1 Introduction

My research broadly focuses on dynamics and differential equations, and in particular bridging the gap between what can be proven mathematically and what can be computed numerically.

For example, standard numerical methods can solve an initial value problem for an ODE and provide *local* error bounds at each step. However a *global* error bound on the final solution requires the cumulative error be quantified. This quickly becomes a nontrivial problem in chaotic systems, where arbitrarily close initial conditions will inevitably diverge, and the difficulties compound in partial differential equations where the phase space is infinite dimensional.

To that end, *rigorous numerics* have been developed to keep track of all the sources of error inherent to numerical calculations. To bridge the gap between rigorous numerics and a *computer assisted proof*, a problem must be translated into a list of the conditions that the computer can check. Most famously used to solve the four color theorem [1], computer assisted proofs have been employed to great effect in dynamics, proving results such as the universality of the Feigenbaum constants [2] and Smale's 14^{th} problem on the nature of the Lorenz attractor [3].

I am particularly interested in developing constructive methods to study *invariant sets and their stability in infinite dimensional dynamical systems.* Analytically, this involves proving theorems with explicitly verifiable hypotheses (e.g. rather than assuming "there exists some $\epsilon > 0$", concretely quantifying how small ϵ must be). Computationally, this involves variety of numerical techniques from fields such as dynamical systems, partial differential equations, global optimization, and applied algebraic topology.

2 Wright's Conjecture on a Nonlinear Delay Differential Equation

In my thesis I proved two half-century old conjectures concerning the delay differential equation known as Wright's equation:

$$x'(t) = -\alpha\left(e^{x(t-1)} - 1\right). \tag{1}$$

First studied in 1955 as a heuristic model of the distribution of primes [4], Wright's equation has come to be known as a canonicial example of a nonlinear scalar delay differential equation (DDE). As with partial differential equations, the initial data for DDEs are functions. In Wright's seminal work he showed that if $\alpha \leq \frac{3}{2}$ then the equilibrium solution $x \equiv 0$ is the global attractor, and made the following conjecture:

Theorem 1 (Wright's Conjecture, 1955)**.** *For every $0 < \alpha \leq \frac{\pi}{2}$ the equilibrium solution $x \equiv 0$ to* (1) *is globally attractive.*

In 1962 Jones [5] proved that for $\alpha > \frac{\pi}{2}$ there exists at least one *slowly oscillating periodic solution (SOPS)* to Wright's equation. That is, a periodic solution $x : \mathbb{R} \to \mathbb{R}$ such that it is positive for at least the length of the time delay, and then negative for at least the length of the time delay. Based on numerical simulations Jones made the following conjecture:

Theorem 2 (Jones' Conjecture, 1962)**.** *For every $\alpha > \frac{\pi}{2}$ there is a unique slowly oscillating periodic solution to* (1)*.*

I proved both of these conjectures in a trio of papers [6–8]. Prior work had proved Wright's conjecture for $\alpha < \frac{\pi}{2} - 2 \times 10^{-4}$ via a computer assisted proof which took months of CPU time, and as authors mention, "*substantial improvement of the theoretical part of the present proof is needed to prove Wright's conjecture fully*" [9]. Hopf bifurcations are canonically analyzed with the method of normal forms, which transforms a given equation into a simpler expression having the same qualitative behavior as the original equation. By an implicit-function-theorem type argument, this transformation is valid in some neighborhood of the bifurcation. However, the proof does not offer any insight into the size of this neighborhood. In [6] with JB van den Berg (VU Amsterdam) we develop an explicit description of a neighborhood wherein the only periodic solutions are those originating from the Hopf bifurcation. The main result of this analysis is the resolution of Wright's conjecture.

In 1991 Xie [10] proved Jones' conjecture for $\alpha \geq 5.67$. He accomplished this by first showing that there is a unique SOPS to (1) if and only if every SOPS is asymptotically stable. By using asymptotic estimates of SOPS for large α, Xie was able to estimate their Floquet multipliers and prove that all SOPS had to be stable. However, at $\alpha = 5.67$ these asymptotic estimates break down.

In [7] with JP Lessard (McGill University) and K Mischaikow (Rutgers University) we used the same basic method as Xie, however we replace the asymptotic estimates with rigorous numerics. We use a branch and bound algorithm to develop pointwise estimates on all the possible SOPS to Wright's equation and then bound their Floquet multipliers. Using these two main steps, we generate a computer assisted proof for $\alpha \in [1.9, 6.0]$ that all SOPS to Wright's equation are asymptotically stable, and thereby unique up to translation.

I finished the proof to Jones' conjecture in [8], proving there is a unique SOPS for $\alpha \in (\frac{\pi}{2}, 1.9]$. While previous work [6] showed that there are no folds in the principal branch of periodic orbits this did not rule out the possibility of *isolas*, that is SOPS far away from the principal branch. To rule out the existence of these isolas, we recast the problem of studying periodic solutions to (1) as the problem of finding zeros of a functional defined on a space of Fourier coefficients, and again employed an infinite dimensional *branch and bound* algorithm. In future work, I plan extend these Fourier-spectral techniques to produce computer-assisted-proofs for an exact count of the number of equilibria to nonlinear parabolic PDEs (e.g. Swift-Hohenberg) on finite intervals, and eventually multidimensional domains.

A corollary to Jones' conjecture is that *all* periodic solutions to Wright's equation, even the rapidly oscillating ones, originate from Hopf bifurcations; there are no isolas of periodic solutions. A conjecture I pose in [8] is the following:

Conjecture 3. *The period length of the SOPS to Wright's equation increases monotonically in* α*.*

This conjecture is of particular interest, because it would imply that the *only* bifurcations of periodic solutions to Wright's equation are Hopf bifurcations. Furthermore, it would imply the so called generalized Wright's conjecture [9], that the global attractor in Wright's equation is the closure of the (finite dimensional) global unstable manifold of the zero solution.

3 Patterns and Stability in Partial Differential Equations

Connecting orbits provide a road map for how a dynamical system transitions between its various fixed points and periodic orbits. Certain kinds of connecting orbits, such as homoclinics from a periodic orbit to itself, can be used to prove the existence of mathematical chaos. In the spatial dynamics of a PDE, a connecting orbit corresponding to a standing wave describes how two homogeneous steady states can coexist with a transition zone in between. In the temporal dynamics of a PDE, a connecting orbit between two nonhomogeneous equilibria describes how perturbations to an unstable equilibrium unfold, and to which stable equilibrium the perturbed state will be attracted. A long term goal of my research program is studying connecting orbits in the temporal dynamics of PDEs and DDEs.

In general, connecting orbits are calculated by solving a boundary value problem between an unstable manifold and a stable manifold. In forthcoming work [11] with JB van den Berg (VU Amsterdam) and J Mireles James (Florida Atlantic University), we present a rigorous computational method for approximating infinite dimensional stable manifolds of non-trivial equilibria for parabolic PDEs. Our approach combines the parameterization method – which can provide high order approximations of finite dimensional manifolds with validated error bounds – together with the Lyapunov-Perron method – which is a powerful technique for proving the existence of (potentially infinite dimensional) invariant and inertial manifolds. As an example, we apply this technique to approximate the stable manifold associated with unstable nonhomogeneous equilibria for the Swift-Hohenberg equation on a finite interval. In future work we plan to rigorously compute saddle to saddle connecting orbits in the Swift-Hohenberg equation, and additionally to develop methods to rigorously approximate stable manifolds of periodic orbits in parabolic PDEs (e.g. Kuromoto Shivashinski). Longer term goals include constructing a computer assisted proof of chaos in a PDE/DDE via a homoclinic tangle, and computing connecting orbits in strongly indefinite problems motivated from Floer homology.

In submitted work [12] with A Takayasu (University of Tsukuba), JP Lessard (McGill University), and H Okamoto (Kyoto University), we prove the existence of finite time blow-up in the PDE $u_t = u_{xx} + u^2$ for $x \in [0,1]$ and periodic boundary conditions. We accomplish this by developing a numerical method to rigorously compute solutions of Cauchy problems. By solving the equation along a contour in the complex plane of time we are able to prove the existence of a branching singularity. Namely, for a contour defined as $\Gamma_\theta = \{z \in \mathbb{C} : z = te^{i\theta}\ t \geq 0\}$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ this PDE can be written as

$$u_t = e^{i\theta}(u_{xx} + u^2), \tag{2}$$

a PDE analogous to the complex Ginzburg-Landau equation with a quadratic nonlinearity. Furthermore, using an approach based on the Lyapunov-Perron method we calculate part of a codimension-0 center-stable manifold of the zero equilibrium. This allows us to prove that our same initial condition which blows up in real time will converge to zero along fixed contours Γ_θ, yielding the global existence of the solution. In current work with JP Lessard and A Takayasu we are working to prove the existence of an infinite family of saddle to center-stable connecting orbits for (2).

Additional difficulties arise when developing rigorous numerics for PDEs on unbounded domains. Recent work on the Maslov index has extended classical results from Sturm-Liouville theory to a much more general setting, thus allowing for spectral stability of nonlinear waves in a variety of contexts to be determined by counting conjugate points. With M Beck (Boston University) we are developing a framework for the computation of conjugate points using rigorous numerics [13]. We apply our method to a parameter-dependent system of bistable equations and show that there exist both stable and unstable standing fronts. In comparison with rigorous numerical methods to compute stability using the Evans function [14], our preliminary results suggest that counting conjugate points is much more efficient.

4 Computational Algebraic Topology

Partial differential equations can be extremely useful in describing patterns in biological and physical systems. However, these patterns can be quite complicated, exhibiting distinct structures at different spatial/temporal scales, and it is usually impossible to completely understand them using analytical methods. Often a coarser but computationally tractable description is needed. In recent years computational topology has become widely recognized as an important tool for quantifying complex structures.

Persistent homology is an algebraic tool that provides a mathematical framework for analyzing the multi-scale structures frequently observed in nature. More specifically, it tracks how the homology groups of a filtration of topological spaces $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_N$ are mapped as one space is included into the next. Similar to classical Morse theory, a filtration can be generated from the sublevel sets of a continuous function $f : X \to \mathbb{R}$ by defining $X_t = f^{-1}((-\infty, t])$. For example, the sublevel set filtration of the distance function to a point cloud in $\{x_i\} \subseteq \mathbb{R}^n$ corresponds to growing ϵ-balls about each point. The 0-dimensional persistent homology tracks when connected components first appear in the filtration, and later merge together; the 1-dimensional persistent homology tracks when loops in the space appear and disappear. Namely, if a specific homology generator first appears in $H_j(X_b)$, and is first mapped to zero by the inclusion induced map $i_* : H_j(X_b) \to H_j(X_d)$, then $[b, d]$ is referred to as the corresponding j-dimensional *persistence interval*. Long persistence intervals are generally considered to correspond to important topological features, whereas short intervals are considered to be noise.

If a point cloud is sampled from a d-dimensional Lebesgue measure, the important persistence intervals will stabilize as the number of points n increases, and the average length of the noisy intervals will decrease. However, the summed-length of all 0-dimensional persistence intervals will grow in proportion with $n^{\frac{d-1}{d}}$. In fact, a fractional dimension can be defined for a measure in terms of the asymptotic growth of the totaled persistence intervals of point samples [15].

In submitted work [16] with B Schweinhart (Ohio State University) we implement an algorithm to estimate the i-dimensional persistent homology dimension ($i = 0, 1, 2$) to study self-similar fractals, chaotic attractors, and an empirical dataset of earthquake hypocenters. We compare the performance of these persistent homology dimensions to classical methods to compute the correlation and box-counting dimensions. In summary, the performance of the 0-dimensional persistent homology dimension is comparable to that of the correlation dimension, and generally better than box-counting, whereas the higher persistent homology dimensions are worse.

When studying multiscale behavior a suitable discretization needs to be chosen. Fine structures may require a fine discretization to accurately describe, whereas a coarser discretization may be sufficient for large regions of space. With M Kramar (University of Oklahoma) we developed a theoretical framework for the algorithmic computation of an arbitrarily good approximation of sublevel-set persistent homology [17]. We implement a rigorous numerical method to compute the persistent homology in the case $f : [0, 1]^2 \to \mathbb{R}$ and provide *a posteriori* bounds of the approximation error introduced.

5 Chaos in Fast-Slow Systems

In ongoing work, I am studying routes to chaos in fast-slow systems. One particular example is the 2D Rulkov map, a phenomenological model of a bursting neuron akin to a logistic map with a slowly varying parameter. Using the recent method of weighted Birkhoff averages [18], we are able to detect the onset of chaos well before the system has a decidedly positive Lyapunov exponent. For a significant region of parameter space, the attractor is technically chaotic, however the slow variable just appears to be noisily following a periodic cycle. In contrast, for larger nonlinearities both the fast and slow variables are markedly chaotic. Standard tools such as Lyapunov exponents are unable to distinguish between these two types of chaos. Together with E Sander (George Mason University) and J Touboul (Brandeis University) we are developing a methodology to distinguish these two types of chaos.

The 2D FitzHugh-Nagumo PDE is often used as a phenomenological model for cardiac dynamics. This model exhibits spiral waves (corresponding to ventricular tachycardia) and complex spatio-temporal behavior (corresponding to ventricular fibrillation). With À Jorba (University of Barcelona) and À Haro (University of Barcelona) we are beginning to study spiral core breakup along a quasiperiodic route to chaos, whereby the core of the spiral wave first begins to *meander*, which is to say that the center of the spiral wave periodically moves around with a second frequency. We aim to develop a numerical method to compute meandering spiral waves and their stable/unstable bundles, similar to methods used to compute invariant tori in fluid flows [19]. In future work, we hope to give a computer assisted proof of chaos in the 2D FitzHugh-Nagumo PDE.

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