Final Exam Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam. Please also consult the review problems for Exams 1 and 2, as these will be useful guides for material covered earlier in the course.

1. Consider the linear operator on $\mathbb{R}^2$ defined by
$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$
   (a) What is the matrix of T with respect to the standard ordered basis for $\mathbb{R}^2$?
   (b) What is the matrix of T with respect to the ordered basis
$$\gamma = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}?$$
   (c) Prove that for any $c \in \mathbb{R}$, the linear operator $T - cI_{\mathbb{R}^2}$ is invertible. Here $I_{\mathbb{R}^2}$ denotes the identity operator on $\mathbb{R}^2$.

2. Suppose that A is a 2×2 symmetric matrix with real entries. Prove that A is similar (over $\mathbb{R}$) to a diagonal matrix.

3. Let $T = L_A : \mathbb{R}^2 \to \mathbb{R}^2$, with
$$A = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}.$$
   Prove that the only subspaces of $\mathbb{R}^2$ that are invariant under T are $\mathbb{R}^2$ itself and $\{0\}$.

4. Let $T = L_A : \mathbb{R}^2 \to \mathbb{R}^2$, with
$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$
   Let
$$W_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$
   (a) Prove that W_1 is invariant under T.
   (b) Prove that there is **no subspace** W_2 which is invariant under T and
$$\mathbb{R}^2 = W_1 \bigoplus W_2.$$

5. Prove the **Triangle Inequality** in an inner product space V: for all $x, y \in V$,
$$||x + y|| \leq ||x|| + ||y||.$$
   Here $|| \cdot ||$ is the associated norm. *Hint:* Use the Cauchy-Schwarz inequality.

6. Define the set of functions
$$S = \{\sin t, \cos t, 1, t\},$$
   and define the subspace U of $C([0, \pi])$ by
$$U = \operatorname{span}(S).$$
   U is thus an inner product space, with inner product
$$\langle f, g \rangle = \int_0^\pi f(t)g(t)\,\mathrm{d}t.$$

(a) Find an orthonormal basis β for U.

(b) Using the orthonormal basis β, find the coordinates of h relative to β, where

$$h(t) = 2t + 1.$$

7. Let V be the vector space of all upper $n \times n$ triangular matrices with coefficients in a field F. Find two non-trvial subspaces U_1 and U_2 such that

$$V = U_1 \bigoplus U_2.$$

8. Give three different bases for $M_{2\times 2}(F)$.

9. Let T be the linear operator defined on $P_2(\mathbb{R})$ defined by

$$T(f(x)) = 2f(x) - f'(x).$$

Find the Jordan canonical form, and Jordan canonical basis, of T.

10. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that

$$\beta = \{v_1, v_2, \dots, v_n\}$$

is a basis for V. Prove that T is an isomorphism if and only if

$$T(\beta) := \{T(v_1), T(v_2), \dots, T(v_n)\}$$

is a basis for W.

11. Does the Jordan form of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

exist? Why or why not?

12. Find the Jordan form and a Jordan canonical basis for

$$B = \begin{pmatrix} 5 & -1 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 12 & -3 \end{pmatrix}.$$

13. **True or False**: the subset

$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 x_2 x_3 = 0\}$$

is a subspace of $\mathbb{R}^3$. Provide justification.

14. **True or False**: there exists a basis $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ of $P_3(F)$ such that none of the polynomials $p_1(x), p_2(x), p_3(x), p_4(x)$ has degree 2. Provide justification.

15. Let V be an inner product space. Suppose that $T \in \mathcal{L}(V)$ is such that

$$\langle T(x), y\rangle = \langle x, T(y)\rangle.$$

Show that every eigenvalue of T is real. *Note:* Such operators T are called **self-adjoint** (or **Hermitian**), and are extremely important in mathematics and science.

16. Let V be an inner product space, and U a finite-dimensional subspace. Recall that we can show that (you should be able to prove this!)

$$V = U \bigoplus U^{\perp}. \tag{1}$$

Define the projection onto U: for $v = u + w \in U \bigoplus U^{\perp}$, define$P : V \to V$ by

$$P(v) = u.$$

Note that due to (1), this is a well-defined transformation, and as a projection is linear.

(a) Show that $P^2 = P$ as linear operators. Recall that $P^2(v) = P(P(v))$.

(b) Show that for all $v \in V$, $||P(v)|| \leq ||v||$.

17. Is the map $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + iz_2 \\ iz_1 + z_2 \end{pmatrix}$$

diagonalizable? If so, find a basis β for $\mathbb{C}^2$ such that $[T]_\beta$ is diagonal.

18. Let W be the subspace of $\mathbb{R}^5$ spanned by

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 4 \\ 7 \\ 2 \\ -1 \end{pmatrix}.$$

Find a basis for the orthogonal complement $W^\perp$ of W.

19. Determine all possible Jordan canonical forms for a linear operator $T : V \to V$ who characteristic polynomial is

$$f_T(t) = (t-2)^3(t-5)^2.$$