Exam 2 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam.

Note: For all problems asking for phase portraits, make sure to include straight-line solutions (if they exist), correct tangents as $\mathbf{t} \to \pm\infty$, the direction of increasing time, and directions of rotation (if this makes sense for the specific matrix). Your phase portrait should include more than just one trajectory, and should give a general idea of the evolution of ANY initial condition.

1. (a) Find the general solution of the system
$$\mathbf{Y}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{Y}.$$
   (b) Draw the phase portrait of the above system.
   (c) Use the general solution from part (b) to solve the IVP $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
   (d) Using the phase portrait and/or the solution found in (c), plot the components $x(t)$ and $y(t)$ (separate or same set of axes, your choice).

2. In the SIR section, we showed that the solution curves of the SIR model with $S(0) \approx 1$ and $I(0) \approx 0$ are graphs of the function
$$I(S) = -S + \frac{\beta}{\alpha} \ln(S) + 1$$
Note that this graph only depends on the ratio $\rho := \frac{\beta}{\alpha}$ of the parameters.
   (a) Determine the maximum value of $I(S)$ in terms of ρ.
   (b) Is the statement "The epidemic cannot get started if $\beta > \alpha$" true or false? Justify your answer.

3. Consider the linear system
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \mathbf{Y}.$$
   (a) Compute all equilibrium solutions for the above system, and plot them in the phase plane.
   (b) Draw the phase portrait for the system.

4. Suppose that $\mathbf{Y}_1(t) = (-e^{-t}, e^{-t})$ is a solution to some linear system $d\mathbf{Y}/dt = A\mathbf{Y}$. If possible, find an **explicit** solution to the the same system with the following initial conditions. If not possible, state why. *Note:* Solutions exist for all initial conditions, but here I am asking you to find a formula, if possible.
   (a) $\mathbf{Y}(0) = (-2, 2)$
   (b) $\mathbf{Y}(0) = (3, 4)$
   (c) $\mathbf{Y}(0) = (0, 0)$
   (d) $\mathbf{Y}(0) = (3, -3)$

5. Consider the system of first-order equations
$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\sin x. \end{aligned}$$
Note that this is a **non-linear** set of equations.

(a) Find an equation $h(x, y) = c$ satisfied by the trajectories. That is, identify the function $h(x, y)$ of which trajectories in the phase are level sets. Note that different c correspond to different initial conditions.

(b) Consider the same system, supplemented with initial conditions

$$x(0) = \frac{\pi}{2}$$
$$y(0) = -1.$$

Find the expression $y = y(x)$ in the phase plane which gives the trajectory of the solution of this IVP. In other words, find the graph of this specific trajectory in the xy-plane.

6. Consider the linear system

$$\mathbf{Y}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{Y}.$$

(a) Find the general solution for the above system.

(b) Draw the corresponding phase portrait.

(c) Use the general solution from part (a) to solve the initial-value problem $\mathbf{Y}(0) = (3, 2)$.

(d) Plot the first component, $x(t)$ of the solution in part (c) as a function of time ($t \in (-\infty, \infty)$). Be sure to include all pertinent information (horizontal asymptotes, relative maxima/minima, etc.).

7. Consider the second-order, constant-coefficient, homogeneous linear differential equation

$$ay'' + by' + cy = 0,$$

where a, b, and c are **positive**. Show that all solutions of this equation tend to 0 as $t \to \infty$. Is this true if $b = 0$?

8. Problem #21, page 379 in the textbook (this is in the review exercises for Chapter 3).

9. (a) Compute the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{Y}.$$

(b) Sketch the phase portrait.

(c) Solve the IVP, with $\mathbf{Y}(0) = (-2, 3)$.

(d) Sketch the $x(t)$ and $y(t)$ graphs for the solution of the initial-value problem found in part (c).

10. Let A be a 2×2 matric. True or false: the linear system $\mathbf{Y}' = A\mathbf{Y}$ can have **exactly** three distinct straight-line solutions.

11. In the absence of damping, the motion of a spring-mass system satisfies the initial-value problem

$$my'' + ky = 0, \quad y(0) = a, \quad y'(0) = b.$$

(a) Show that the kinetic energy ($\frac{1}{2}mv^2$) *initially* imparted to the mass is $mb^2/2$, and that the potential energy *initially* stored in the spring is $ka^2/2$, so that the total energy initially is $(ka^2 + mb^2)/2$.

(b) Solve the IVP.

(c) Using the solution from part (b), determine the total energy in the system *at any time* t. How does this relate to a well-known physical principle?

12. Sketch the phase portrait for the system

$$\mathbf{Y}' = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

13. Consider the matrix $A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$.

(a) Find all eigenvalues and a corresponding eigenvector for A.

(b) Find an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

(c) Compute e^{At} using your result from part (b). Simplify to obtain a single 2×2 matrix.

(d) Solve the IVP

$$\mathbf{Y}' = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \mathbf{Y},$$
$$\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

14. (a) Find the general solution of the linear system

$$\mathbf{Y}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{Y}$$

(b) Use the above to solve the IVP with initial conditions

$$\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) Plot the phase portrait for this system.