Linear systems of differential equations

1. Linearity properties

In this section we discuss the most important general properties of first order linear systems of ordinary differential equations. This is not a thorough presentation of the entire subject; rather, we want to summarize the essential **general** properties of the problem, and in particular those aspects related to the **linearity** of the equations. In doing so we will emphasize the parallels with the solution of algebraic linear equations, discussed in a separate set of notes.

The system we will study is

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{b}(t), \tag{1}$$

where $\mathbf{y}$ and $\mathbf{b}$ are vector functions of t, with $\mathbf{b}$ known and $\mathbf{y}$ unknown—$\mathbf{y}$ is the *dependent variable*—and A is a known matrix function of t:

$$\mathbf{y} = \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \mathbf{b} = \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix} \quad A = A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}.$$

Note that A is a **square** matrix. Throughout we assume that all the functions $a_{ij}(t)$ and $f_i(t)$ are continuous functions of t for t in some interval $(\alpha, \beta) = \{\, t \mid \alpha < t < \beta \,\}$—more succinctly, we assume that $A(t)$ and $\mathbf{b}(t)$ are continuous on (α, β)—and look for a solution or solutions $\mathbf{y}(t)$ defined on this interval.

The homogeneous problem. Let us first consider the homogeneous case in which $\mathbf{b} = 0$, so that our system becomes

$$\mathbf{y}' = A(t)\mathbf{y}. \tag{2}$$

Just as for a homogeneous system of (algebraic) linear equations we have

> **Principle 1: Linearity Principle.** If $\mathbf{y}^{(1)}(t)$, $\mathbf{y}^{(2)}(t)$, $\dots$, $\mathbf{y}^{(k)}(t)$ are solutions of (2), and c_1, c_2, $\dots$, c_k are constants, then
>
> $$\mathbf{y}(t) = c_1\mathbf{y}^{(1)}(t) + c_2\mathbf{y}^{(2)}(t) + \cdots + c_k\mathbf{y}^{(k)}(t) \tag{3}$$
>
> is also a solution of (2).

The linearity principle tells us how to build new solutions from solutions we already have; $\mathbf{y}(t)$ in (3) is called a *linear combination* or *superposition* of the solutions $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(k)}$. The principle is of fundamental importance; let us verify it. Since we want to check a claim that $\mathbf{y}$ is a solution of (2), we just plug $\mathbf{y}$ into (2) and see if the equation is satisfied. We need to use the linearity of differentiation and of matrix multiplication:

$$\mathbf{y}' = (c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \cdots + c_k\mathbf{y}^{(k)})' = c_1\mathbf{y}^{(1)\prime} + c_2\mathbf{y}^{(2)\prime} + \cdots + c_k\mathbf{y}^{(k)\prime}, \tag{4}$$

$$A\mathbf{y} = A(c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \cdots + c_k\mathbf{y}^{(k)}) = c_1A\mathbf{y}^{(1)} + c_2A\mathbf{y}^{(2)} + \cdots + c_kA\mathbf{y}^{(k)}. \tag{5}$$

Then since $\mathbf{y}^{(i)\prime} = A\mathbf{y}^{(i)}$ for each i,

$$\mathbf{y}' = c_1\mathbf{y}^{(1)\prime} + c_2\mathbf{y}^{(2)\prime} + \cdots + c_k\mathbf{y}^{(k)\prime} = c_1A\mathbf{y}^{(1)} + c_2A\mathbf{y}^{(2)} + \cdots + c_kA\mathbf{y}^{(k)} = A\mathbf{y}, \tag{6}$$

that is, $\mathbf{y}$ satisfies (2).

Remark 1: A *vector space* V is a set of "vectors" (which may in fact be vectors in the traditional sense, functions, or something else) such that if $\mathbf{v}_1, \ldots \mathbf{v}_k$ are vectors in V then the linear combination $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is defined and belongs to V. (One also requires that these linear combinations obey the usual algebraic rules, but that will go without saying in our context.) Thus one may summarize Principle 1 by saying that **the space of solutions of the homogeneous system (2) forms a vector space.** We will call this vector space V.

The inhomogeneous problem. We now consider general case of (1), in which $\mathbf{b}$ may or may not be zero. Again the result is parallel to the result for algebraic systems.

Principle 2: Extended linearity principle. Suppose that $\mathbf{Y}(t)$ is some (particular) solution of the inhomogeneous equation (1). Then every solution of this equation is of the form

$$\mathbf{y}(t) = \mathbf{Y}(t) + \mathbf{y}_h(t), \tag{7}$$

where $\mathbf{y}_h(t)$ is a solution of the homogeneous system (2). Moreover, every vector function of the form $\mathbf{y}(t) = \mathbf{Y}(t) + \mathbf{x} = y_h(t)$ is indeed a solution.

To verify this principle, one checks two things: first, that (7) is a solution of (1), and second, that if $\hat{\mathbf{y}}$ is any solution of (1), then $\hat{\mathbf{y}} - \mathbf{Y}$ satisfies the homogeneous equation (2). Both of these are checked by substituting the purported solutions into the relevant equations.

2. Existence and Uniqueness

The basic existence and uniqueness result for linear systems is much easier to state and understand than the corresponding results for general, nonlinear, systems (or even for nonlinear equations with one unknown). We state it for the inhomogeneous system (1), since the homogeneous system is a special case of the inhomogeneous one.

Principle 3: Existence and uniqueness. Suppose, as assumed above, that the coefficients $A(t)$ and $\mathbf{b}(t)$ in (1) are continuous on the interval (α, β). Then the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{b}(t), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{8}$$

where t_0 is a point in the interval (α, β) and $\mathbf{y}_0$ an arbitrary vector, has a unique solution $y(t)$ defined for all t with $\alpha < t < \beta$.

Remark 2: This existence and uniqueness principle shows that linear equations behave quite differently from nonlinear ones. Recall a simple example of a nonlinear problem in one unknown:

$$\frac{dy}{dt} = y^2, \quad y(0) = 1, \qquad \Rightarrow \qquad y(t) = \frac{1}{t-1}.$$

The solution blows up as t approaches 1, but there is no simple way we can tell this from looking at the equation, and indeed if we change the initial condition the blow-up will happen at some different value of t. On the other hand, for the linear IVP

$$\begin{aligned} \frac{dx}{dt} &= \frac{x}{t} - 3y + 2 \\ \frac{dy}{dt} &= \sin t\, x + t^2 y + \ln(5-t), \end{aligned} \qquad , \qquad \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \end{pmatrix},$$

corresponding to $A = \begin{pmatrix} 1/t & -3 \\ \sin t & t^2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ \ln(5-t), \end{pmatrix}$, and $\mathbf{y}_0 = \begin{pmatrix} 3 \\ 17 \end{pmatrix}$, we see by inspection that A and $\mathbf{b}$ are both continuous on the interval $(0, 5)$, so that a solution of the IVP must exist, and be unique, for $0 < t < 5$.

3. Building solutions as linear combinations

In this section we consider only the homogeneous problem (2). Principle 1 suggests that one might be able to find a fundamental set of solutions from which all others could be built as linear combinations. Our next result shows that it is indeed possible to construct all solutions in this way from a set of (at most) n basic solutions.

> **Principle 4: Spanning set.** One can find n solutions $\mathbf{z}^{(1)}(t), \ldots, \mathbf{z}^{(n)}(t)$ such that every solution $\mathbf{y}(t)$ of the system (2) of n linear equations has the form
>
> $$\mathbf{y}(t) = c_1\mathbf{z}^{(1)}(t) + c_2\mathbf{z}^{(2)}(t) + \cdots + c_n\mathbf{z}^{(n)}(t).$$

In Remark 1 we pointed out that the set of all solutions to (2) forms a vector space V; in this language, Principle 4 says that one may find n solutions which *span* the space of all solutions. It is important to realize that the "n" in this principle—the number of solutions in the spanning set—is the same as the number n of equations in the system we are solving.

To verify Principle 4 it is convenient to introduce the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ which form the columns of the $n \times n$ identity matrix:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{u}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form the *standard basis* in $\mathbb{R}^n$; they have the property that for any vector $\mathbf{v}$ in $\mathbb{R}^n$ with components $v_1, \ldots, v_n$, $\mathbf{v} = v_1\mathbf{u}_1 + \cdots + v_n\mathbf{u}_n$.

Now fix some point t_0 in the interval (α, β) and let $\mathbf{z}^{(k)}(t)$, for $k = 1, \ldots, n$, be the solution of (2) which satisfies the initial condition

$$\mathbf{z}^{(k)}(t_0) = \mathbf{u}_k. \tag{9}$$

We want to show that these solutions $\mathbf{z}^{(k)}$ form a spanning set of solutions, as in Principle 4. To see this, let $\mathbf{y}(t)$ be any solution of (2) and let $\mathbf{v}$ denote the vector $\mathbf{y}(t_0)$. Then define

$$\tilde{\mathbf{y}}(t) = v_1\mathbf{z}^{(1)}(t) + \cdots + v_n\mathbf{z}^{(n)}(t). \tag{10}$$

Here we have built $\tilde{\mathbf{y}}(t)$ as a linear combination of the solutions $\mathbf{z}^{(1)}(t), \ldots, \mathbf{z}^{(n)}(t)$, with coefficients $v_1, \ldots, v_n$ which are just the components of the vector $\mathbf{v} = \mathbf{y}(t_0)$. Now $\tilde{\mathbf{y}}(t)$ is a solution of (2), by Principle 1; moreover,

$$\tilde{\mathbf{y}}(t_0) = v_1\mathbf{z}^{(1)}(t_0) + \cdots + v_n\mathbf{z}^{(n)}(t_0) = v_1\mathbf{u}_1 + \cdots \mathbf{v}_n\mathbf{u}_n = \mathbf{v}.$$

Thus $\mathbf{y}(t)$ and $\tilde{\mathbf{y}}(t)$ are both solutions of the IVP

$$\mathbf{y}' = A(t)\mathbf{y}, \qquad \mathbf{y}(t_0) = \mathbf{v},$$

and by the uniqueness part of Principle 3, $\mathbf{y}(t) = \tilde{\mathbf{y}}(t)$ for all t in (α, β). In view of (10), this shows that $\mathbf{y}(t)$ is a linear combination of the solutions $\mathbf{z}^{(k)}(t)$.

One should also realize that the spanning set $\mathbf{z}^{(1)}(t), \ldots, \mathbf{z}^{(n)}(t)$ which we have just constructed is only one of many such sets. One could produce others by a different choice of t_0 or by replacing the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in the initial conditions $\mathbf{z}^{(k)}(t_0) = \mathbf{u}_k$ by any n vectors which form a basis for $\mathbb{R}^n$.

4. Linear independence of functions

Now we turn to the question of whether or not we really need n different solutions in order to build every solution—that is, whether we might find some set of n' solutions, with $n' < n$, which also spans the space of all solutions. To discuss this we need to first discuss linear independence.

We assume familiarity with the notion of linear independence of vectors in $\mathbb{R}^n$; for a review, see the posted notes on linear algebra. Briefly, vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_k$ in $\mathbb{R}^n$ are linearly independent if it is impossible to express one of them as a linear combination of the remaining ones; otherwise, they are *linearly dependent*. Alternatively, these vectors are linearly independent if a linear combination of them can be the zero vector,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}, \tag{11}$$

only if all the coefficients are zero: $c_1 = c_2 = \cdots = c_k = 0$. Now we discuss a slightly different concept, that of linear independence of *functions*.

Consider a collection of functions defined on some interval (α, β); these could either be scalar-valued functions $f_1(t), \ldots, f_k(t)$, say

$$f_1(t) = 1, \quad f_2(t) = \sin^2 t, \quad f_3(t) = \cos^2 t, \tag{12}$$

or could be vector-valued functions $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(k)}(t)$, all with the same number of components, say

$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad \mathbf{y}^{(2)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}. \tag{13}$$

(We will usually use the notation of vector-valued functions in general discussions, to avoid separate consideration of several cases. One can of course think of scalar functions as vector functions having only one component.) We say that the collection is *linearly independent on* (α, β) if no one of these functions may be expressed as a linear combination, with constant coefficients, of the others. Alternatively, the collection is linearly independent if whenever a linear combination of the functions, with constant coefficients, is zero for all t in (α, β), then necessarily all the coefficients are zero:

$$c_1\mathbf{y}^{(1)}(t) + \cdots + c_k\mathbf{y}^{(k)}(t) = \mathbf{0} \quad \text{for all } t \text{ in } (\alpha, \beta) \qquad \Rightarrow \qquad c_1 = \cdots = c_k = 0. \tag{14}$$

Equivalence of these two definitions of linear independence is proved just as was the corresponding equivalence for vectors, above. A set of functions which is not linearly independent is *linearly dependent.*

Example 3: (a) The three (scalar) functions $f_1(t) = 1$, $f_2(t) = \sin^2 t$, and $f_3(t) = \cos^2 t$ of (12) are linearly *dependent* on any interval (α, β), since we always have $1 - \sin^2 t - \cos^2 t = 0$, that is, $c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) = 0$ for $c_1 = 1$, $c_2 = c_3 = -1$.

(b) On the other hand, the three functions $g_1(t) = 1$, $g_2(t) = \sin t$, and $g_3(t) = \cos t$ *are* linearly *independent* on any interval (α, β). Here is one way to see this, assuming that the interval contains at least the points 0, $\pi/2$, and π. Suppose that $c_1 g_1(t) + c_2 g_2(t) + c_3 g_3(t) = 0$ for all t; then plugging in successively these three values of t gives

$$c_1 + c_3 = 0, \quad (t = 0); \qquad c_1 + c_2 = 0, \quad (t = \pi/2); \qquad c_1 - c_3 = 0, \quad (t = \pi),$$

and the only solution of these equations is $c_1 = c_2 = c_3 = 0$.

(c) The two vector functions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ of (13) are linearly independent on any interval (α, β). For if $c_1\mathbf{y}^{(1)}(t) + c_2\mathbf{y}^{(2)}(t) = 0$ for all t then one may plug in any two distinct values of t in the interval, say $t = r$ and $t = s$, to find

$$c_1\mathbf{y}^{(1)}(r) + c_2\mathbf{y}^{(2)}(r) = c_1 \begin{pmatrix} 1 \\ r \end{pmatrix} + c_2 \begin{pmatrix} e^r \\ re^r \end{pmatrix} = \begin{pmatrix} c_1 + c_2 e^r \\ r(c_1 + c_2 e^r) \end{pmatrix} = 0$$
$$c_1\mathbf{y}^{(1)}(s) + c_2\mathbf{y}^{(2)}(s) = c_1 \begin{pmatrix} 1 \\ s \end{pmatrix} + c_2 \begin{pmatrix} e^s \\ se^s \end{pmatrix} = \begin{pmatrix} c_1 + c_2 e^s \\ s(c_1 + c_2 e^s) \end{pmatrix} = 0$$

In particular, $c_1 + c_2 e^r = 0$ and $c_1 + c_2 e^s = 0$, and since $r \neq s$ these equations imply $c_1 = c_2 = 0$.

(d) Consider again the vector functions $\mathbf{y}^{(1)}(t)$ and $\mathbf{y}^{(2)}(t)$ of (13). From (c) we know that these are independent as functions on any interval. However, for *fixed* t, say $t = t_0$, the two vectors $\mathbf{y}^{(1)}(t_0) = \begin{pmatrix} 1 \\ t_0 \end{pmatrix}$ and $\mathbf{y}^{(2)}(t_0) = e^{t_0} \begin{pmatrix} 1 \\ t_0 \end{pmatrix}$ are linearly *dependent*, since they are proportional (with proportionality constant e^{t_0}).

Remark 4: We have a systematic method of determining whether or not a set of vectors is linearly independent, since this reduces to solving a system of linear equations. There is no corresponding simple method for checking linear dependence or independence of functions. However, the method used in Example 6(b,c), of plugging in a few points, is very useful to show linear independence. Sometimes differentiating the linear relation $\sum c_i \mathbf{y}^{(i)} = \mathbf{0}$ is useful. To show linear dependence one can, for examples considered in these notes, usually see by inspection how to express one of the functions in as a linear combination of some of the others.

5. The dimension of the space of solutions

The next result is important but too special to be called a principle. We'll call it just a fact.

Fact 1: The solutions $\mathbf{z}^{(1)}(t), \ldots, \mathbf{z}^{(n)}(t)$ constructed during the proof of Principle 4, that is, the solutions which satisfy the initial conditions $\mathbf{z}^{(k)}(t_0) = \mathbf{u}_k$, $k = 1, \ldots, n$ (see (9)), are linearly independent.

To see that this is true we go back to the definition of linear independence summarized in (14). Suppose that the constants $c_1, \ldots, c_n$ are such that $c_1\mathbf{z}^{(1)}(t) + \cdots + c_n\mathbf{z}^{(n)}(t) = \mathbf{0}$ for all t. Setting $t = t_0$ and using $\mathbf{z}^{(k)}(t_0) = \mathbf{u}_k$ we then have

$$\mathbf{c}_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

and this says that $c_1 = c_2 = \cdots = c_n = 0$, verifying (14).

If we now put Principle 4 together with Fact 1 we see that the n solutions $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(n)}$ of (2) constructed from the initial conditions (9) *span* the space V of solutions of (2) and are *linearly independent*. This means that they form a *basis* of the space V, and that V is an n-dimensional vector space. Standard facts from linear algebra then tell us that every basis contains n solutions and that any set of n linearly independent solutions forms a basis and thus spans the space. To summarize:

Principle 5: The general solution of a homogeneous system. Suppose that $\mathbf{y}^{(1)}$, $\mathbf{y}^{(2)}$, $\ldots$, $\mathbf{y}^{(n)}$ are n solutions of (2) which are linearly independent on the interval (α, β). Then every solution $\mathbf{y}$ of (2) is of the form

$$\mathbf{y}(t) = c_1\mathbf{y}^{(1)}(t) + c_2\mathbf{y}^{(2)}(t) + \cdots + c_n\mathbf{y}^{(n)}(t) \tag{15}$$

for some constants c_1, c_2, $\ldots$, c_n. Equivalently, we may say that every initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(t_0) = \mathbf{y}_0$ has a (unique) solution of the form (15).

It is important to realize that there are three conditions here on the vector functions $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ which are necessary to guarantee that the general solution has the form (15). First, the $\mathbf{y}^{(i)}$ must themselves be solutions of (2). Second, there must be exactly n solutions: *the same number of solutions as the number of components in the vectors, i.e., as the number of different first order equations represented in vector form in (2).* Third, these n solutions must be *linearly independent.*

Suppose now that we have some set $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(n)}(t)$ of solutions of (2). To apply Principle 5 it is necessary to determine whether this set of solutions is linearly independent. As indicated above (Remark 4 on page 6) there is in general no simple method to determine whether a set of function is or is not linearly independent. However, when the functions in questions are all solutions of (2) there is such a method; one uses the *Wronskian* of these solutions, which we define shortly. To understand its use, however, it is convenient to first make a slight digression.

Given any n *vector functions* $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(n)}(t)$, which may or may not be solutions of (2), one may ask not only whether or not they are linearly independent but also, *for a fixed* t_0 whether or not the *vectors* $\mathbf{y}^{(1)}(t_0), \ldots, \mathbf{y}^{(n)}(t_0)$ are linearly independent. The questions are related:

Fact 2: If the functions $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(n)}(t)$ are such that the vectors $\mathbf{y}^{(1)}(t_0), \ldots, \mathbf{y}^{(n)}(t_0)$ are linearly independent for some t_0 then the functions are also linearly independent.

To see this, note that a linear relation on the functions,

$$c_1\mathbf{y}^{(1)}(t) + \cdots + c_n\mathbf{y}^{(n)}(t) = \mathbf{0} \quad \text{for all } t,$$

would certainly imply a linear relation on the vectors, just by taking $t = t_0$ above:

$$c_1\mathbf{y}^{(1)}(t_0) + \cdots + c_n\mathbf{y}^{(n)}(t_0) = \mathbf{0}.$$

Linear independence of the vectors would imply that $c_1 = \cdots = c_n = 0$, verifying (14).

In general the converse of Fact 2 does not hold, that is, linear independence of the functions does not imply that their (vector) values at any point must be linearly independent. This is shown by Example 3 (c,d). But when the functions are solutions of the homogeneous linear system (2) the situation is very simple:

Principle 6: Equivalence of different linear independence conditions. Suppose that $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(n)}(t)$ are solutions of the linear system (2) on the interval (α, β). Then the following three statements are equivalent, that is, all are true if any one is true:

(i) The *functions* $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(n)}(t)$ are linearly independent.

(ii) For *every* $t_0 \in (\alpha, \beta)$ the *vectors* $\mathbf{y}^{(1)}(t_0), \ldots, \mathbf{y}^{(n)}(t_0)$ are linearly independent.

(iii) For *some* $t_0 \in (\alpha, \beta)$ the *vectors* $\mathbf{y}^{(1)}(t_0), \ldots, \mathbf{y}^{(n)}(t_0)$ are linearly independent.

We want to explain briefly why Principle 6 holds. It is clear that if (ii) is true then so is (iii), that is, (ii) implies (iii). Fact 2 tells us that (iii) implies (i), so if we can prove that (i) implies (ii), we will know that the three statements are equivalent. Now if $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(n)}(t)$ are linearly independent then, by Principle 5, for any $t_0 \in (\alpha, \beta)$ and any vector $\mathbf{y}_0$ we can solve the initial value problem $\mathbf{y}' = A(t)\mathbf{y}$, $\mathbf{y}(t_0) = \mathbf{y}_0$ with a solution of the form $\sum_{k=1}^{n} c_k \mathbf{y}^{(k)}$; this means that for any $\mathbf{y}_0$ the linear equations

$$\sum_{k=1}^{n} c_k \mathbf{y}^{(k)}(t_0) = \begin{pmatrix} y_1^{(1)}(t_0) & y_1^{(2)}(t_0) & \cdots & y_1^{(n)}(t_0) \\ y_2^{(1)}(t_0) & y_2^{(2)}(t_0) & \cdots & y_2^{(n)}(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(1)}(t_0) & y_n^{(2)}(t_0) & \cdots & y_n^{(n)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{02} \\ \vdots \\ \mathbf{y}_{0n} \end{pmatrix} \tag{16}$$

have a solution. But now standard facts from linear algebra tell us that this implies that the columns of the matrix in (16) are linearly independent.

6. The Wronskian

Principle 6 is in some ways a remarkable statement: since (iii) implies (i), we can determine whether or not n solutions $\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(n)}(t)$ of (2) are linearly independent checking whether or not the vectors $\mathbf{y}^{(1)}(t_0), \ldots, \mathbf{y}^{(n)}(t_0)$ are linearly independent, and we can make the latter check at any point t_0. To do this we introduce the *Wronskian determinant*, also called just the *Wronskian*. The Wronskian is the determinant of the matrix whose columns are the vectors $\mathbf{y}^{(k)}(t)$:

$$W(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)})(t) = W(t) = \begin{vmatrix} y_1^{(1)}(t) & y_1^{(2)}(t) & \cdots & y_1^{(n)}(t) \\ y_2^{(1)}(t) & y_2^{(2)}(t) & \cdots & y_2^{(n)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(1)}(t) & y_n^{(2)}(t) & \cdots & y_n^{(n)}(t) \end{vmatrix}.$$

By elementary linear algebra, the vectors $\mathbf{y}^{(1)}(t_0), \ldots, \mathbf{y}^{(n)}(t_0))$ are linearly independent if and only if $W(t_0) \neq 0$. Thus we can re-express Principle 6 in terms of the Wronskian:

Principle 7: The Wronskian. Suppose that we have n functions $\mathbf{y}^{(1)}(t)$, ..., $\mathbf{y}^{(n)}(t)$, all of which are solutions of the homogeneous equation (2). Then either

(i) $W(t) = 0$ for all t in (α, β), in which case the vector functions $\mathbf{y}^{(1)}(t)$, ..., $\mathbf{y}^{(n)}(t)$ are linearly dependent on (α, β) and, for any fixed point t_0 in (α, β), the (constant) vectors $\mathbf{y}^{(1)}(t_0)$, ..., $\mathbf{y}^{(n)}(t_0)$ are linearly dependent, or

(ii) $W(t) \neq 0$ for all t in (α, β), in which case the vector functions $\mathbf{y}^{(1)}(t)$, ..., $\mathbf{y}^{(n)}(t)$ are linearly independent on (α, β) and, for any fixed point t_0 in (α, β), the (constant) vectors $\mathbf{y}^{(1)}(t_0)$, ..., $\mathbf{y}^{(n)}(t_0)$ are linearly independent.

In particular, Principle 7 implies that if we find n solutions of the homogeneous problem (2) whose Wronskian does not vanish (and it suffices to check the Wronskian at one point) then these can be used to build the general solution, as in (15).

Example 5: Consder the system $\mathbf{y}' = A\mathbf{y}$ with $A = \begin{pmatrix} 4 & 3 \\ -2 & -3 \end{pmatrix}$. Using eigenvalues and eigenvectors one can find two solutions of this system

$$\mathbf{y}^{(1)}(t) = e^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \mathbf{y}^{(2)} = e^{-2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

The Wronskian of these two solutions is

$$W(t) = \begin{vmatrix} -3e^{3t} & -e^{-2t} \\ e^{3t} & 2e^{-2t} \end{vmatrix} = -6e^t - (-2)e^t = 5e^t.$$

Since $W(t) \neq 0$ the solutions $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are linearly independent, and every solution has the form $\mathbf{y}(t) = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$.

Remark 6: There is a traditional way to understand, without using any linear algebra, why at least part of Principle 7 holds—specifically, to show that the Wronskian $W(t)$ is zero either for *all* $t \in (\alpha, \beta)$ or for *no* $t \in (\alpha, \beta)$. Let $g(t) = a_{11}(t) + \cdots + a_{nn}(t)$ be the sum of the diagonal elements of the matrix $A(t)$ ($g(t)$ is called the *trace* of the matrix $A(t)$). One can then show that $W(t)$ satisfies the differential equation

$$W'(t) = g(t)W(t); \tag{17}$$

this is a little messy in general but is Exercise 3.1.35 in Blanchard, Devaney, and Hall for the case of $A(t)$ a constant, 2×2 matrix. But we know how to solve (17), which is just a first order linear equation; the solution is

$$W(t) = Ce^{\int g(t)\,dt},$$

with C a constant. But if $C = 0$ then $W(t) = 0$ for all t, while if $C \neq 0$ then $W(t) \neq 0$ for all t, and this dichotomy is exactly what we wanted to show.

7. Solving the inhomogeneous equation

We finally return to the case of inhomogeneous equations to make a brief observation: combining Principle 2 with Principle 5 we at once obtain

Principle 8: The general solution of an inhomogeneous system. Suppose that $\mathbf{y}^{(1)}$, $\mathbf{y}^{(2)}$, ..., $\mathbf{y}^{(n)}$ are n solutions of the homogeneous equation (2), linearly independent on the interval (α, β), and that $\mathbf{Y}(t)$ is some (particular) solution of the inhomogeneous equation (1). Then every solution of (1) is of the form

$$\mathbf{y}(t) = \mathbf{Y}(t) + c_1\mathbf{y}^{(1)}(t) + \cdots + c_n\mathbf{y}^{(n)}(t). \tag{18}$$