GOSTS Mini-Talk

Introduction to Determinacy

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Basic Setup

- The basic "infinite game" setup is looking at games of length ω where players **I** and **II** take turns playing elements of ω.
- There are other, longer notions of games, but these are harder to study, or else too easy to lose/win.

I:	n_0		n_2		$\cdots$		n_m	
II:		n_1		n_3		$\cdots$		

- The *play* is $x = \langle n_k : k \in \omega \rangle \in \mathcal{N}$.
- What distinguishes different games is the winning conditions. For each $A \subseteq \mathcal{N}$, we have a different game $G(A)$ where **I** attempts to get $x \in A$ and **II** tries to get $x \notin A$.
- I.e. in $G(A)$, **I** wins iff $x \in A$. (So there are no ties.)

Basic Setup

- Each player is also playing their own real: $x = y * z$ where $y = \langle x(2n) : n \in \omega \rangle$ and $z = \langle x(2n+1) : n \in \omega \rangle$.
- If **I** uses a strategy τ, we frequently write $\tau * z$ for the play where **I** plays with τ and **II** plays $z \in \mathcal{N}$.

I:	$\tau(\emptyset)$		$\tau(z(0))$		$\tau(z(0), z(1))$		$\cdots$	
II:		$z(0)$		$z(1)$		$z(2)$		$\cdots$

- We similarly consider $z * \tau$ if τ is a strategy for **II**.

Basic Setup

- Note that rules don't matter.
- if we want to consider the game where **I** and **II** has to play in a certain tree constructed from possible moves, T, then we just consider

$$A' = (A \cup \{x \in \mathcal{N} : \textbf{II} \text{ broke a rule}^1\}) \setminus \{x \in \mathcal{N} : \textbf{I} \text{ broke a rule}\},$$

- In the game with rules, $G(A, T)$, a player has a winning strategy iff that same player has a winning strategy for $G(A')$.

[1] "**I** broke a rule" iff there's an initial segment $x \restriction n \in T$ but $x \restriction (n+1) \notin T$ and n is even, and similarly for **II**.

Motivation for Accepting Determinacy

Some motivation from semantics:

- We know that $\neg\forall$ is really the same as $\exists\neg$.
- A winning strategy for **I** in $G(A)$ basically says

$$\exists n_0\ \forall n_1\ \exists n_2\ \forall n_3\ \cdots(\langle n_k : k \in \omega\rangle \in A).$$

- The negation of this "should" allow us to "push" the negations through:

$$\begin{aligned} &\neg\exists n_0\ \forall n_1\ \exists n_2\ \cdots\ (\langle n_k : k \in \omega\rangle \in A) \\ \text{iff}\quad &\forall n_0\ \exists n_1\ \forall n_2\ \cdots\ (\langle n_k : k \in \omega\rangle \notin A) \end{aligned}$$

- Writing the *game quantifier* $\Game A$ to mean **I** has a winning strategy in $G(A)$, the above says

$$\neg\Game A \quad \text{iff} \quad \Game\neg A$$

(here $\neg A$ is the complement of A.) This is equivalent to AD.

Closed games

Theorem (Closed Determinacy)

Let $A \subseteq \mathcal{N}$ be closed. Therefore $G(A)$ is determined.

Proof.

- If **II** doesn't have a winning strategy, **I** plays defensively to avoid **II** winning.
- If **II** can force a win no matter what **I** does, then **II** can force a win.
- Since **II** can't force a win at the first stage, there must be some move n_0 by **I** such that **II** still can't force a win after this move.
- But then **I** just continues in this same way, choosing a move that ensures **II** can't force a win.
- The resulting play x is in A since A is closed (we've built up x from things nearby in A since **II** doesn't always win). Thus **I** wins. $\dashv$

Other games

Corollary (Open Determinacy)

Let $A \subseteq \mathcal{N}$ be open. Therefore $G(A)$ is determined.

$\mathcal{N} \setminus A$ is closed, and the game $G(\mathcal{N} \setminus A)$ is basically the same as $G(A)$ except that we have switched the players and added a turn at the beginning.

Result

Let $A \subseteq \mathcal{N}$ be countable. Therefore $G(A)$ is determined.

Proof.

Let $A = \{a_n : n \in \mathbb{N}\}$. Whatever **I** plays, **II** on their mth turn will play

$$\begin{cases} 0 & \text{if the } m\text{th digit of } a_m \text{ isn't } 0 \\ 1 & \text{otherwise.} \end{cases}$$

It follows that the real **I** and **II** build up, x, is different from every $a_m \in A$ and thus **I** loses. ⊣

Determinacy

Theorem (Borel Determinacy)

Let $A \subseteq \mathcal{N}$ be Borel. Therefore $G(A)$ is determined.

Definition

The *Axiom of Determinacy* (AD) is the axiom that $G(A)$ is determined for all $A \subseteq [0, 1]$.

- We know that AD is incompatible with AC, but AD is in fact much stronger.
- We can investigate three games corresponding to the standard, interesting properties for sets of reals:
  - The perfect set property and the game $G^*(A)$
  - The Baire property and the Banach-Mazur game $G^{**}(A)$
  - Lebesgue measurability and the covering game
- The result is that if these games are determined for a set A, then A has the perfect set property etc.
- Hence AD implies every set is Lebesgue measurable, etc., and thus AC fails (which can also be shown fairly directly too).

The Perfect Set Property

The easiest variant game to look at is $G^*(A)$ for $A \subseteq \mathcal{C} = {}^\omega 2$, which is *severely* biased towards **I**.

I:	$\sigma_0 \in {}^{<\omega}2$		σ_1		$\cdots$
II:		$n_0 \in 2$		n_1	$\cdots$

As usual, **I** wins iff the resulting play, $x = \sigma_0{}^\frown n_0{}^\frown \sigma_1{}^\frown n_1 \cdots \in A$.

Lemma

*If **I** wins $G^*(A)$, then A contains a perfect set.*

Proof.

- A contains a perfect set iff there's a $C \subseteq A$ which is the continuous 1-1 image of $\mathcal{C}$.
- Let τ be a winning strategy for **I**. Define $f : \mathcal{C} \to A$ by $f(x) = \tau * x$.
- This f is continuous as $f(x)$ is built up from x.
- f is injective because $x \neq y$ implies $x(n) \neq y(n)$ for some n so that $f(x)(2n+1) \neq f(y)(2n+1)$. So im $f \subseteq A$ is perfect. $\dashv$

The Perfect Set Property

The harder thing to prove is that if **II** wins, then A is countable. Clearly the converse holds.

Lemma

*If **II** wins $G^*(A)$, then A is countable.*

Proof.

- Let τ win for **II**. Say a real $y \in \mathcal{C}$ is *rejected* at a partial play p (where **II** just played) iff no matter what σ **I** plays, $y \not\prec p \frown \sigma^\frown \tau(p, \sigma)$.
- Basically y is rejected p iff playing according to τ ensures y is not the resulting play.
- Since τ wins for **II**, every $y \in A$ is rejected at some stage $p \in {}^{<\omega}2$.
- Moreover, for every $p \in {}^{<\omega}2$, there's only one $y \in A$ that's rejected there. (As we will prove.)
- Mostly this is because we can only play 0s or 1s.

The Perfect Set Property

Lemma

*If **II** wins $G^*(A)$, then A is countable.*

Proof.

- If $y \in A$ is rejected at p and **I** plays some σ with $p^\frown\sigma \lhd y$, then **II** rejecting y means $\tau(p,\sigma)$ is the opposite value of $y(\mathrm{lh}(p)+\mathrm{lh}(\sigma))$. This determines $y \restriction \mathrm{lh}(p)+\mathrm{lh}(\sigma)+1$ and so we can consider **I** playing σ followed by this value, and this determines $y \restriction \mathrm{lh}(p)+\mathrm{lh}(\sigma)+2$, and so on.
- But this gives a surjection from ${}^{<\omega}2$ to A so A is countable. $\dashv$

This tells us if AD holds then every subset of $\mathcal{C}$ has the perfect set property. (Technically we need to play an analogous game where **I** plays $n \in \omega$ coding $\sigma \in {}^{<\omega}2$ and the winning conditions are translated to $\mathcal{N}$, but this is just a formal concern.)

The Covering Game

The covering game is an attempt to show the following.

Result

(AD) *If every measurable $X \subseteq Y$ is Lebesgue null, then Y is Lebesgue null.*

This is false in the world of AC, since Y might be non-measurable.

In an attempt to show this, the covering game's purpose is to cover Y with a set of measure $\varepsilon > 0$. Assuming we can always do this, then Y will have measure 0.

Why does this tell us every set is measurable? For any set X, there's a minimal (modulo null sets) A: $X \subseteq A$ and any measurable A' with $X \subseteq A' \subseteq A$ has $A \setminus A'$ as null. If we consider $A \setminus X$, then every measurable subset of this is null, which tells us $A \setminus X$ is null and thus $A \setminus (A \setminus X) = X$ is measurable.

The Covering Game

What is the covering game?

Let $Y \subseteq [0,1]$ and $\varepsilon > 0$ be given. The idea behind the covering game $G(Y,\varepsilon)$ is that **I** plays a number in Y and **II** tries to cover it with small sets.

I:	$n_0 \in 2$		n_1		n_2		$\cdots$	
II:		H_0		H_1		H_2		$\cdots$

Where **I** builds up an element $x = \sum_{i\in\omega} \frac{n_i}{2^{n+1}} \in \mathbb{R}$, and where H_i is a union of finitely many intervals of rational endpoints of measure $\varepsilon/2^{2(i+1)}$.

I wins $G(Y,\varepsilon)$ iff $x \in Y$ and $x \notin \bigcup_{n\in\omega} H_n$.

This is often re-phrased in terms of **II** playing $m_i \in \omega$ where m_i is the index of H_i in the enumeration of these countably many intervals. But it's easier to think about playing the unions of intervals directly.

The Covering Game

Lemma

*Suppose every measurable subset of Y is null. Let $\varepsilon > 0$ be given. Therefore **I** doesn't have a winning strategy for $G(Y, \varepsilon)$.*

Proof.

- If σ wins for **I**, again define the continuous function from $\mathcal{N}$ to $\mathbb{R}$ by $f(z) = \sigma * z$ where z is the play by **II** (H_i is regarded as the n_ith union of finitely many [...] of measure $\varepsilon/2^{2(i+1)}$ and $z = \langle n_k : k \in \omega \rangle$).
- This f is again continuous because it's built from z.
- The image of a continuous function, e.g. $f"\mathcal{N}$, is $\underset{\sim}{\mathbf{\Sigma}}^1_1$ and so Lebesgue measurable.
- But if **I** wins, $f"\mathcal{N} \subseteq Y$ means $f"\mathcal{N}$ is null.
- But every null set can be completely covered by a play by **II**, meaning **I** couldn't win. ⊣

The Covering Game

Lemma

*Suppose every measurable subset of $Y \subseteq [0,1]$ is null. Let $\varepsilon > 0$ be given. Therefore **I** doesn't have a winning strategy for $G(Y, \varepsilon)$.*

So what happens if **II** wins?

Lemma

*Let $Y \subseteq [0,1]$ and $\varepsilon > 0$. Suppose **II** has a winning strategy for $G(Y, \varepsilon)$. Therefore the outer-measure of Y is at most ε.*

Proof.

- Let τ win for **II**. For each $p \in {}^{<\omega}2$ a play by **I**, let $H(p)$ be the set **II** plays in response (using τ).
- Every $x \in Y$ will thus be in $\bigcup_{p \lhd x} H(p)$ and so $Y \subseteq \bigcup_{p \in {}^{<\omega}2} H(p) = \bigcup_{n \in \omega} \bigcup_{p \in {}^{n}2} H(p)$.
- The measure of $\bigcup_{p \in {}^{n}2} H(p)$ is at most $2^n \cdot (\varepsilon / 2^{2n}) = \varepsilon / 2^n$.
- So the measure of $\bigcup_{p \in {}^{<\omega}2} H(p)$ is at most $\sum_{n \in \omega} \varepsilon / 2^n = \varepsilon$.
- Being covered by this set, the outer-measure of Y is at most ε. ⊣

The Covering Game

Lemma

*Suppose every measurable subset of $Y \subseteq [0, 1]$ is null. Let $\varepsilon > 0$ be given. Therefore **I** doesn't have a winning strategy for $G(Y, \varepsilon)$.*

Lemma

*Let $Y \subseteq [0, 1]$ and $\varepsilon > 0$. Suppose **II** has a winning strategy for $G(Y, \varepsilon)$. Therefore the outer-measure of Y is at most ε.*

- As a result, if AD holds, **II** always wins $G(Y, \varepsilon)$ if every measurable subset of $Y \subseteq [0, 1]$ is null.
- This means the outer-measure of Y must be $\leq \varepsilon$ for every $\varepsilon > 0$, i.e. Y must be null.
- As discussed before, this implies every subset of $[0, 1]$ (and hence $\mathbb{R}$) is measurable.

Banach–Mazur Games

Just as the covering game is best understood as players playing things *other* than natural numbers, we again can consider players playing sets. For $\mathcal{N}$, this means cones.

I:	$\mathcal{N}_{\sigma_0}$		$\mathcal{N}_{\sigma_2}$		$\cdots$	
II:		$\mathcal{N}_{\sigma_1}$		$\mathcal{N}_{\sigma_3}$		$\cdots$

Such that $\sigma_0 \lhd \sigma_1 \lhd \sigma_2 \lhd \cdots$ building up an $x = \bigcup_{n\in\omega} \sigma_n \in \mathcal{N}$.

As usual, **I** wins $G^{**}(A)$ iff $x \in A$, where $A \subseteq \mathcal{N}$.

Recall some definitions:

- A set is nowhere dense iff its complement contains an open dense set.
- A set is meagre iff it's the union of countably many open dense sets.
- A set has the Baire property iff it's symmetric difference with some open set is meagre.

Banach–Mazur Games

An alternative characterization of nowhere dense sets is pretty useful, and not difficult to show.

Result

A set $X \subseteq \mathcal{N}$ is nowhere dense iff for every open $U \neq \emptyset$, there's a $\emptyset \neq V \subseteq U$ with $V \cap X = \emptyset$.

How does the Banach–Mazur game help us? The general idea behind the Banach–Mazur game $G^{**}(A)$ is that

- **II** wins $G^{**}(A)$ iff A is meagre.
- **I** wins $G^{**}(A)$ iff some $\mathcal{N}_\sigma \setminus A$ is meagre.

So if AD holds, then (non-trivially) every A has the Baire property.

Banach–Mazur Games

Result

A set $X \subseteq \mathcal{N}$ is nowhere dense iff for every open $U \neq \emptyset$, there's a $\emptyset \neq V \subseteq U$ with $V \cap X = \emptyset$.

Result

***II** wins $G^{**}(A)$ iff A is meagre.*

Proof.

- Suppose $A = \bigcup_{n\in\omega} A_n$ where each A_n is nowhere dense.
- If **I** has played $\mathcal{N}_{\sigma_n}$ thus far, then using the above result, **II** should play an open $\emptyset \neq V \subseteq \mathcal{N}_\sigma$ disjoint from A_n. WLOG, $V = \mathcal{N}_{\tau_n}$ for some $\tau_n \in {}^{<\omega}\omega$.
- This gives a winning strategy for **II** just by diagonalizing through the A_ns.

Banach–Mazur Games

Result

A set $X \subseteq \mathcal{N}$ is nowhere dense iff for every open $U \neq \emptyset$, there's a $\emptyset \neq V \subseteq U$ with $V \cap X = \emptyset$.

Result

***II** wins $G^{**}(A)$ iff A is meagre.*

Proof.

- So suppose τ wins for **II**. Again, we can define what it means for an $x \in A$ to be rejected at stage $p \lhd x$ (where **II** just played): no matter what **I** plays, **II**'s move using τ will disagree with x.
- As before, we get that every $x \in A$ is rejected at some stage p.
- The perfect set game had $R_p = \{x \in A : x \text{ is rejected at } p\}$ be a singleton.
- We instead get that $R_p \subseteq \mathcal{N}_p$ is nowhere dense. (You can't be rejected at one stage and then a later stage too.)
- Thus $A = \bigcup_{p \in {}^{<\omega}\omega} R_p$ is meagre. ⊣

Banach–Mazur Games

Result

II *wins* $G^{**}(A)$ *iff* A *is meagre.*

Corollary

I *wins* $G^{**}(A)$ *iff* $\mathcal{N}_\sigma \setminus A$ *is meagre for some* $\sigma \in {}^{<\omega}\omega$*.*

This just follows by taking the first move σ by **I**, and then using the strategy for **II** in $G^{**}(\mathcal{N}_\sigma \setminus A)$.

- So how does this give us the Baire property for A?
- It doesn't.
- We need to assume the determinacy of $G^{**}(A \setminus S)$ for a certain set S.

Banach–Mazur Games

For $A \subseteq \mathcal{N}$, define

$$S = \bigcup \{\mathcal{N}_\sigma : \sigma \in {}^{<\omega}\omega \wedge \mathcal{N}_\sigma \setminus A \text{ is meagre}\}.$$

Basically, this is the best approximation of $\mathcal{N} \setminus A$ modulo meagre sets. In particular, S is open and $S \setminus A$ is meagre.

Result

If $G^{**}(A \setminus S)$ is determined, then A has the Baire property.

Proof.

- If **I** wins, then $\mathcal{N}_\sigma \setminus (A \setminus S)$ is meagre for some σ.
- But then $\mathcal{N}_\sigma \setminus A \subseteq \mathcal{N}_\sigma \setminus (A \setminus S)$ is meagre, meaning $\mathcal{N}_\sigma \subseteq S$.
- But then $\mathcal{N}_\sigma \setminus (A \setminus S) = \mathcal{N}_\sigma$ isn't meagre, a contradiction.
- Thus **II** wins, meaning $A \setminus S$ is meagre.
- But since $S \setminus A$ is meagre, $A \triangle S = (S \setminus A) \cup (A \setminus S)$ is meagre. Since S is open, A has the Baire property. ⊣

Unfolding

Thus under AD we have

- Every set has the perfect set property.
- Every set is Lebesgue measurable.
- Every set has the Baire property.

Unfortunately, we can't use full determinacy with ZFC. But often we don't need full determinacy to get nice consequences.

In particular, just from the determinacy of closed games, we get the following which took a long time to show otherwise.

- Every $\undertilde{\Sigma}^1_1$-set has the perfect set property.
- Every $\undertilde{\Sigma}^1_1$-set has the Baire property.

Showing these isn't too difficult and uses an idea called "unfolding".

Unfolding

Recall the perfect set game: $G^*(A)$ for $A \subseteq \mathcal{C} = {}^\omega 2$, biased towards **I**.

I:	$\sigma_0 \in {}^{<\omega}2$		σ_1		$\cdots$
II:		$n_0 \in 2$		n_1	$\cdots$

As usual, **I** wins iff the resulting play, $x = \sigma_0{}^\frown n_0{}^\frown \sigma_1{}^\frown n_1 \cdots \in A$.

For $A \in \underset{\sim}{\Sigma}^1_1$, we have that $A = \exists^{\mathcal{N}} B$ for some closed $B \subseteq \mathcal{C} \times \mathcal{N}$. So let's have **I** not only try to get $x \in \exists^{\mathcal{N}} B$, but also have **I** must find a $y \in \mathcal{N}$ such that $\langle x, y \rangle \in B$.

I:	$\sigma_0, y(0)$		$\sigma_1, y(1)$		$\cdots$
II:		$n_0 \in 2$		n_1	$\cdots$

Where x is as before, and $y = \langle y(n) : n \in \omega \rangle$. We say **I** wins $G^*_u(B)$ iff $\langle x, y \rangle \in B$.

Unfolding

I:	$\sigma_0, y(0)$	$\sigma_1, y(1)$	$\cdots$
II:	$n_0 \in 2$	n_1	$\cdots$

I wins $G^*_u(B)$ iff $\langle x, y\rangle \in B$. By closed determinacy, $G^*_u(B)$ is determined if $B \subseteq \mathcal{C} \times \mathcal{N}$ is closed.

Lemma

*If **I** wins $G^*_u(B)$, then $\exists^{\mathcal{N}} B$ has a perfect subset.*

Proof.

The proof is just as before: a winning strategy gives a function $f : \mathcal{C} \to B$ according to how the game is played. We can regard $f = \langle f_0, f_1\rangle \in {}^{\mathcal{C}}\mathcal{C} \times {}^{\mathcal{C}}\mathcal{N}$, and disregarding the second component still yields that $f_0 : \mathcal{C} \to \mathcal{C}$ is continuous, injective, and $f_0``\mathcal{C} \subseteq \exists^{\mathcal{N}} B$ is a perfect subset. ⊣

Unfolding

I:	$\sigma_0, y(0)$	$\sigma_1, y(1)$	$\cdots$
II:	$n_0 \in 2$	n_1	$\cdots$

I wins $G_u^*(B)$ iff $\langle x, y\rangle \in B$.

Lemma

*If **II** wins $G_u^*(B)$, then $\exists^{\mathcal{N}} B$ is countable.*

Proof.

If **II** wins with τ, then playing with τ means that everything is rejected at some stage $p \in {}^{<\omega}2 \times {}^{<\omega}\omega$. For any given $y(n)$, there can be only one possible $x \in \mathcal{C}$ rejected at the nth stage p. So the $x \in \mathcal{C}$ rejected at p is a countable set, and thus $\exists^{\mathcal{N}} B$ is countable. ⊣

Corollary

Closed determinacy implies $\mathsf{PSP}(\utilde{\boldsymbol{\Sigma}}^1_1)$. *More generally,* $\mathsf{Det}(\utilde{\boldsymbol{\Pi}}^1_n)$ *implies* $\mathsf{PSP}(\utilde{\boldsymbol{\Sigma}}^1_{n+1})$ *for* $n < \omega$.