GOSTS Descriptive Set Theory

Prewellorderings, Scales, and Uniformization

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2021-02-10

Previously

- Last time we showed one direction of the following equivalences:

Theorem

The following are equivalent:

1. $\omega_1^{\mathsf{L}[x]} < \omega_1$.
2. $\mathsf{PSP}(\Sigma_2^1(x))$.
3. $\mathsf{PSP}(\Pi_1^1(x))$.

- Namely, (1) implies (2), and (2) implies (3).
- To get (3) implies (1), we need concepts related to *prewellorders*,
- Basically, a way to talk about the complexity of functions $f : X \to \mathrm{Ord}$ for $X \subseteq \mathcal{N}$.
- This motivates the notion of a *scale* as well as bounds on the lengths of prewellorders.

Adequate Pointclasses

Because we'll be talking a bit in the abstract, it helps to talk about properties common to the bold and lightface pointclasses pointclasses:

Definition

A pointclass $\Gamma \subseteq \mathcal{P}(\mathcal{N})$ is *adequate* iff

- Γ contains all computable relations;
- Γ is closed under computable substitution/preimages;
- Γ is closed under finite unions and intersections; and
- Γ is closed under bounded quantification (over ω).

- So clearly all borel, arithmetical, projective, and analytical pointclasses are adequate.
- Basically, simple operations don't change complexity.

Defining Prewellorders

Definition

A *prewellorder* is a relation $\leqslant$ that is transitive, total, and well-founded.

- This is a *pre*wellorder in the following sense.

Result

Let $\leqslant$ be a prewellorder on X. Define the equivalence relation $x \approx y$ iff $x \leqslant y \leqslant x$. Therefore $\leqslant/\approx$ is a well-order on $\{[x]_\approx : x \in X\}$.

- So what do prewellorders look like?
- Basically just well-orders with clusters of loops, but which don't fundamentally change the order.
- So how do we get examples of prewellorders?

Defining Prewellorders

Definition

A *prewellorder* is a relation $\lesssim$ that is transitive, total, and well-founded.

Definition

A *norm* on a set X is a function $\varphi : X \to \mathrm{Ord}$.

Result (19 C • 5)

Every prewellorder has a norm (its rank function). Moreover, every norm $\varphi : X \to \mathrm{Ord}$ *gives a prewellorder* $x \lesssim y$ *iff* $\varphi(x) \leq \varphi(y)$.

- Note that the norm associated with a prewellorder isn't unique.
- For example, $\varphi = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$ and $\varphi' = \{\langle 0, 0\rangle, \langle 1, 4\rangle\}$ give the same prewellorder.

Defining Prewellorders

Result (19 C • 5)

Every prewellorder has a norm (its rank function). Moreover, every norm $\varphi : X \to \mathrm{Ord}$ gives a prewellorder $x \lesssim y$ iff $\varphi(x) \leq \varphi(y)$.

- Also note that this gives lots of trivial prewellorders: any constant function $\varphi : X \to \{\alpha\}$.
- On the other hand, assuming φ is injective, we get a well-order.
- Any set has lots of prewellorders, even in ZF.
- But we are interested in prewellorders with definability restrictions.

Γ-norms

Definition

Let $\Gamma \subseteq \mathcal{P}(\mathcal{N})$ be a pointclass. $X \subseteq \mathcal{N}$ has a Γ-norm iff there's a norm $\varphi : X \to \mathrm{Ord}$ such that $\leq_\varphi$ and $<_\varphi$ are both in Γ, defined by

$$\begin{array}{lll} x \leq_\varphi y & \text{iff} & x \in X \wedge (y \in X \to \varphi(x) \leq \varphi(y)) \\ x <_\varphi y & \text{iff} & x \in X \wedge (y \in Y \to \varphi(x) < \varphi(y)) \end{array}$$

- Let's consider the complexity here: $x \leq_\varphi x$ iff $x \in X$ so having a Γ-norm implies $X \in \Gamma$ (if Γ is adequate).
- We know the constant 0 function $\varphi = \mathrm{const}_0$ is a norm on every set, but it may not have the best complexity:
- $\mathrm{const}_0(x) < \mathrm{const}_0(y)$ is always false so that $x <_\varphi y$ iff $x \in X$ and $y \notin X$.
- This requires complexity $\Gamma \wedge \neg\Gamma$.

Corollary

If $\Delta \subseteq \mathcal{P}(\mathcal{N})$ is adequate with $\neg\Delta = \Delta$, then every $X \in \Delta$ has a Δ-norm. E.g. each Δ^1_n has this.

Γ-norms

Definition

$\Gamma \subseteq \mathcal{P}(\mathcal{N})$ has the *prewellordering property*, $\mathsf{PWO}(\Gamma)$, iff every $X \in \Gamma$ has a Γ-norm.

Corollary

- $\mathsf{PWO}(\Delta^0_\alpha)$ *and* $\mathsf{PWO}(\Delta^1_n)$ *for every* $\alpha < \omega_1$, $n < \omega$ *(and their boldface counterparts).*
- $\mathsf{PWO}(\Sigma^1_0)$ *(and* $\mathsf{PWO}(\utilde{\Sigma}^0_1)$*)*

Proof.

- $A \in \Sigma^1_0 = \Sigma^0_1$ is $\bigcup_{n\in\omega} \mathcal{N}_{f(n)}$ for some computable f.
- Define $\varphi(x)$ as the least $n \in \omega$ with $x \in \mathcal{N}_{f(n)}$.
- This is a Σ^1_0-norm: each cone is Δ^0_1

$$x \leq_\varphi y \quad \text{iff} \quad x \in A \wedge \exists n \in \omega\, (x \in \mathcal{N}_{f(n)} \wedge \forall m < n\, (y \notin \mathcal{N}_{f(m)}))$$
$$x \leq_\varphi y \quad \text{iff} \quad x \in A \wedge \exists n \in \omega\, (x \in \mathcal{N}_{f(n)} \wedge \forall m \leq n\, (y \notin \mathcal{N}_{f(m)})).$$

⊣

The Prewellordering Property

Definition

$\Gamma \subseteq \mathcal{P}(\mathcal{N})$ has the *prewellordering property*, $\mathsf{PWO}(\Gamma)$, iff every $X \in \Gamma$ has a Γ-norm.

A much harder pointclass to show PWO for is Π^1_1.

Theorem

$\mathsf{PWO}(\Pi^1_1)$

To prove this, recall the following nice properties for Π^1_1-sets.

Lemma

Every $X \in \Pi^1_1$ is the computable preimage of WO.

This is quite nice for us, because we already showed that WO has a norm on it: $x \mapsto \|x\|$ where $\|x\|$ is the length of the well-order coded by $x \in \mathrm{WO}$.

Lemma (19 B • 8 and 19 C • 8)

$\psi(x) = \|x\|$ *is a* Π^1_1*-norm on* WO.

The Prewellordering Property

Lemma

Every $X \in \Pi^1_1$ is the computable preimage of WO, *and $\psi(x) = \|x\|$ is a Π^1_1-norm on* WO.

Theorem

$\mathsf{PWO}(\Pi^1_1)$

Proof.

As a result, we can define a Π^1_1-norm for $X \in \Pi^1_1$ as follows:

- Let $X = f^{-1}``\mathrm{WO}$ for $f : \mathcal{N} \to \mathcal{N}$ computable.
- Define $\varphi = \|f \restriction X\|$. We get that

$$\begin{aligned} x \leq_\varphi y &\leftrightarrow f(x) \leq_\psi f(y) \\ &\leftrightarrow f(x) \in \mathrm{WO} \wedge (f(y) \in \mathrm{WO} \to \|f(x)\| \leq \|f(y)\|) \\ &\leftrightarrow x \in X \wedge (y \in X \to \varphi(x) \leq \varphi(y)). \end{aligned}$$

⊣

The Prewellordering Property

This also lifts to Σ^1_2.

Corollary

$\mathsf{PWO}(\Sigma^1_2)$ *(and* $\mathsf{PWO}(\underset{\sim}{\Sigma}^1_2)$*)*

The idea is actually quite easy and nice, having the following form.

Theorem

If Γ *is adequate and* X *has a* Γ*-norm, then* $\exists^{\mathcal{N}} X$ *has a* $\exists^{\mathcal{N}}\forall^{\mathcal{N}}\Gamma$*-norm. So* $\mathsf{PWO}(\Gamma)$ *implies* $\mathsf{PWO}(\exists^{\mathcal{N}}\forall^{\mathcal{N}}\Gamma)$.

Proof.

- Let φ be a Γ-norm for X.
- Define $\psi : \exists^{\mathcal{N}} X \to \mathrm{Ord}$ just by $\psi(x) = \min\{\varphi(x, y) : \langle x, y\rangle \in X\}$.
- This is a $\exists^{\mathcal{N}}\forall^{\mathcal{N}}\Gamma$-norm:

$$\begin{aligned} x \leq_\psi y &\quad \text{iff} \quad \exists z\, \forall t\, (\langle x, z\rangle \leq_\varphi \langle y, t\rangle) \\ x <_\psi y &\quad \text{iff} \quad \exists z\, \forall t\, (z = t \vee \langle x, z\rangle <_\varphi \langle y, t\rangle). \end{aligned}$$

⊣

The Prewellordering Property

Question

We know $\mathsf{PWO}(\Sigma^1_2)$. *Does this imply* $\mathsf{PWO}(\Sigma^1_1)$ *just because* $\Sigma^1_1 \subseteq \Sigma^1_2$*?*

- The answer here is no: every Σ^1_1-set has a Σ^1_2-norm, but not necessarily a Σ^1_1-norm.
- In fact, these three pointclasses (Σ^1_0, Π^1_1, and Σ^1_2) are the only three analytical pointclasses we can show PWO for under ZFC:

$$\begin{array}{cccccc}
\Sigma^1_0 & \Sigma^1_1 & \Sigma^1_2 & \Sigma^1_3 & \Sigma^1_4 & \\
 & & & & & \cdots \\
\Pi^1_0 & \Pi^1_1 & \Pi^1_2 & \Pi^1_3 & \Pi^1_4 &
\end{array}$$

The Prewellordering Property

ZFC + "V = L":

$$\begin{array}{cccccc}
\Sigma^1_0 & \Sigma^1_1 & \Sigma^1_2 - & \Sigma^1_3 - & \Sigma^1_4 - & \\
 & & & & & \cdots \\
\Pi^1_0 & \Pi^1_1 & \Pi^1_2 & \Pi^1_3 & \Pi^1_4 &
\end{array}$$

ZFC + PD:

$$\begin{array}{cccccc}
\Sigma^1_0 & \Sigma^1_1 & \Sigma^1_2 & \Sigma^1_3 & \Sigma^1_4 & \\
 & & & & & \cdots \\
\Pi^1_0 & \Pi^1_1 & \Pi^1_2 & \Pi^1_3 & \Pi^1_4 &
\end{array}$$

- In particular (assuming Con(ZFC + PD)), PWO(Σ^1_n) is independent for odd $n > 2$.
- For even $n > 2$, Harrington has some (unpublished) work which gives a model where PWO(Σ^1_n) and PWO(Π^1_n) both fail.

The Prewellordering Property

- One question that arises from this picture is can we have both $\mathsf{PWO}(\Sigma^1_n)$ and $\mathsf{PWO}(\Pi^1_n)$?
- The answer is "no", and this is a result of properties more topological in nature.

Theorem (19 D • 2, 4, 6, and 7)

Let Γ be an adequate pointclass. Therefore

- $\mathsf{PWO}(\Gamma)$ *implies* Γ *has the reduction property.*
- Γ *has the reduction property implies* $\neg\Gamma$ *has the separation property.*
- *If* Γ *has a universal set,* Γ *cannot have both the reduction and separation properties.*
- *Every* σ*-algebra (e.g.* Δ^1_n*) has both the reduction and separation properties.*

- What these properties *are* exactly isn't too important for us.
- They at least establish $\neg(\mathsf{PWO}(\Gamma) \wedge \mathsf{PWO}(\neg\Gamma))$ for $\Gamma = \Sigma^1_n$.
- They also tell us we shouldn't be considering Δ^1_n, since they trivially have these properties.

Introducing Scales

Theorem

The following are equivalent:

1. $\omega_1^{\mathsf{L}[x]} < \omega_1$.
2. $\mathsf{PSP}(\Sigma^1_2(x))$.
3. $\mathsf{PSP}(\Pi^1_1(x))$.

- Let's get back on track.
- The main way we'll show (3) implies (1) above is to use Π^1_1-uniformization (aka *Kondô's theorem*).
- In general, the way to show Γ-uniformization is with the scale property on Γ.
- As usual, we can prove the scale property on Π^1_1 and Σ^1_2, but we can't go beyond this in ZFC alone.
- Determinacy axioms can push this further with the same alternating pattern as with PWO.

Introducing Scales

Definition

Let $X \subseteq \mathcal{N}$. A *scale* on X is a sequence $\vec{\varphi} = \langle \varphi_n : n < \omega \rangle$ such that

- each φ_n is a norm on X;
- for all convergent $x \in {}^{\omega}X$, if each $\varphi_n \circ x$ is eventually constant then
  - $\lim x \in X$ and
  - (lower semi-continuity) $\varphi_n(\lim x) \leq \lim(\varphi_n \circ x)$ for each $n < \omega$.

Without lower semi-continuity, $\vec{\varphi}$ is called a *semi-scale*.

- As before, we are interested in scales with definability restrictions.
- We can easily get scales on any X we want by using AC.
- $f : X \to |X|$ a bijection with $\varphi_n = f$ for every n works.
- If $x \in {}^{\omega}X$ is convergent and $f \circ x$ is eventually constant then x is eventually constant. So $\lim x \in \operatorname{im} x \subseteq X$.
- Lower semi-continuity is also easy in this case: $f(\lim x) = \lim(f \circ x)$.
- As before, we are interested in scales with definability restrictions.

Introducing Scales

- So what are some less trivial ways to get scales?
- How could scales possibly be useful?

Result (19 E • 3)

For $X \subseteq \mathcal{N}$ and $\kappa \geq \aleph_0$, the following are equivalent.

- *X has a scale $\vec{\varphi}$ where $\varphi_n(x) < \kappa$ for all $n < \omega$, $x \in X$.*
- *X has a semi-scale where $\vec{\varphi}$ where $\varphi_n(x) < \kappa$ for all $n < \omega$, $x \in X$.*
- *X is κ-suslin.*

- Suslin representations of sets are very important because they allow us to ask questions about trees instead of whatever weird reals we have.
- We will only prove the downward direction. The upward isn't very interesting to me.

Introducing Scales

Result (19 E • 3)

If X has a semi-scale bounded by κ then X is κ-suslin.

Proof.

- A scale is a semi-scale so one direction is obvious.
- If $\vec{\varphi}$ is a semi-scale, consider the tree building up sequences of elements and their norms: $\langle \tau, \rho \rangle \in {}^{<\omega}\omega \times {}^{<\omega}\kappa$ is in T iff there's an $x \in X$ with
  - $\tau \lhd x$;
  - $\rho = \langle \varphi_n(x) : n < \mathrm{lh}(\tau) \rangle$.
- This is a tree over $\omega \times \kappa$. So $\mathfrak{p}[T]$ is κ-suslin.
- $\langle x, \langle \varphi_n(x) : n < \omega \rangle \rangle \in [T]$ for any $x \in X$ so $X \subseteq \mathfrak{p}[T]$.

Introducing Scales

Result (19 E • 3)

If X has a semi-scale bounded by κ then X is κ-suslin.

Proof.

- $\langle \tau, \rho \rangle \in T$ iff there's an $x \in X$ with
  - $\tau \lhd x$;
  - $\rho = \langle \varphi_n(x) : n < \mathrm{lh}(\tau) \rangle$.
- To show $\mathfrak{p}[T] \subseteq X$, if $x \in \mathfrak{p}[T]$, then we get a sequence $\langle x_n \in X : n < \omega \rangle$ witnessing $x \restriction n \in \mathfrak{p}T$.
- $x \restriction n = x_n \restriction n$ so $\langle x_n \in X : n < \omega \rangle$ converges to x with $\langle \varphi_n(x_m) : m < \omega \rangle$ eventually determined and thus constant.
- Hence $\lim x_n = x \in X$. $\dashv$

Introducing Scales

- The motivating result for PWO was that WO had a Π^1_1-norm given by $x \mapsto \|x\|$.
- We get a similar result for scales.

Result (19 E • 4)

There is a scale on WO*. In fact, the relations on triples* $\langle x, y, n\rangle \in \mathcal{N}^2 \times \omega$ *are both* Π^1_1*:*

$$\begin{aligned} x \le_{\varphi_n} y \quad &\textit{iff} \quad x \in \mathrm{WO} \wedge (y \in \mathrm{WO} \to \varphi_n(x) \le \varphi_n(y)) \\ x <_{\varphi_n} y \quad &\textit{iff} \quad x \in \mathrm{WO} \wedge (y \in \mathrm{WO} \to \varphi_n(x) \le \varphi_n(y)). \end{aligned}$$

- What are φ_n? They code $\|x\|$ and its initial segment "up to n".
- More precisely, we can consider

$$(E_x)_{<n} = \{\langle a, b\rangle : a \; E_x \; b \; E_x \; n \ne b\}.$$

- These $(E_x)_{<n}$ tell us how E_x is built up.
- So we define $\varphi_n(x) = \mathrm{code}(\|E_x\|, \|(E_x)_{<n}\|)$.
- Showing $\langle \varphi_n : n < \omega\rangle$ is in fact a scale is a boring, technical process.

Γ-Scales

- This motivates the concept of a Γ-scale.

Definition

Let Γ be a pointclass. A Γ-scale is a scale $\vec{\varphi}$ on a set X such that the relations on $\langle x, y, n\rangle \in \mathcal{N}^2 \times \omega$ are in Γ:

$$\begin{aligned} x \leq_{\varphi_n} y &\quad \text{iff} \quad x \in X \wedge (y \in X \to \varphi_n(x) \leq \varphi_n(y)) \\ x <_{\varphi_n} y &\quad \text{iff} \quad x \in X \wedge (y \in X \to \varphi_n(x) \leq \varphi_n(y)). \end{aligned}$$

- So for adequate pointclasses, X having a Γ-scale implies $X \in \Gamma$ by $x \in X$ iff $x \leq_{\varphi_0} x$.
- We can also show that every Π^1_1-set has a Π^1_1-scale in the same way as with PWO.

Γ-Scales

Theorem

Every Π^1_1-set has a Π^1_1-scale, i.e. the scale property holds for Π^1_1.

Proof.

- Let $A \in \Pi^1_1$, $A = f^{-1}$"WO for some computable $f : \mathcal{N} \to \mathcal{N}$.
- Let $\vec{\psi}$ a Π^1_1-scale on WO.
- We can form $\langle \psi_n \circ f : n < \omega \rangle$ and get a Π^1_1-scale on A. $\dashv$

- We can also show the scale property for Σ^1_2.
- Indeed, we get the same restrictions as with PWO: after Σ^1_2,
  - the scale property is independent of ZFC;
  - PD gives a full zig-zag pattern; and
  - $\mathrm{V} = \mathrm{L}$ gives an initial zig followed by a line.
- Again, we mostly care about scales for now to get uniformization.

Uniformization

Definition

Let $X \subseteq A \times B$. A *uniformization* is a function $f \subseteq X$ such that $\text{dom}(f) = \text{dom}(X)$.
For Γ a pointclass, Γ-uniformization is the statement that every $X \in \Gamma$ has a uniformization $f \in \Gamma$.

Let's prove Π^1_1-uniformization now that we have the scale property.

Theorem

Π^1_1-uniformization holds (and $\underset{\sim}{\mathbf{\Pi}}^1_1$-uniformization).

- Let $A \in \Pi^1_1$ be arbitrary and let $A"x = \{y \in \mathcal{N} : \langle x, y \rangle \in A\}$.
- To get a uniformization, we need to find a unique $y \in A"x$ defined from x.
- Uniqueness is the easy part. The hard part is showing existence and ensuring the resulting f is Π^1_1 with $\text{dom}(f) = \text{dom}(A)$.
- That is where scales come in.

Uniformization

Theorem

Π^1_1-uniformization holds (and $\underset{\sim}{\Pi}^1_1$-uniformization).

- Let $x \in A$ be arbitrary and $\vec{\varphi}$ a Π^1_1-scale on $A \in \Pi^1_1$.
- Set $f(x) = y$ iff $\langle x, y \rangle \in A$ and for all $z \in \mathcal{N}$ and all $n < \omega$,

  If $(*)$ $\langle x, z \rangle \in A$, $z \restriction n = y \restriction n$, and $\varphi_m(x, z) = \varphi_m(x, y)$ for all $m < n$,

  Then $(**)$ $y(n) < z(n)$, or else $y(n) = z(n) \wedge \varphi_n(x, y) \leq \varphi_n(x, z)$.
- Basically, y is lexicographically least among those whose first norms agree with y's initial first segments.
- Let's show that this indeed uniquely defines a y (if there exists one).
- Suppose this holds for two $y_1, y_2 \in \mathcal{N}$ and let N be their first disagreement.
- $(*)$ therefore holds inductively for $n \leq N$ (consider $(*)$ with $n = 0$ and then $(**)$ with $n = 0$ to get $\varphi_n(x, y_1) = \varphi_n(x, y_2)$)
- $(**)$ thus holds. But applying $(*)$ with $z = y_1$ and $y = y_2$ gives $y_2(N) < y_1(N)$ and vice versa yields $y_1(N) < y_2(N)$, a contradiction.

Uniformization

Theorem

Π^1_1-uniformization holds (and $\underset{\sim}{\Pi}{}^1_1$-uniformization).

- Set $f(x) = y$ iff $\langle x, y\rangle \in A$ and for all $z \in \mathcal{N}$ and all $n < \omega$,

  If $(*)$ $\langle x, z\rangle \in A$, $z \restriction n = y \restriction n$, and $\varphi_m(x,z) = \varphi_m(x,y)$ for all $m < n$,
  Then $(**)$ $y(n) < z(n)$, or else $y(n) = z(n) \wedge \varphi_n(x,y) \leq \varphi_n(x,z)$.
- It's not hard to see that $(*)$ is Σ^1_1 and $(**)$ is Π^1_1.
- Thus f (if well-defined) will be $\forall^{\mathcal{N}}(\neg\Sigma^1_1 \vee \Pi^1_1) = \Pi^1_1$.
- So we need to show there *is* such a y, which is much harder.
- We use the scale to do the heavy lifting in the construction and ensure the limit (i.e. end result) is in A.
- Start out with $A_0 = A"x$.
- At stage $n+1$, we first consider the set of all $y \in A_n$ with minimal $\varphi_n(x,y)$ (among other elements in A_n).
- Then we thin out that set to only consider the ys with minimal $y(n)$. The result is A_{n+1}.
- We now want a $y \in \bigcap_{n<\omega} A_n$.

Uniformization

Theorem

Π^1_1-uniformization holds (and $\mathbf{\Pi}^1_1$-uniformization).

- Start out with $A_0 = A"x$.
- At stage $n+1$, we first consider the set of all $y \in A_n$ with minimal $\varphi_n(x, y)$ (among other elements in A_n).
- Then we thin out that set to only consider the ys with minimal $y(n)$. The result is A_{n+1}.
- Any sequence $\langle y_n \in A_n : n < \omega\rangle$ is necessarily convergent since we're deciding more and more when moving from A_n to A_{n+1}.
- Such a sequence also has $\langle \varphi_n(y_k) : k < \omega\rangle$ as eventually constant because again we're deciding more of the norm moving from A_n to A_{n+1}.
- As a scale, it follows that $y = \lim_{n\to\infty} y_n \in A$ and an easy induction shows it satisfies the definition of $f(x) = y$.
- Hence f is well-defined and a Π^1_1-uniformization of A. $\dashv$

What was all this about again?

- Again, we have the same picture with uniformization as with PWO, scales, and other properties: Σ^1_2 is the limit of what can be shown in ZFC.
- The zig-zag (or lackthereof in L) of PD also holds with uniformization.
- This is quite useful with determinacy because uniformization is a choice-like principle telling us not only that we can pick out an element, but we can do so in a simply definable way.
- Let's return to the main goal here:

Theorem

The following are equivalent:

1. *For every $x \in \mathcal{N}$, $\mathsf{L}[x] \models$ "ω_1^{V} is inaccessible".*
2. $\mathsf{PSP}(\underset{\sim}{\mathbf{\Pi}}^1_1)$.

What was all this about again?

- Restated, the main goal is the following.

Theorem

$\mathsf{PSP}(\Pi^1_1(x))$ *implies* $\omega_1^{\mathsf{L}[x]} < \omega_1$ *for every* $x \in \mathcal{N}$.

Proof.

- Let $x \in \mathcal{N}$ and suppose $\omega_1^{\mathsf{L}[X]} = \omega_1$, aiming to show $\neg\mathsf{PSP}(\Pi^1_1(x))$.
- For each $\alpha < \omega_1^{\mathsf{L}[x]}$, let $f(\alpha)$ be the $<_{\mathrm{L}[x]}$-least real in $\mathrm{WO} \cap \mathrm{L}[x]$ coding $\langle \alpha, \in \rangle$.
- Set $X = f"\omega_1^{\mathsf{L}[x]}$ so that every ordinal is coded by an element of X.
- We actually considered this set already to show that $\mathsf{L} \models$ "$\neg\mathsf{PSP}(\Sigma^1_2)$".
- The relevant theorems generalize to show
  1. $X \in \Sigma^1_2(x)$ is uncountable (since $\aleph_1^{\mathsf{L}[x]} = \aleph_1$), and
  2. X has no uncountable $\undertilde{\Sigma}^1_1$-subset (by the Boundedness lemma).

What was all this about again?

Theorem

$\mathsf{PSP}(\Pi^1_1(x))$ *implies* $\omega_1^{\mathsf{L}[x]} < \omega_1$ *for every* $x \in \mathcal{N}$.

Proof.

- $X = f``\omega_1^{\mathsf{L}[x]}$ where $f(\alpha) \in \mathrm{WO} \cap \mathrm{L}[x]$ codes α.
  1. $X \in \Sigma^1_2(x)$ is uncountable (since $\aleph_1^{\mathsf{L}[x]} = \aleph_1$), and
  2. X has no uncountable $\underset{\sim}{\mathbf{\Sigma}}^1_1$-subset (by the Boundedness lemma).
- Let $X = \mathfrak{p}Y$ for $Y \in \Pi^1_1(x)$.
- By $\Pi^1_1(x)$-uniformization, we get a Π^1_1-function $f \subseteq Y$ with $\mathrm{dom}(f) = X$.
- Hence $|f| = |X| = \aleph_1$.
- f also can't have any perfect subset. To see this, firstly any perfect subset must be closed and uncountable.
- But any closed $g \subseteq f$ has $\mathrm{dom}(g) \in \underset{\sim}{\mathbf{\Sigma}}^1_1$ and is thus countable since $\mathrm{dom}(g) \subseteq X$ and (2) holds.
- Thus $\neg\mathsf{PSP}(\Pi^1_1(x))$. $\dashv$

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