Exam 1 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam.

1. Let V be a vector space, and $W_1, \ldots, W_m$ be subspaces of V. Show that $W := W_1 \cap W_2 \cap \cdots \cap W_m$ is a subspace of V.

2. Prove that if $(v_1, \ldots, v_n)$ spans V, then so does the list
$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n).$$

3. True or False: The mapping $T : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$T(x, y) = xy$$
is linear. Provide justification.

4. Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be the map defined by
$$F(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4).$$
   (i) Show that F is linear.
   (ii) Find a basis for the null space $N(F)$ in $\mathbb{R}^4$ and the image $R(T)$ in $\mathbb{R}^3$.
   (iii) Find the rank and the nullity of F.

5. Consider the vector space of polynomials with coefficients in $\mathbb{R}$ of degree at most n, $P_n(\mathbb{R})$. Let $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ be a set of polynomials such that $\deg(p_k(x)) = k$. Show that $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ forms a basis for $P_n(\mathbb{R})$.

6. Fix
$$M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$
in $M_{2\times 2}(\mathbb{R})$. Let $T_M : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the map defined by
$$T_M(A) = AM - MA.$$
   (i) Show that T_M is linear.
   (ii) Find a basis and the dimension of the kernel $N(T_M)$.

7. Fix a vector $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ in $\mathbb{R}^2$.
   (i) Find the coordinates of v, $[v]_\beta$, with respect to the standard ordered basis $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$.
   (ii) Let $\tilde{\beta} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Clearly $\tilde{\beta}$ is a basis for $\mathbb{R}^2$ (why?). Find $[v]_{\tilde{\beta}}$.
   (iii) Show that $\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$.
   (iv) Find the coordinates of v with respect to γ, $[v]_\gamma$.

8. Let V be a vector space such that $\dim(V) = 1$. If $T : V \to V$ is a linear map from V into V, show that there exists an $a \in F$ such that
$$T(v) = av$$
for all $v \in V$. *Hint:* What does T do to the basis $\{v_1\}$ of V?

9. True or False: If V is a vector space *other than the zero vector space*, then V contains a subspace W such that $W \neq V$. Provide justification.

10. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

11. (i) Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that

$$V = W_1 \bigoplus W_2.$$

(ii) Let $V = \mathbb{R}^2$ and $W_1 = \{(a, 0) \mid a \in \mathbb{R}\}$. Give examples of **two different subspaces** W_2 and $W_2^{'}$ such that

$$V = W_1 \bigoplus W_{,2}$$
$$V = W_1 \bigoplus W_2^{'}.$$

12. Suppose U and W are subspaces of $\mathbb{R}^8$ such that

$$\begin{aligned} \dim(U) &= 3, \\ \dim(W) &= 5, \\ U + W &= \mathbb{R}^8. \end{aligned}$$

Show that $\mathbb{R}^8 = U \bigoplus W$. *Hint:* First find $U \cap W$.

13. Find the dimension of $\mathbb{C}^3$ over $\mathbb{R}$. Recall that

$$\mathbb{C}^3 = \{(z_1, z_2, z_3) \mid z_1, z_2, z_3 \in \mathbb{C}\}.$$

Hint: Find a basis.

14. For a field F, define F^∞ as the set of all sequences of elements of F:

$$F^\infty = \{(x_1, x_2, \ldots) \mid x_i \in F \text{ for } i = 1, 2, \ldots\}.$$

Vector addition and scalar multiplication is defined componentwise as usual. Show that F^∞ is a vector space over F. Do not forgot to find the zero vector.

15. Define

$$T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$$

by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + d) + (2d)x + bx^2.$$

Let β and γ be the following ordered bases for $M_{2\times 2}(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively:

$$\begin{aligned} \beta &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \\ \gamma &= \left\{1, x, x^2\right\}. \end{aligned}$$

Compute the matrix of T with respect to β and γ, $[T]_\beta^\gamma$.