here is (4).

A • 9. Theorem

Let $\mathbb{P}$ be a poset appropriate for forcing. For $p \in \mathbb{P}$, write $p^* \in \mathbb{P}$ for an arbitrary $p^* \leqslant p$ (an arbitrary point in time after p). Let φ be a formula with parameters in $\mathrm{V}^{\mathbb{P}}$. Therefore,

1. $p \Vdash \varphi$ iff every $p^* \Vdash \varphi$;
2. $p \Vdash$ "$\neg\varphi$" iff every $p^* \nVdash \varphi$, i.e. you can conclude it's false iff you will never discover that it's true;
3. $p \Vdash$ "$\varphi \wedge \psi$" iff $p \Vdash \varphi$ and $p \Vdash \psi$;
4. if $p \Vdash$ "$\exists x\ \varphi(x)$" then there is some $p^* \leqslant p$ and τ where $p^* \Vdash$ "$\varphi(\tau)$"; and
5. if $p \Vdash \varphi$, and φ is logically equivalent to ψ, then $p \Vdash \psi$;

Proof .:.

1. This follows from Lemma A • 5 and Corollary A • 7: $p \Vdash \varphi$ implies $p \Vdash^* \varphi$, which implies every $p^* \Vdash^* \varphi$, which implies every $p^* \Vdash \varphi$. The converse follows similarly.
2. This follows from Corollary A • 7 and Definition A • 4.
3. This follows from Definition 2 B • 2.
4. Suppose $p \Vdash$ "$\exists x\ \varphi(x)$". Let $p \in G$ which is $\mathbb{P}$-generic over V. Since $\mathrm{V}[G] \vDash$ "$\exists x\ \varphi(x)$", there is some $\pi \in \mathrm{V}^{\mathbb{P}}$ where $\mathrm{V}[G] \vDash \varphi(\pi_G)$ and thus some condition of G forces this: $q \in G$ has $q \Vdash$ "$\varphi(\pi)$". As G is a filter, there is some common extension $p^* \leqslant p, q$ which then forces "$\varphi(\pi)$".
5. If φ is logically equivalent to ψ, then any generic extension $\mathrm{V}[G] \vDash$ "$\varphi \leftrightarrow \psi$" so if $p \in G$ and $\mathrm{V}[G] \vDash \varphi$, then clearly $\mathrm{V}[G] \vDash \psi$.

[viii] $\mathcal{D} = \{D \in \mathrm{V} : \mathrm{V} \vDash$ "D is dense in $\mathbb{P}$"$\} \subseteq \mathrm{V}$ must also be countable if V is

Appendix A. ZFC *in the Generic Extension*

Our goal is now to prove Theorem 4 B • 6. Doing this amounts mainly to finding the right $\mathbb{P}$-names for certain sets that a particular axiom of ZFC claims the existence of. Now ostensibly, we could just apply a result like Corollary A • 8 and begin by, for each axiom φ of ZFC, finding an element of the poset that forces ZFC. This isn't exactly easy to do if, say, $\mathbb{1}_{\mathbb{P}} \Vdash \mathsf{ZFC}$ (which will be the case if $\mathrm{V} \vDash \mathsf{ZFC}$). Mostly this is because while $\Vdash$ is defined, that doesn't mean it's computable, as whether certain sets are dense isn't always immediate.

We now collect together the implications of how much set theory V satisfies on how much set theory $\mathrm{V}[G]$ satisfies. As a bit of notation, we refer to V as the *ground model* while $\mathrm{V}[G]$ is the *generic extension*. Also, P refers to the powerset axiom while ZF^- refers to $\mathsf{ZF} - \mathsf{P} + \mathsf{Col}$, where Col is the axiom scheme of collection. Collection is strictly stronger than replacement, as there is a complicated forcing where replacement holds in the generic extension, but not collection [3].

A • 1. Theorem

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Therefore,

- $\mathrm{V} \vDash \mathsf{ZF}^-$ implies $\mathrm{V}[G] \vDash \mathsf{ZF} - \mathsf{P}$;
- $\mathrm{V} \vDash \mathsf{ZF}$ implies $\mathrm{V}[G] \vDash \mathsf{ZF}$;
- $\mathrm{V} \vDash \mathsf{ZFC}$ implies $\mathrm{V}[G] \vDash \mathsf{ZFC}$.

To prove these inequalities, we need to prove various closure properties of the generic extension given by appropriate names in the ground model. The existence of these names follows from the amount of set theory the ground model satisfies.

Note that I will often use "a name", "a $\mathbb{P}$-name", and "a name in V" all for the same thing: an element of $\mathrm{V}^{\mathbb{P}}$ for some given element of $\mathrm{V}[G]$.

A • 2. Theorem

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Suppose $\mathrm{V} \vDash \mathsf{ZF} - \mathsf{P}$. Therefore $\mathrm{V}[G] \vDash \mathsf{ZF} - \mathsf{P} - \mathsf{Rep}$, where Rep is the axiom scheme of replacement.

Proof .:.

- The axioms of extensionality and foundation follow from the fact that $\mathrm{V}[G]$ is transitive.
- The empty set axiom follows from the fact that $\emptyset \in \mathrm{V}$ so that $\check{\emptyset} \in \mathrm{V}$ and thus $\check{\emptyset}_G = \emptyset \in \mathrm{V}[G]$.
- Pairing follows easily: for $x, y \in \mathrm{V}[G]$, let $\dot{x}, \dot{y} \in \mathrm{V}^{\mathbb{P}}$ be two $\mathbb{P}$-names for x and y respectively: $\dot{x}_G = x$ and $\dot{y}_G = y$. Consider the $\mathbb{P}$-name in V
$$\tau = \{\langle \dot{x}, \mathbb{1}_{\mathbb{P}} \rangle, \langle \dot{y}, \mathbb{1}_{\mathbb{P}} \rangle\}.$$
Since any filter has $\mathbb{1}_{\mathbb{P}} \in G$, $\tau_G = \{\dot{x}_G, \dot{y}_G\} = \{x, y\}$. Hence $\mathrm{V}[G]$ is closed under pairs, and so as a transitive set, $\mathrm{V}[G] \vDash \mathsf{Pair}$.
- Comprehension requires some work. Let φ be a formula, and $x \in \mathrm{V}[G]$. We'd like to show $\{y \in x : \mathrm{V}[G] \vDash \text{“}\varphi(x, y, \vec{w})\text{”}\} \in \mathrm{V}[G]$ for any parameters $\vec{w} \in \mathrm{V}[G]$.
So let $\vec{\dot{w}}$ be $\mathbb{P}$-names for the parameters, and $\dot{x}$ a $\mathbb{P}$-name for x. For any $y \in x$ (with name $\dot{y} \in \mathrm{dom}(\dot{x})$) such that $\mathrm{V}[G] \vDash \text{“}\varphi(x, y, \vec{w})\text{”}$, there is some $p \in G$ such that $p \Vdash \text{“}\dot{y} \in \dot{x} \wedge \varphi(\dot{x}, \dot{y}, \vec{\dot{w}})\text{”}$. So consider the set
$$\tau = \{\langle \sigma, p \rangle \in \mathrm{dom}(\dot{x}) \times \mathbb{P} : p \Vdash \text{“}\sigma \in \dot{x} \wedge \varphi(\dot{x}, \sigma, \vec{\dot{w}})\text{”}\}.$$
Thus any $\sigma_G \in \tau_G$ has $\langle \sigma, p \rangle \in \tau$ with $p \in G$, and hence $\mathrm{V}[G] \vDash \text{“}\sigma \in x \wedge \varphi(x, \sigma, \vec{w})\text{”}$. The argument given above shows that any $y \in x$ with $\mathrm{V}[G] \vDash \text{“}\varphi(x, y, \vec{w})\text{”}$ has some $\dot{y}$ and $p \in G$ with $\langle \dot{y}, p \rangle \in \tau$. Hence this τ witnesses this arbitrary instance of comprehension, and thus $\mathrm{V}[G] \vDash \mathsf{Comp}$.

- For union, for $X \in \mathrm{V}[G]$, we need to show $\bigcup X \in \mathrm{V}[G]$. Because comprehension holds, we only need to show there is some $Y \in \mathrm{V}[G]$ with $\bigcup X \subseteq Y$, because then we can consider in $\mathrm{V}[G]$ the set $\{y \in Y : \exists x \in X\ (y \in x)\} = \bigcup X$.
  So let $\dot{X}$ be a name for X. Consider the set
  $$\tau = \{\langle \sigma, p\rangle : \exists \langle \sigma', p'\rangle \in \dot{X}\ (\langle \sigma, p\rangle \in \mathrm{dom}(\sigma'))\}.$$
  This clearly works as $x \subseteq \tau_G$ not only for $\langle \dot{x}, p\rangle \in \dot{X}$ with $p \in G$ but for *all* x with $\dot{x} \in \mathrm{dom}(x)$. Hence $\bigcup \dot{X}_G = \bigcup X \subseteq \tau_G$. Thus $\mathrm{V}[G] \vDash \mathsf{Union}$.
- For infinity, just note that the name $\check{\omega} = \{\langle \check{n}, \mathbb{1}_{\mathbb{P}}\rangle : n \in \omega\} \in \mathrm{V}^{\mathbb{P}}$ witnesses that $\omega \in \mathrm{V}[G]$.

With the addition of powerset in the ground model, we also get powerset in the generic extension.

A • 3. Lemma

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Suppose $\mathrm{V} \vDash \mathsf{ZF}$. Therefore $\mathrm{V}[G] \vDash \mathsf{P}$

Proof .:.

Let $x \in \mathrm{V}[G]$. We need to show that $\mathcal{P}(x) \cap \mathrm{V}[G] \in \mathrm{V}[G]$, meaning that there is a set that collects every subset of x that is in $\mathrm{V}[G]$. By Theorem A • 2, $\mathrm{V}[G] \vDash \mathsf{Comp}$ so it suffices to find $Y \in \mathrm{V}[G]$ with $\mathcal{P}(X) \cap \mathrm{V}[G] \subseteq Y$, since then we just consider $\mathcal{P}(x) \cap \mathrm{V}[G] = \{y \in Y : y \subseteq x\} \in \mathrm{V}[G]$.

So let $\dot{x}$ be a name for x. If $\sigma_G \subseteq x$ in $\mathrm{V}[G]$, then there is some $p \in \mathbb{P}$ with $p \Vdash$ "$\sigma_G \subseteq \dot{x}$". So consider
$$\tau = \{\langle \sigma, p\rangle \in \mathcal{P}(\mathrm{dom}(\dot{x}) \times \mathbb{P}) \times \mathbb{P} : p \Vdash \text{“}\sigma \subseteq x\text{”}.\}$$
This is a set in V by the powerset axiom in V. If $\langle \sigma, p\rangle \in \tau$ with $p \in G$, then $\mathrm{V}[G] \vDash$ "$\sigma_G \subseteq x$" and thus $\tau_G \subseteq \mathcal{P}(x) \cap \mathrm{V}[G]$. Recall Result 6 C • 2, which says that any $y \in \mathrm{V}[G]$ with $y \subseteq x$ has a name $\dot{y} \in \mathcal{P}(\mathrm{dom}(\dot{x}) \times \mathbb{P})$. So if $\mathrm{V}[G] \vDash$ "$y \subseteq x$", there is some $p \in G$ with $p \Vdash$ "$\dot{y} \subseteq \dot{x}$", and so $\langle \dot{y}, p\rangle \in \tau$ has $y = \dot{y}_G \in \tau_G$. Hence $\tau_G = \mathcal{P}(x) \cap \mathrm{V}[G]$ witnesses this instance of powerset, and so $\mathrm{V}[G] \vDash \mathsf{P}$. ⊣

This shows that $\mathrm{V} \vDash \mathsf{ZF}$ implies $\mathrm{V}[G] \vdash \mathsf{ZF} - \mathsf{Rep}$. In order to confirm replacement we need the axiom scheme of collection in the ground model. This follows from powerset and replacement, but without powerset, we might not have the axiom scheme of collection. So when we jump from $\mathrm{V} \vDash \mathsf{ZF} - \mathsf{P}$ to $\mathrm{V} \vDash \mathsf{ZF}$, we can confirm two axioms in $\mathrm{V}[G]$: P and Rep.

First we introduce the axiom scheme of collection, and then we show this follows from ZF. We introduce this axiom, because it is used in the proof that $\mathrm{V} \vDash \mathsf{ZF}$ implies $\mathrm{V}[G] \vDash \mathsf{ZF}$. Of course, we could just proof the particular instance(s) we need during the proof, but this isn't exactly instructive.

A • 4. Definition

The axiom scheme of collection (Col) states the following: if φ is a relation on a set D, then there is a set containing φ-relatives of each $x \in D$. Symbolically, Col consists of all formula of the form
$$\forall \vec{w}, D\ (\forall x \in D\ \exists y\ \varphi(x, y, D, \vec{w}) \to \exists R\ \forall x \in D\ \exists y \in R\ \varphi(x, y, D, \vec{w})).$$
where φ is a formula.

The $\vec{w}$ just allow parameters. Note that this clearly stronger than replacement, which requires φ to define a function over D:
$$\forall \vec{w} \forall D\ (\forall x \in D\ \exists! y\ \varphi(x, y, \vec{w}) \to \exists R\ \forall x \in D\ \exists y \in R\ \varphi(x, y, \vec{w})).$$

A • 5. Lemma

$\mathsf{ZF} \vdash \mathsf{Col}$

Proof .:.

For each φ, $\vec{w}$, and D, consider the collection of all relatives of elements in D:
$$R' = \{y : \exists x \in D\ \varphi(x, y, D, \vec{w})\}.$$
Note that this is potentially a proper class. But with powerset, we can consider
$$R = \{y : \exists x \in D\ (\varphi(x, y, D, \vec{w}) \wedge \forall z\ (\varphi(x, z, D, \vec{w}) \to \mathrm{rank}(z) \geq \mathrm{rank}(y)))\}.$$

This will be a set, because we've defined a function $f : D \to \mathrm{V}$ where $f(x)$ is the least rank of a y with $\varphi(x, y, D, \vec{w})$. This yields $f"D \subseteq \mathrm{Ord}$ as a set of ordinals, and thus $R \subseteq \mathrm{V}_{\sup f"D}$ yields that R is a set by comprehension. ⊣

The above idea (considering only the elements of least rank) has been dubbed "Scott's trick".

A • 6. Lemma

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Suppose $\mathrm{V} \vDash \mathsf{ZF} - \mathsf{P} + \mathsf{Col}$. Therefore $\mathrm{V}[G] \vDash \mathsf{Rep}$.

Proof .:.

Let φ be a formula with parameters in $\mathrm{V}[G]$, $D \in \mathrm{V}[G]$. Suppose

$$\mathrm{V}[G] \vDash \text{“}\forall x \in D\ \exists! y\ \varphi(x, y, D)\text{”}. \tag{$*$}$$

We need to find a $\mathbb{P}$-name for the range of φ restricted to D. Note that there is some $p_D \in \mathbb{P}$ forcing $(*)$ (translated with parameters as $\mathbb{P}$-names).

Consider the formula $\psi(p, \sigma, \tau)$ stating:

$$p \in \mathbb{P} \wedge \left(p \Vdash \text{“}\varphi(\sigma, \tau, \dot{D})\text{”} \vee \neg\exists\pi\ p \Vdash \text{“}\varphi(\sigma, \pi, \dot{D})\text{”}\right).$$

In V, for each $\langle\sigma, p\rangle \in \mathrm{dom}(\dot{D}) \times \mathbb{P}$, there is a $\tau \in \mathrm{V}^{\mathbb{P}}$ where $\psi(p, \sigma, \tau)$ holds (τ can be anything if $\neg\exists\pi\ p \Vdash$ “$\varphi(\sigma, \pi, \dot{D})$”). By collection in V, there is a set $R \subseteq \mathrm{V}^{\mathbb{P}}$ where each $\langle\sigma, p\rangle \in \mathrm{dom}(\dot{D}) \times \mathbb{P}$ has a $\tau \in R$. As $R \subseteq \mathrm{V}^{\mathbb{P}}$ is a set, $\rho = R \times \mathbb{P}$ is a $\mathbb{P}$-name.

To see that $\rho_G \in \mathrm{V}[G]$ satisfies our requirements, suppose $\mathrm{V}[G] \vDash$ “$x \in D$”. We can take $x = \sigma_G$ for $\sigma \in \mathrm{dom}(\dot{D})$. Since $(*)$ holds, there is some y where $\mathrm{V}[G] \vDash$ “$\varphi(x, y, D)$”. This is forced by some $p \in \mathbb{P}$: $p \Vdash$ “$\varphi(\sigma, \dot{y}, \dot{D})$”. Hence there is a $\tau \in R$ where $p \Vdash$ “$\varphi(\sigma, \tau, \dot{D})$”, and thus $\mathrm{V}[G] \vDash$ “$\tau_G \in \rho_G \wedge \varphi(x, \tau_G, D)$”, yielding the result. This shows this arbitrary instance of replacement holds in $\mathrm{V}[G]$, and thus $\mathrm{V}[G] \vDash \mathsf{Rep}$. ⊣

So we can conclude $\mathrm{V} \vDash \mathsf{ZF}$ implies $\mathrm{V}[G] \vDash \mathsf{ZF}$. The last thing to consider is choice. There are multiple versions of choice, but we will consider one that's easy to use. In particular, we're using the version that says every set is covered by an ordinal.[ix]

A • 7. Theorem

Let $\mathbb{P} \in \mathrm{V}$ be a poset. Let G be $\mathbb{P}$-generic over V. Suppose $\mathrm{V} \vDash \mathsf{ZFC}$. Therefore $\mathrm{V}[G] \vDash \mathsf{ZFC}$.

Proof .:.

We have by the previous lemmas that $\mathrm{V}[G] \vDash \mathsf{ZF}$. So it suffices to show $\mathrm{V}[G] \vDash \mathsf{AC}$, and so it suffices to show that for any $x \in \mathrm{V}[G]$, there is an $f \in \mathrm{V}[G]$ and an $\alpha \in \mathrm{Ord} \cap \mathrm{V}[G]$ where $\mathrm{V}[G] \vDash$ “$f : \alpha \to x$ is surjective”. Let $\dot{x}$ be a name for x. By AC in V, there is a surjection $F : \alpha \to \mathrm{dom}(\dot{x})$ for some $\alpha \in \mathrm{Ord} \cap \mathrm{V}$. Thus

$$f = \{\langle\langle\!\langle \check{\xi}, F(\xi)\rangle\!\rangle, \mathbb{1}_{\mathbb{P}}\rangle : \xi < \alpha\}$$

works. (Here, $\langle\!\langle a, b\rangle\!\rangle$ is a name for $\langle a_G, b_G\rangle$, an in particular is $\left\{\left\langle\{\langle a, \mathbb{1}_{\mathbb{P}}\rangle\}, \mathbb{1}_{\mathbb{P}}\right\rangle, \left\langle\{\langle a, \mathbb{1}_{\mathbb{P}}\rangle, \langle b, \mathbb{1}_{\mathbb{P}}\rangle\}, \mathbb{1}_{\mathbb{P}}\right\rangle\right\}$.) If we consider f_G, we have that any element is of the form $\langle\xi, F(\xi)_G\rangle$ where $F(\xi) \in \mathrm{dom}(\dot{x})$ and $\xi < \alpha$. And so we can regard f_G as a function from α, and It should be clear that $\mathrm{V}[G] \vDash$ “$x \subseteq \mathrm{im}(f_G)$”. Hence this version of AC holds in $\mathrm{V}[G]$. ⊣

Choice in the generic extension also allows us to confirm a stronger result than in Theorem A • 9. In particular, we don't need to extend p to find a particular name for a witness.

A • 8. Result (Maximum Principle)

Let $\mathbb{P} \in \mathrm{V}$ be a poset with $p \in \mathbb{P}$. Suppose $\mathrm{V} \vDash \mathsf{ZFC}$. Therefore, if $p \Vdash$ “$\exists x\ \varphi(x)$”, then there is some $\sigma \in \mathrm{V}^{\mathbb{P}}$ where $p \Vdash$ “$\varphi(\sigma)$”.

[ix] We get an injection from $g : x \to \alpha$ just by setting $g(y)$ to be the least β with $f(\beta) = y$ where $f : \alpha \to x$ is the surjection. This yields a well-order of x. Note that in ZF, we have that x can always be surjected onto various ordinals, but the reverse is equivalent to x having a well-order.

Proof .:.

Suppose $p \Vdash$ "$\exists x\, \varphi(x)$". This means $D = \{p^* \leqslant p : \exists \sigma \in V^{\mathbb{P}}\ (p^* \Vdash \text{"}\varphi(\sigma)\text{"})\}$ is dense below p. So in V, for each $q \in D$, let σ_q be such a name. To introduce some terminology, an antichain is a subset of $\mathbb{P}$ in which any two elements are incompatible. There is an antichain $\mathcal{A} \subseteq D$ that is maximal in the antichains contained in D (such an antichain exists by Zorn's lemma). Consider the name

$$\sigma = \{\langle \tau, q^* \rangle : \exists r, q\ (q^* \leqslant q \in \mathcal{A} \wedge q^* \leqslant r \wedge \langle \tau, r \rangle \in \sigma_q)\}.$$

It follows that $p \Vdash$ "$\varphi(\sigma)$". To see this, let G be $\mathbb{P}$-generic over V with $p \in G$.

$G \cap \mathcal{A} \neq \emptyset$. To see this, closing $\mathcal{A}$ downwards yields a set A' dense below p (any condition without an extension into this set is incompatible with any element of $\mathcal{A}$, meaning we can add it to $\mathcal{A}$ to yield a larger antichain, contradicting maximality). Thus $G \cap A' \neq \emptyset$ so there is some $p \in G$ with $p \leqslant a$ for some $a \in \mathcal{A}$. Since G is closed upwards, $a \in G \cap \mathcal{A}$.

Therefore, there is some $a \in G \cap \mathcal{A}$. Since any two elements of $\mathcal{A}$ are incompatible while any two elements of G are compatible, $|G \cap \mathcal{A}| = 1$. Thus

$$\sigma_G = \{\langle \tau, a^* \rangle : a^* \leqslant a \wedge \exists r\ (a^* \leqslant r \wedge \langle \tau, r \rangle \in \sigma_a)\}_G = \{\langle \tau, r \rangle \in \sigma_a : r \text{ is compatible with } a\}_G.$$

Since all $r \in G$ are already compatible with a, this is just $\{\tau_G : \exists r \in G\ (\langle \tau, r \rangle \in \sigma_a)\} = (\sigma_a)_G$. Since $a \Vdash$ "$\varphi(\sigma_a)$", we thus have $V[G] \vDash$ "$\varphi(\sigma_G)$", and so $p \Vdash$ "$\varphi(\sigma)$". $\dashv$

One may note that this is actually *equivalent* to AC holding in V.