Exam 2 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam.

1. Suppose that V is finite dimensional vector space over a field F, and $T : V \to V$ is linear. Prove that

$$T(x) = aI_v(x),$$

for some $a \in F$, i.e. T is a scalar multiple of the identity if and only if $ST = TS$ for all $S \in \mathcal{L}(V)$.

2. Recall that definition of the eigenspace corresponding to λ of a linear operator T:

$$E_\lambda = \{v \in V \,|\, T(v) = \lambda v\}.$$

That is, E_λ is the union of $\{0\}$ and the set of eigenvectors of T corresponding to λ. Show that E_λ is invariant subspace with respect to T.

3. Let $A \in M_{n\times n}(F)$. Recall that the trace of A, denoted $\mathrm{tr}(A)$, is the sum of the main diagonal of A:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

(i) Show that if $A \in M_{n\times m}(F), B \in M_{m\times n}(F)$, then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

(ii) Use the result of (a) to show that if $A, B \in M_{n\times n}(F)$ are **similar**, then

$$\mathrm{tr}(A) = \mathrm{tr}(B).$$

Thus, the trace of a matrix is invariant under similarity (i.e. change of coordinates).

4. As a matrix over $\mathbb{R}$, find the rank and nullity of

$$\begin{pmatrix} 2 & 0 & 1 & -3 & 4 \\ -1 & 2 & 4 & 0 & 3 \\ 3 & -2 & -3 & -3 & 1 \end{pmatrix}.$$

4. Consider the vector space of "traceless" 2×2 matrices over $\mathbb{R}$:

$$V = \{A \in M_{2\times 2}(\mathbb{R}) \,|\, \mathrm{tr}(A) = 0\}.$$

Is V isomorphic to $\mathbb{R}^4$? Prove it is or why it is not.

5. Let $A \in M_{n\times n}(F)$ such that $A^2 = 0$, where 0 denotes the $n \times n$ zero-matrix:

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Show that A is not invertible.

6. Consider the two ordered bases for $\mathbb{R}^2$:

$$\beta = \left\{ \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\},$$
$$\beta' = \left\{ \begin{pmatrix} 0 \\ 10 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\}.$$

Find the change of coordinate matrix Q that changes β'-coordinates into β-coordinates.

7. Define the linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by

$$T(f(x)) = \begin{pmatrix} f(-1) \\ f(0) \\ f(1) \end{pmatrix}.$$

Is T invertible? If so, find T^{-1}. Note that T^{-1} will be (if it exists) a map from $\mathbb{R}^3$ to $P_2(\mathbb{R})$; I am not looking just for a matrix representation (although it may be useful to compute as an intermediate step).

8. Let

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and define $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$ by

$$T(A) = \begin{pmatrix} \mathrm{tr}(A) \\ \mathrm{tr}(A^T) \\ \mathrm{tr}(EA) \\ \mathrm{tr}(AE) \end{pmatrix}.$$

Repeat Problem 7 for this transformation.

9. Show that $\lambda = 0$ is an eigenvalue of A if and only if A is not invertible.

10. Prove that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable.

11. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear operator defined by

$$T(f(x)) = f(x) + f(2)x.$$

Find the eigenvalues of T and an ordered basis for $P_3(\mathbb{R})$ such that $[T]_\beta$ is diagonal. Write $P_3(\mathbb{R})$ as the direct sum of T-invariant subspaces.

12. For the linear operator defined in Problem 11, find the matrix $[T]_\gamma$, where γ is the standard order basis of $P_3(\mathbb{R})$. Find a matrix Q such that

$$[T]_\beta = Q^{-1}[T]_\gamma Q,$$

where again $[T]_\beta$ is as in Problem 11.

13. Let T be an invertible linear transformation on a finite-dimensional vector space V. Show that if T is diagonalizable, then so is T^{-1}.

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = 4x$. Find the matrix representation of T with respect to the standard ordered basis β of $\mathbb{R}^2$. Is T diagonalizable? *Hint:* Find the matrix with respect to a more convenient basis γ ($[T]_\gamma$), and then use a change of coordinates matrix Q to find $[T]_\beta$.

15. **True or False**: For $A \in M_{n\times n}(F)$, $A^2 = I$ implies that $A = I$ or $A = -I$. Provide justification (i.e. a proof if true, or a counterexample if false).

16. Let $T \in \mathcal{L}(V, W)$ be an isomorphism between two vector spaces over F. Prove that if U is a subspace of V, then $\dim(T(U)) = \dim(U)$. Recall that

$$T(U) := \{T(u) \mid u \in U\}$$

is the image of U in W under T.