# Math 654
# Introduction to Mathematical Fluid Dynamics

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# Lecture 1: Vectors, Tensors, and Operators

## 1 Vectors: Notation and Operations

Given a vector $\mathbf{x} \in \mathbb{R}^3$, we can write it with respect to the canonical basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ as $\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. In this manner, we can define vector fields as

$$\mathbf{v}(x, y, z) = u(x, y, z)\hat{\mathbf{i}} + v(x, y, z)\hat{\mathbf{j}} + w(x, y, z)\hat{\mathbf{k}}.$$

Note that sometimes the canonical basis is written as $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, and similarly

$$\begin{aligned} \mathbf{x} &= x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3, \\ \mathbf{v}(x_1, x_2, x_3) &= v_1(x_1, x_2, x_3)\hat{\mathbf{e}}_1 + v_2(x_1, x_2, x_3)\hat{\mathbf{e}}_2 + v_3(x_1, x_2, x_3)\hat{\mathbf{e}}_3. \end{aligned}$$

In this way we easily generalize to $\mathbb{R}^d$, with a vector field being

$$\mathbf{v}(x_1, \ldots, x_d) = \sum_{i=1}^{d} v_i(x_1, \ldots, x_d)\hat{\mathbf{e}}_i$$

or just

$$\mathbf{v}(\mathbf{x}) = v_i(\mathbf{x})\hat{\mathbf{e}}_i$$

using "Einstein notation."[1] The function $v_i$ is commonly referred to as the "$i$th component" of the vector field.

We have the following operations on pairs of vectors.

**Definition 1.** The *dot product* (or *inner product*) of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = v_i w_i.$$

We can arrive at this with the following formalism. First, define the dot product on the canonical basis as

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then write $\mathbf{v} = v_i\hat{\mathbf{e}}_i$ and $\mathbf{w} = w_j\hat{\mathbf{e}}_j$, and define

$$\mathbf{v} \cdot \mathbf{w} = v_i w_j (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j).$$

Carrying out the implied summation yields the earlier definition.

**Definition 2.** The *outer product* of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ is a linear self-mapping of $\mathbb{R}^d$ defined via

$$\mathbf{v}\mathbf{w} = v_i w_j \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j,$$

where $\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$ is the linear self-mapping of $\mathbb{R}^d$ having as matrix representation in the canonical bases

$$(\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j)_{mn} = \delta_{im}\delta_{jn}.$$

So, the outer product of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ produces the linear map with the canonical matrix representation

$$(\mathbf{v}\mathbf{w})_{mn} = v_m w_n,$$

a so-called "dyadic tensor." This brings us to the next topic.

[1] A handy notation where repeated indices imply summation.

## 2 Tensors

**Definition 3.** A *2-tensor* on $\mathbb{R}^d$ is a bilinear form on $\mathbb{R}^d$. Specifically, $T : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a 2-tensor if it satisfies

1. Component-wise additivity:
$$\begin{aligned} T(\mathbf{v}+\mathbf{v}', \mathbf{w}) &= T(\mathbf{v}, \mathbf{w}) + T(\mathbf{v}', \mathbf{w}) \\ T(\mathbf{v}, \mathbf{w}+\mathbf{w}') &= T(\mathbf{v}, \mathbf{w}) + T(\mathbf{v}, \mathbf{w}'), \end{aligned}$$
2. Component-wise homogeneity:
$$\begin{aligned} T(\alpha\mathbf{v}, \mathbf{w}) &= \alpha T(\mathbf{v}, \mathbf{w}) \\ T(\mathbf{v}, \beta\mathbf{w}) &= \beta T(\mathbf{v}, \mathbf{w}), \end{aligned}$$
given $\alpha, \beta \in \mathbb{R}$.

**Proposition 1.** *The set of 2-tensors on $\mathbb{R}^d$ is isomorphic to the set of linear self-maps of $\mathbb{R}^d$.*

In other words, 2-tensors are matrices; we pursue this idea throughout the rest of this section. First, just as the set of linear self-maps of $\mathbb{R}^d$ forms a linear space, the set of 2-tensors on $\mathbb{R}^d$ forms a linear space. Moreover, one can easily show that $\{\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\}_{1\le i,j\le d}$ is a basis. Given a 2-tensor $T$, we use this basis to write

$$T = T_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j,$$

so $T$ has the canonical matrix representation

$$(T)_{ij} = T_{ij}.$$

We have a way to write the operation of $T$ on $\mathbf{v} \in \mathbb{R}^d$ using tensor notation:

$$\begin{aligned} T\cdot\mathbf{v} &= (T_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j)\cdot(v_k\hat{\mathbf{e}}_k) \\ &= T_{ij}v_k\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k \\ &= T_{ij}v_k\delta_{jk} \\ &= (T_{ij}v_j)\,\hat{\mathbf{e}}_i. \end{aligned}$$

The key thing to notice is that $\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k = \delta_{jk}$, which follows from the definition of $\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$. Composition of 2-tensors is just composition of linear maps; in tensor notation,

$$\begin{aligned} T\cdot T' &= T_{ij}T'_{kl}\,(\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j)\cdot(\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l) \\ &= T_{ij}T'_{kl}\delta_{jk}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l \\ &= T_{ij}T'_{jl}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l. \end{aligned}$$

Again, the key step here is $(\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j)\cdot(\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l) = \delta_{jk}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_l$. Notice that the canonical matrix representation of $T\cdot T'$ is the matrix product of the representations of $T$ and $T'$, a sensible proposition.

Square matrices can be symmetric or anti-symmetric, and so can 2-tensors:

**Definition 4.** $T$, a 2-tensor on $\mathbb{R}^d$, is said to be *symmetric* if $T_{ij} = T_{ji}$, and is said to be *anti-symmetric* if $T_{ij} = -T_{ji}$.

Generally, a 2-tensor can be written as a sum of its symmetric and anti-symmetric parts. Indeed,

$$T_{ij} = S_{ij} + A_{ij}$$

where

$$\begin{aligned} S_{ij} &= \frac{1}{2}\left(T_{ij} + T_{ji}\right), \\ A_{ij} &= \frac{1}{2}\left(T_{ij} - T_{ji}\right). \end{aligned}$$

Just like matrices, we have a trace operation.

**Definition 5.** Given a 2-tensor $T$ on $\mathbb{R}^d$, its *trace* is

$$\mathrm{Tr}\,(T) = T_{ii} = \sum T_{ij}\delta_{ij}.$$

And we may "contract" tensors as well.

**Definition 6.** Let $T'$ be another 2-tensors on $\mathbb{R}^d$, then the *tensor contraction* of $T$ and $T'$ is

$$T : T' = T_{ij}T'_{ij}.$$

Note that just as with the inner product, we can also define tensor contraction formally by writing

$$\begin{aligned} T : T' &= T_{ij}T'_{kl}\left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k\right)\left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_l\right) \\ &= T_{ij}T'_{kl}\delta_{ik}\delta_{jl} \end{aligned}$$

and summing up over all indices.

*Remark.* We've seen quite a few definitions regarding 2-tensors, motivated by linear algebra. We can generalize the idea of a tensor by considering mappings $T : \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{k \text{ products}} \to \mathbb{R}$ which are multilinear (i.e., linear in each component). These so-called $k$-tensors do not have analogs in linear algebra, but are sometimes visualized as "multi-index arrays."

**Example.** (Cross-product) We all know that the cross-product of two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ can be written as a formal determinant

$$\mathbf{v} \times \mathbf{w} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

By defining a tensor known as the "Levi-Civita symbol,"

$$\epsilon_{ijk} = \begin{cases} 1 & \{i,j,k\} \text{ a cyclic perm. of } \{1,2,3\} \\ -1 & \{i,j,k\} \text{ an anti-cyclic perm. of } \{1,2,3\} \\ 0 & \text{otherwise} \end{cases},$$

we can write the cross-product in a compact way:

$$\left(\mathbf{v} \times \mathbf{w}\right)_i = \epsilon_{ijk}v_jw_k.$$

## 3 Derivative Operators

Given coordinates $x_1, \ldots, x_d$ in $\mathbb{R}^d$ we know how to take derivatives. It will be convenient to make the following notation. Let $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable, and write

$$\frac{\partial f}{\partial x_i} = \partial_{x_i} f = \partial_i f = f_{,i}$$

to mean the partial derivative of $f$ w.r.t. the $i$th coordinate. In such a way, we have Leibniz's rule:

$$(fg)_{,i} = f_{,i} g + f g_{,i}.$$

**Definition 7.** The *gradient operator* is the differential operator

$$\operatorname{grad}(\cdot) = \hat{\mathbf{e}}_i \partial_i (\cdot),$$

and is usually written as $\nabla$.

*Remark.* In Cartesian coordinates,

$$\nabla = \hat{\mathbf{e}}_i \partial_i = \partial_i \hat{\mathbf{e}}_i,$$

but watch out in other coordinate systems! For example, this does not hold in polar coordinates. For the rest of this lecture, we'll work in Cartesian coordinates, unless otherwise specified.

For differentiable $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\nabla f = (\partial_i f)\, \hat{\mathbf{e}}_i,$$

and for differentiable $\mathbf{v}(\mathbf{x})$,

$$\begin{aligned} \nabla \mathbf{v} &= (\hat{\mathbf{e}}_i \partial_i)(\hat{\mathbf{e}}_j v_j) \\ &= \frac{\partial v_j}{\partial x_i} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \\ &= v_{j,i} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j. \end{aligned}$$

Composing the gradient operator is allowed, e.g.,

$$\nabla \nabla f = \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \frac{\partial^2 f}{\partial x_i \partial x_j} = \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j f_{,ij}.$$

And we can even take the gradient of a tensor, e.g.,

$$(\nabla T)_{ijk} = T_{jk,i}$$

for some differentiable 2-tensor $T$.

**Definition 8.** The *divergence operator* is the differential operator on vector fields defined by

$$\operatorname{div} \mathbf{v} = \operatorname{Tr}(\nabla \mathbf{v}) = v_{i,i},$$

and is usually written as $\nabla \cdot \mathbf{v}$. Similarly, we have the *curl operator* which is defined by

$$\operatorname{curl} \mathbf{v} = \epsilon_{ijk} \partial_j v_k,$$

and is usually written as $\nabla \times \mathbf{v}$.

With these definitions, we can begin to develop some useful identities.

**Proposition 2.** *Given a vector field* $\mathbf{v}$*,*

$$\nabla \times \nabla \times \mathbf{v} = \nabla (\nabla \cdot \mathbf{v}) - \triangle \mathbf{v}.$$

*Remark.* This is used to define the Laplacian in general coordinates for vector fields – a non-trivial job in arbitrary coordinate systems because derivatives of unit vectors must be considered.

*Proof.* On the left, we have

$$\epsilon_{ijk} \partial_j (\epsilon_{klm} \partial_l v_m) = \epsilon_{ijk} \epsilon_{klm} v_{m,jl}.$$

It's easy to show that

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl},$$

so the left hand side becomes

$$\begin{aligned}
(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) v_{m,jl} &= \delta_{il} v_{j,jl} - v_{i,jj} \\
&= v_{j,ji} - v_{i,jj} \\
&= \partial_i (v_{i,i}) - \triangle v_i
\end{aligned}$$

which is exactly the $i$th component of the right hand side. □

**Definition 9.** The *vorticity* of a vector field is exactly its curl:

$$\omega = \nabla \times \mathbf{v}.$$

In components,

$$\omega_i = \epsilon_{ijk} v_{k,j}.$$

Now consider a 2-tensor $\Omega$ with components

$$\begin{aligned}
\Omega_{ij} &= \epsilon_{ijk} \omega_k \\
&= \epsilon_{ijk} \epsilon_{klm} v_{m,l} \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) v_{m,l} \\
&= v_{j,i} - v_{i,j} \\
&= 2 \cdot (\text{anti-symmetric part of } v_{j,i}).
\end{aligned}$$

This tensor has canonical matrix representation

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

**Proposition 3.** *Given a vector field* $\mathbf{v}$*,*

$$\nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = \mathbf{v} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{v} + (\nabla \cdot \mathbf{v}) \omega.$$

*Proof.* The $i$th component on the left hand side is

$$\begin{aligned} \epsilon_{ijk}\partial_j \left(v_l \partial_l v_k\right) &= \epsilon_{ijk}\partial_j \left(v_{k,l} v_l\right) \\ &= \epsilon_{ijk} v_{k,lj} v_l + \epsilon_{ijk} v_{k,l} v_{l,j}. \end{aligned}$$

The first term in the sum becomes

$$\begin{aligned} \epsilon_{ijk} v_{k,lj} v_l &= v_l \partial_l \left(\epsilon_{ijk} v_{k,l}\right) \\ &= \left(\mathbf{v}\cdot\nabla\omega\right)_i . \end{aligned}$$

Before we deal with the second term, we note that the following hold:

$$\begin{aligned} \epsilon_{ijk} v_{l,k} v_{l,j} &= 0, \\ v_{k,l} v_{l,j} - v_{l,k} v_{l,j} &= \epsilon_{lkm}\omega_m v_{l,j}. \end{aligned}$$

So, the second term becomes

$$\begin{aligned} \epsilon_{ijk} v_{k,l} v_{l,j} &= \epsilon_{ijk}\left(v_{k,l} v_{l,j} - v_{l,k} v_{l,j}\right) \\ &= \epsilon_{ijk}\epsilon_{lkm}\omega_m v_{l,j} \\ &= \left(\delta_{im}\delta_{jl} - \delta_{il}\delta_{im}\right)\omega_m v_{l,j} \\ &= \omega_i v_{j,j} - \omega_j v_{i,j} \\ &= \left(-\omega\cdot\nabla\mathbf{v} + \left(\nabla\cdot\mathbf{v}\right)\omega\right)_i . \end{aligned}$$

Hence, the result. $\square$

## Lecture 2: Convective Derivatives and Conservation Equations

What do we mean by a "fluid?" A state of matter that cannot support shear stresses, as opposed to a solid. We'll primarily consider "ideal" fluids (non-viscous, etc.).

Our basic setup is in $\mathbb{R}^3$. Consider a subset $\Omega \subset \mathbb{R}^3$ which contains fluid matter.

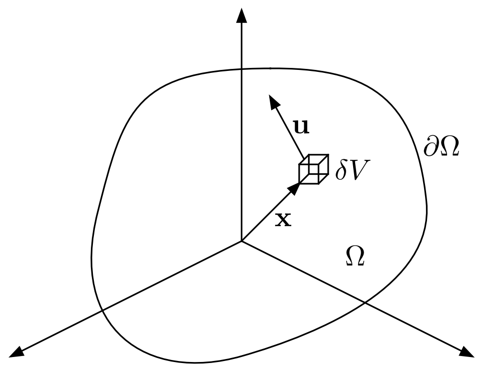



Figure 0.1: A subset of $\mathbb{R}^3$ containing fluid.

We can consider several ideas:

- Density, $\rho(\mathbf{x}, t)$
- Mass (infinitesimal), $\delta m = \rho(\mathbf{x}, t)\delta V$
- Velocity field $\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{i}}u + \hat{\mathbf{j}}v + \hat{\mathbf{k}}w = \hat{\mathbf{e}}_i u_i = (u_x, u_y, u_z)$
- Pressure field, $p(\mathbf{x}, t)$.

**Definition 1.** Consider the force into a fluid element across the face with outward pointing normal $\hat{\mathbf{n}}$ and area $\delta a$ to be

$$-\hat{\mathbf{n}}p\delta a,$$

which implicitly defines the scalar quantity $p(\mathbf{x}, t)$. This quantity is called the *pressure*.

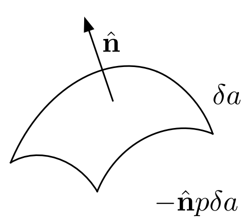



Figure 0.2: Force on an infinitesimal area $\delta a$.

The dynamics of these variables (all presumed to be smooth functions) are related by

- Conservation of mass
- Newton's 2nd Law
- Thermodynamics.

# 1 The Convective Derivative

This is a fundamental kinematic principle. Consider $f(\mathbf{x}, t)$. There are two ways of computing the rate of change of this quantity with respect to time. First, we could fix position and calculate the time derivative:

$$\left(\frac{df}{dt}\right)_{\text{fixed position}} = \frac{\partial f}{\partial t} = \lim_{\delta t \to 0} \frac{f(\mathbf{x}, t+\delta t) - f(\mathbf{x}, t)}{\delta t}.$$

Or, we may consider calculating the rate of change of $f$ while moving along with fluid elements:

$$\left(\frac{df}{dt}\right)_{\text{moving}} = \lim_{\delta t \to 0} \frac{f(\mathbf{x}+\mathbf{u}(\mathbf{x}, t)\delta t) - f(\mathbf{x}, t)}{\delta t},$$

to first order (which suffices in the limit). "Moving with the flow" means

$$\dot{\mathbf{X}}(t) = \mathbf{u}\left(\mathbf{X}(t), t\right),$$

where $\mathbf{X}(t)$ is the path we take through the fluid. In this manner, we find

$$\begin{aligned} \left(\frac{df}{dt}\right)_{\text{moving}} &= \frac{d}{dt} f\left(\mathbf{X}(t), t\right)\Big|_{\mathbf{X}(t)=\mathbf{X}} \\ &= \frac{\partial f}{\partial t}(\mathbf{x}, t) + \dot{\mathbf{X}}_i \frac{\partial f}{\partial x_i}(\mathbf{x}, t) \\ &= \frac{\partial f}{\partial t} + \mathbf{u}.\nabla f. \end{aligned}$$

We'll denote the convective derivative by $\frac{Df}{Dt}$.

**Proposition.** *The convective derivative is a differential operator, i.e., it satisfies the product rule, the chain rule, etc.*

# 2 Conservation Equations

## 2.1 Conservation of Mass

Consider a fluid element, having mass $\delta m = \rho \delta V$. As the fluid flows, the location, volume, and shape of this element may change. Conservation of mass means

$$\begin{aligned} 0 &= \frac{D\delta m}{Dt} \\ &= \frac{D\rho}{Dt}\delta V + \rho \frac{D\delta V}{Dt} \\ &= \left(\frac{\partial \rho}{\partial t} + \mathbf{u}.\nabla \rho\right)\delta V + \rho \frac{D\delta V}{Dt}. \end{aligned}$$

Suppose $\delta V = \delta x \delta y \delta z$, then

$$\frac{D\delta V}{Dt} = \left(\frac{D\delta x}{Dt}\right)\delta y \delta z + \dots .$$

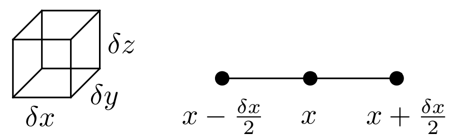



Figure 2.1: Infinitesimal volume element.

The rate of change of this line segment depends on the difference in fluid velocity at its endpoints. So,

$$\begin{aligned}
\frac{D\delta x}{Dt} &= u_1(x+\delta x/2, y, z, t) - u_1(x - \delta x/2, y, z, t) \\
&= u_1(x,y,z,t) + \frac{\delta x}{2}\frac{\partial u_1}{\partial x}(x,y,z,t) + \dots - u_1(x,y,z,t) + \frac{\delta x}{2}\frac{\partial u_1}{\partial x}(x,y,z,t) + \dots \\
&= \frac{\partial u_1}{\partial x}\delta x
\end{aligned}$$

and similarly for $\delta y, \delta z$. Hence,

$$\frac{D\delta V}{Dt} = (\nabla . \mathbf{u})\,\delta V,$$

and so conservation of mass becomes

$$0 = \left(\frac{\partial \rho}{\partial t} + \mathbf{u}.\nabla\rho\right)\delta V + \rho\,(\nabla.\mathbf{u})\,\delta V.$$

We conclude that

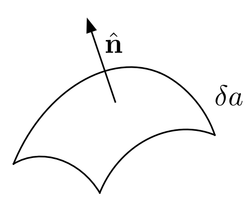
$$0 = \frac{\partial \rho}{\partial t} + \mathbf{u}.\nabla\rho + \rho\,(\nabla.\mathbf{u})\,,$$

and after applying a vector identity,

$$0 = \frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{u}).$$

This is called the **continuity equation**.

*Remark.* The mass flux of a flow field is given by $\rho\mathbf{u}$.




$$(\rho\mathbf{u})\,.\hat{\mathbf{n}}\delta a = \tfrac{\text{mass}}{\text{time}} \text{ flowing through } \delta a$$

Figure 2.2: Mass flux through an infinitesimal area $\delta a$.

*Remark.* The continuity equation is linear in $\mathbf{u}$, so it would seem thus far that things are straightforward.

### 2.2 Conservation of Momentum (Newton's 2nd Law)

Recall that $\mathbf{F} = \frac{d}{dt}\mathbf{p}$. Applied to a fluid element,

$$\begin{aligned}\frac{D}{Dt}(\delta m\mathbf{u}) &= \left(\frac{D}{Dt}\delta m\right)\mathbf{u} + \delta m\left(\frac{D}{Dt}\mathbf{u}\right)\\ &= \delta m\left(\frac{D}{Dt}\mathbf{u}\right)\end{aligned}$$

due to conservation of mass. Continuing,

$$\begin{aligned}\frac{D}{Dt}(\delta m\mathbf{u}) &= \delta m\left(\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right)\left(\hat{\mathbf{i}}u + \hat{\mathbf{j}}v + \hat{\mathbf{k}}w\right)\\ &= \delta m\left(\partial_t\mathbf{u} + \mathbf{u}.\nabla\mathbf{u}\right)\\ &= \delta\mathbf{F}\end{aligned}$$

by Newton's 2nd law.

**Definition 2.** An *ideal fluid* is one in which there are no shear stresses.

Note that this definition permits normal forces, which we account for in the pressure field.

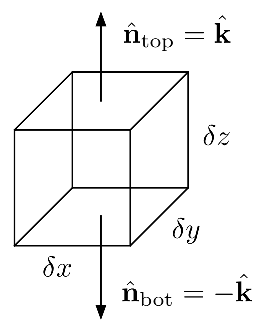



Figure 2.3: Normal forces on a fluid element.

The net force in the $z$-direction is

$$\begin{aligned}\hat{\mathbf{k}}\delta F_3 &= -\hat{\mathbf{n}}_{\text{top}}p_{\text{top}}\delta x\delta y - \hat{\mathbf{n}}_{\text{bot}}p_{\text{bot}}\delta x\delta y\\ &= \hat{\mathbf{k}}\delta x\delta y(-p_{\text{top}} + p_{\text{bot}}).\end{aligned}$$

Taylor expanding, we find

$$\begin{aligned}-p_{\text{top}} + p_{\text{bot}} &= -p(x,y,z+\delta z/2,t) + p(x,y,z-\delta z/2,t)\\ &= -p(x,y,z,t) - \frac{\delta z}{2}\frac{\partial p}{\partial z}(x,y,z,t) + p(x,y,z,t) - \frac{\delta z}{2}\frac{\partial p}{\partial z}(x,y,z,t)\\ &= -\frac{\partial p}{\partial z}\delta z.\end{aligned}$$

Hence,

$$\hat{\mathbf{k}}\delta F_3 = -\hat{\mathbf{k}}\left(\frac{\partial}{\partial z}p\right)\delta V.$$

Similarly, we have

$$\begin{aligned}\hat{\mathbf{i}}\delta F_1 &= -\hat{\mathbf{i}}\left(\frac{\partial}{\partial x}p\right)\delta V,\\ \hat{\mathbf{j}}\delta F_2 &= -\hat{\mathbf{j}}\left(\frac{\partial}{\partial y}p\right)\delta V.\end{aligned}$$

So,

$$\delta m\left(\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u}\right) = \delta\mathbf{F} = -\delta V \cdot \nabla p.$$

Dividing through by $\delta m$, we find

$$\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p.$$

The is the **momentum equation** for ideal fluid motion.

## 2.3 Thermodynamic Closure

We have

$$\begin{aligned}\partial_t \rho + \nabla.(\rho\mathbf{u}) &= 0,\\ \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} &= -\frac{1}{\rho}\nabla p.\end{aligned}$$

But since there are $d+2$ variables $(\rho, u_1, \ldots, u_d, p)$, we need a relation between $p$ and $\rho$. For example, we could take $p = (k_B T)\rho$ for an isothermal gas. There are closures for polytropic gases: $p = K\rho^{n/(n+1)}$, for different values of $n$. Once we select a thermodynamic closure, we obtain a system of three dynamical equations for fluid motion. These are known as the **Euler equations** for an ideal fluid.

## 2.4 The Incompressible Limit

There are simplifications to be made. One is to consider so-called "incompressible" fluids (e.g., water, or air for low Mach number.) Suppose $\rho = \rho_0 + r(\mathbf{x}, t)$ with $\rho_0$ constant, and further suppose $p = p(\rho)$. Our plan is to linearize the momentum equation for ideal fluids (i.e., take $\|\mathbf{u}\|$ to be "small," so neglect terms which are order $\|\mathbf{u}\|r$ (??), and take $|r|$ to be "small," i.e., $|r|/\rho_0 \ll 1$.) Linearizing, we find

$$\begin{aligned}\partial_t r + \rho_0\nabla.\mathbf{u} &= 0,\\ \partial_t \mathbf{u} + \frac{1}{\rho}\nabla p(\rho) &= 0.\end{aligned}$$

The second equation becomes

$$\begin{aligned}0 &= \partial_t \mathbf{u} + \frac{1}{\rho_0 + r}p'(\rho)\nabla r\\ &= \partial_t \mathbf{u} + \frac{1}{\rho_0}p'(\rho_0)\nabla r,\end{aligned}$$

so we arrive at the system

$$\begin{aligned}
\partial_t r + \rho_0 \nabla.\mathbf{u} &= 0,\\
\partial_t \mathbf{u} + \frac{1}{\rho_0} p'(\rho_0)\nabla r &= 0.
\end{aligned}$$

Taking the time derivative of the first equation and operating with $\rho_0\nabla.(\cdot)$ on the second yields

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} r + \rho_0 \nabla.\partial_t \mathbf{u} &= 0,\\
\rho_0 \nabla.\partial_t \mathbf{u} + p'(\rho_0)\triangle r &= 0.
\end{aligned}$$

Subtracting, we arrive at

$$\partial_t^2 r - p'(\rho_0)\triangle r = 0,$$

which is a wave equation for acoustic waves. If we call $c^2 = p'(\rho_0)$, then the wave speed is

$$c = \sqrt{\frac{\partial p}{\partial \rho}}.$$

For a very compressible fluid, $\partial\rho/\partial p$ is large. For a nearly incompressible fluid, $\partial\rho/\partial p$ is small. In the limit $\partial\rho/\partial p \to 0$, we reach incompressibility. Note that the sign of $\partial\rho/\partial p$ is important (otherwise we'd have complex wave speeds, a rather unfortunate situation).

**Definition 3.** An *incompressible* fluid is one in which $\partial\rho/\partial p = 0$ or equivalently in which $c = \infty$.

For an incompressible, ideal fluid, we have the **incompressible Euler equations**:

$$\begin{aligned}
\rho &= \textbf{constant} > 0,\\
\nabla.\mathbf{u} &= 0,\\
\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla p &= 0.
\end{aligned}$$

Now, there are $d+1$ variables and $d+1$ equations.

# Lecture 3: Boundary Conditions and Helmholtz-Hodge

Last time, we discussed ideal fluids. We applied conservation of mass and momentum, and discussed thermodynamic closure. For low Mach number, we arrived at the **incompressible Euler equations**:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla p &= \frac{1}{\rho}\mathbf{f}(\mathbf{x},t), \\ \rho &= \text{constant}, \\ \nabla.\mathbf{u} &= 0. \end{aligned}$$

Here, $\mathbf{f}$ is an externally applied body force (force/volume). Sometimes, we set $\rho = 1$ for simplicity, which is just a rescaling of mass units. We have $d$ equations and $d$ unknowns. Note that $p$ is determined implicitly; we'll see how this works later.

## 1 Boundary Conditions

### 1.1 Velocity

Suppose the fluid lies in a domain $\Omega$, with rigid boundary $\partial\Omega$ which is at rest.

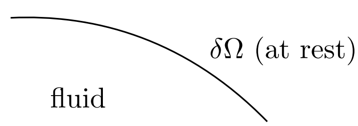



Figure 1.1: Rigid boundary.

For inviscid solutions, no-slip does not apply. However, suppose $\hat{\mathbf{n}}$ is an outward pointing normal on $\partial\Omega$. Then, the boundary condition is simply that no fluid flows across the boundary, i.e.,

$$\hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} = 0.$$

Or suppose the fluid lies in a infinite domain $\Omega$. Then, we often take the far-field to be at rest, i.e.,

$$\|\mathbf{u}\| \to 0 \text{ as } \|\mathbf{x}\| \to \infty.$$

Or we could specify $\mathbf{u}$ at infinity.

A mathematically convenient scenario is to consider periodic boundary conditions.

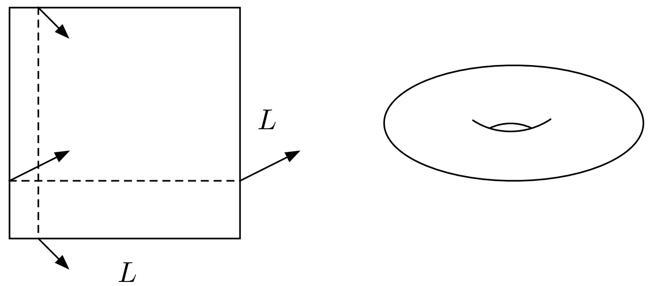



Figure 1.2: Periodic boundary conditions.

This corresponds to considering periodic solutions in an infinite domain,

$$\begin{aligned}\mathbf{u}(\mathbf{x}+\hat{\mathbf{i}}L,t) &= \mathbf{u}(\mathbf{x},t),\\ \mathbf{u}(\mathbf{x}+\hat{\mathbf{j}}L,t) &= \mathbf{u}(\mathbf{x},t).\end{aligned}$$

Topologically, this means we consider flows on a two- or three-torus.

On the other hand, we can consider physically motivated boundary conditions, such as pipes or channels. On the top or bottom of the channel, we'd have rigid boundary conditions; if the channel is long enough, then we can consider solutions which are periodic along the length of the channel.

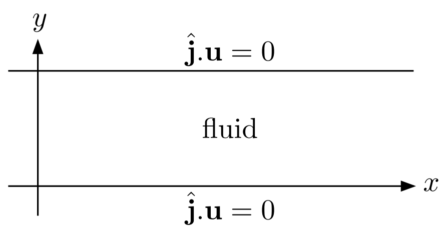



Figure 1.3: Fluid in a periodic channel.

## 1.2 Pressure

How do we evolve the pressure of the fluid? There is no explicit evolution equation in our setup. But take the divergence of the momentum equation to get

$$\partial_t\left(\nabla.\mathbf{u}\right)+\nabla.\left(\mathbf{u}.\nabla\mathbf{u}\right)+\frac{1}{\rho}\nabla.\nabla p=\frac{1}{\rho}\nabla.\mathbf{f}.$$

Enforcing the divergence-free condition, we arrive at

$$-\triangle p=\rho\nabla.\left(\mathbf{u}.\nabla\mathbf{u}\right)-\nabla.\mathbf{f}.$$

This is a Poisson equation for the pressure. Now we need boundary conditions for $p$.

For a periodic domain, we also have

$$\begin{aligned} p(\mathbf{x}+\hat{\mathbf{i}}L,t) &= p(\mathbf{x},t), \\ p(\mathbf{x}+\hat{\mathbf{j}}L,t) &= p(\mathbf{x},t). \end{aligned}$$

This is convenient, as the Poisson equation is solvable by separation of variables when the boundary conditions are periodic. On the other hand, if velocity is specified at $\infty$, then we also must specify $p$ at $\infty$. What about rigid boundaries?

Suppose we have a domain $\Omega$ with rigid boundary $\partial\Omega$ which is at rest. Dot the momentum equation with the outward pointing normal on $\partial\Omega$ to get

$$\partial_t\left(\hat{\mathbf{n}}.\mathbf{u}\right)+\left(\mathbf{u}.\nabla\mathbf{u}\right).\hat{\mathbf{n}}+\frac{1}{\rho}\hat{\mathbf{n}}.\nabla p=\frac{1}{\rho}\hat{\mathbf{n}}.\mathbf{f}.$$

Evaluating at the boundary, we see that

$$\hat{\mathbf{n}}.\left.\nabla p\right|_{\partial\Omega}=-\rho\left(\mathbf{u}.\nabla\mathbf{u}\right).\hat{\mathbf{n}}+\hat{\mathbf{n}}.\mathbf{f}.$$

This is an inhomogeneous Neumann boundary condition. This is also good for solving the elliptic problem posed above for the pressure.

There are simplifications. Suppose $\mathbf{f}=0$ and suppose $\partial\Omega$ is flat, e.g., $\hat{\mathbf{n}}=-\hat{\mathbf{k}}$.

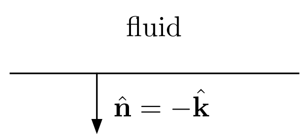



Figure 1.4: Flat boundary.

Then,

$$\left(\mathbf{u}.\nabla\mathbf{u}\right).\hat{\mathbf{n}}=-\mathbf{u}.\nabla u_3=0.$$

So for no body forces and flat boundary,

$$\hat{\mathbf{n}}.\left.\nabla p\right|_{\partial\Omega}=0,$$

a homogeneous Neumann boundary condition.

If the boundary is curved, this is no longer true. Consider, though, that at the boundary, $\mathbf{u}.\hat{\mathbf{n}}=0$. Hence,

$$\begin{aligned} 0 &= \mathbf{u}.\nabla\left(\mathbf{u}.\hat{\mathbf{n}}\right) \\ &= \left(\mathbf{u}.\nabla\mathbf{u}\right).\hat{\mathbf{n}}+\mathbf{u}.\left(\nabla\hat{\mathbf{n}}\right).\mathbf{u}. \end{aligned}$$

So even if the boundary is curved, so long as $\mathbf{f}$ is zero, we arrive at

$$\hat{\mathbf{n}}.\left.\nabla p\right|_{\partial\Omega}=\rho\left(\mathbf{u}.\nabla\hat{\mathbf{n}}\right).\mathbf{u}.$$

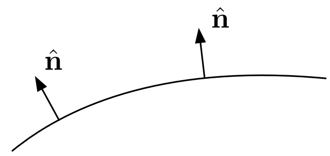



Figure 1.5: Boundary with curvature.

This last condition encodes the curvature of the boundary.

### 1.3 Initial Value Problem

In summary, we have

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla p &= \frac{1}{\rho}\mathbf{f}(\mathbf{x}, t),\\ \rho &= \text{constant},\\ \nabla.\mathbf{u} &= 0,\end{aligned}$$

along with the implicit equation for the pressure,

$$-\triangle p = \rho\nabla.\left(\mathbf{u}.\nabla\mathbf{u}\right) - \nabla.\mathbf{f}.$$

As the boundary conditions we have

$$\begin{aligned}\mathbf{u}.\hat{\mathbf{n}} &= 0 \quad \text{(or periodic b.c.)},\\ \hat{\mathbf{n}}.\left.\nabla p\right|_{\partial\Omega} &= -\rho\left(\mathbf{u}.\nabla\mathbf{u}\right).\hat{\mathbf{n}} + \hat{\mathbf{n}}.\mathbf{f}.\end{aligned}$$

So the initial value problem is the following:

Given

- domain, b.c.'s, $\mathbf{f}$
- initial conditions, $\mathbf{u}(\mathbf{x}, 0)$

Ask: is there a solution to the evolution equations for $t > 0$? If so, does the smoothness of the solution depend in a regular way on the smoothness of the initial conditions?

## 2 Another View of the Euler Equations

If there was an applied $\delta\mathbf{F} = \mathbf{f}\delta V$, we would like to say that $\delta m \cdot \mathbf{a} = \delta\mathbf{F}$, i.e.,

$$\frac{d}{dt}\mathbf{u} - \frac{1}{\rho}\mathbf{f} = 0.$$

Taking the divergence, we get

$$\frac{\partial}{\partial t}\left(\nabla.\mathbf{u}\right) = -\nabla.\left(\mathbf{u}.\nabla\mathbf{u}\right) + \frac{1}{\rho}\nabla.\mathbf{f}.$$

This is a problem! Consider that the right hand side does not have to be zero, so the divergence-free condition on $\mathbf{u}$ can fail in our naive approach. To fix this, we want to take only the component of the acceleration that preserves $\nabla.\mathbf{u} = 0$.

**Theorem.** *(Helmholtz-Hodge Decomposition) Suppose $\Omega$ is a simply-connected domain. A vector field $\mathbf{v}$ on $\Omega$ can be decomposed uniquely into the sum of a divergence-free part that is parallel to $\partial\Omega$, and of a remainder that is a gradient.*

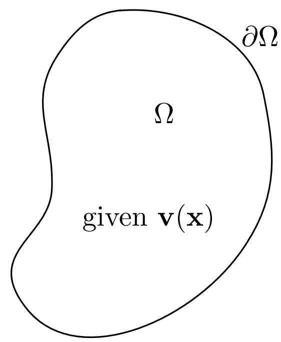



Figure 2.1: Problem domain.

Consider a simply-connected domain $\Omega$ with boundary $\partial\Omega$. If we're given $\mathbf{v}(\mathbf{x})$, the theorem says that

$$\mathbf{v} = \mathbf{u} + \nabla q$$

for $\mathbf{u}(\mathbf{x}), q(\mathbf{x})$ and for $\nabla.\mathbf{u} = 0$ and $\hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} = 0$.

*Proof.* Define $q(\mathbf{x})$ as the unique (up to constant) solution of the Poisson equation

$$\begin{aligned} \triangle q &= \nabla.\mathbf{v}, \\ \hat{\mathbf{n}}.\ \nabla q|_{\partial\Omega} &= \hat{\mathbf{n}}.\ \mathbf{v}|_{\partial\Omega}\,. \end{aligned}$$

Then, define $\mathbf{u} = \mathbf{v} - \nabla q$. Check that

$$\nabla.\mathbf{u} = \nabla.\mathbf{v} - \triangle q = 0$$

and that

$$\hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} = \hat{\mathbf{n}}.\ \mathbf{v}|_{\partial\Omega} - \hat{\mathbf{n}}.\ \nabla q|_{\partial\Omega} = 0.$$

□

*Remark.* This operation defines a linear map $\mathbf{v} \mapsto \mathbf{u}$. Call this map $\mathbb{P}$, so that $\mathbb{P}\{\mathbf{v}\} = \mathbf{u}$. Then $\mathbb{P} = \mathbb{P}^2$, i.e. $\mathbb{P}$ is a projection operator on the space of vector fields. (Proven by application of the theorem twice.) In this grain, we may write

$$\mathbf{v} = \mathbb{P}\{\mathbf{v}\} + (\mathbf{v} - \mathbb{P}\{\mathbf{v}\})\,.$$

Note that the first and second terms here are orthogonal, given the right inner product.

Using this projection operator, we may write the incompressible Euler equations as

$$\mathbb{P}\left\{\frac{d}{dt}\mathbf{u}\right\} = \frac{1}{\rho}\mathbb{P}\left\{\mathbf{f}\right\},$$

or just

$$\partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} - \nabla q = \frac{1}{\rho}\mathbb{P}\{\mathbf{f}\}.$$

Taking the divergence, we get

$$\nabla.\left(\mathbf{u}.\nabla \mathbf{u}\right) - \triangle q = 0.$$

This is the view taken in many numerical algorithms. Since the projection operator is linear, we often find

$$\partial_t u + \mathbb{P}\left\{\mathbf{u}.\nabla \mathbf{u}\right\} = \frac{1}{\rho}\mathbb{P}\{\mathbf{f}\}$$

in the literature.

## Lecture 4: Projection Formulation, Energy, and Vorticity

We're discussing the incompressible Euler equations:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \frac{1}{\rho}\nabla p &= \frac{1}{\rho}\mathbf{f}^{\text{ext}}, \\ \nabla.\mathbf{u} &= 0, \end{aligned}$$

given $\mathbf{u}_0(\mathbf{x})$ and b.c.'s. This is a version of Newton's 2nd Law, in the limit where neighboring fluid elements don't press each other hard enough to compress the fluid.

## 1 Projection Formulation of the Euler Equations

Last time, we saw that the Euler equations are a projection of $\frac{d}{dt}\mathbf{u} = \frac{1}{\rho}\mathbf{f}$ onto divergence-free vector fields. Explicitly, this is

$$\mathbb{P}\left(\frac{d}{dt}\mathbf{u} = \frac{1}{\rho}\mathbf{f}\right)$$

where $\mathbb{P}$ is the projection operator provided by Helmholtz-Hodge. This theorem said that if $\Omega$ is a simply-connected domain with boundary $\partial\Omega$ and if $\mathbf{v}(\mathbf{x})$ is a vector field on $\Omega$, then

$$\mathbf{v}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \nabla q$$

uniquely, where $\nabla.\mathbf{u} = 0$ and $\hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} = 0$. In the proof, we found $q$ by solving the Poisson problem

$$\begin{aligned} \triangle q &= \nabla.\mathbf{v}, \\ \hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} &= \hat{\mathbf{n}}.\ \mathbf{v}|_{\partial\Omega}. \end{aligned}$$

**Example.** Let $\Omega = [0,1]^2$ and $\mathbf{v}(\mathbf{x}) = \hat{\mathbf{j}}\cos(2\pi x)\sin(2\pi y)$.

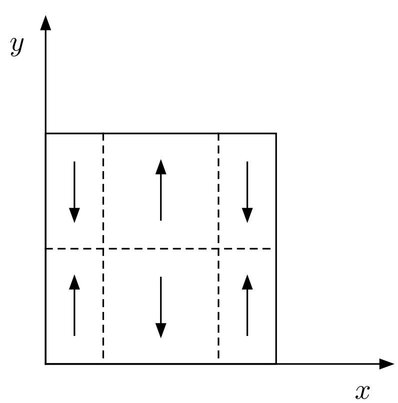



Figure 1.1: $\mathbf{v}(\mathbf{x})$

Then

$$\nabla.\mathbf{v} = 2\pi\cos(2\pi x)\cos(2\pi y),$$

but note

$$\nabla\times\mathbf{v} = \hat{\mathbf{k}}(-2\pi\sin(2\pi x)\sin(2\pi y)$$

which is not identically zero. What is the divergence-free part? We need

$$\begin{aligned}\triangle q &= \nabla.\mathbf{v} = 2\pi\cos(2\pi x)\cos(2\pi y),\\ \hat{\mathbf{n}}.\,\nabla q|_{\partial\Omega} &= \hat{\mathbf{n}}.\,\mathbf{v}|_{\partial\Omega} = 0.\end{aligned}$$

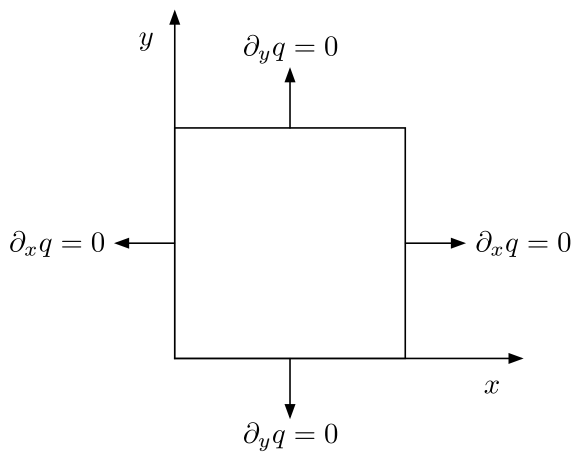



Figure 1.2: Boundary conditions.

The solution of this problem is

$$q(\mathbf{x}) = -\frac{1}{4\pi}\cos(2\pi x)\cos(2\pi y),$$

so

$$\nabla q = \frac{1}{2}\hat{\mathbf{i}}\sin(2\pi x)\cos(2\pi y) + \frac{1}{2}\hat{\mathbf{j}}\cos(2\pi x)\sin(2\pi y).$$

Hence, the divergence-free part is exactly

$$\begin{aligned}\mathbf{u} &= \mathbf{v} - \nabla q\\ &= -\frac{1}{2}\hat{\mathbf{i}}\sin(2\pi x)\cos(2\pi x) + \frac{1}{2}\hat{\mathbf{j}}\cos(2\pi x)\sin(2\pi y).\end{aligned}$$

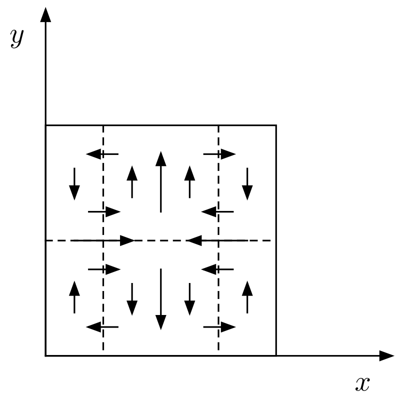



Figure 1.3: $\mathbf{u}$, the divergence-free part of $\mathbf{v}$.

This vector field looks quite different from $\mathbf{v}$.

Back to the equations:

$$\begin{aligned} \mathbb{P}\left(\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u}\right) &= \frac{1}{\rho}\mathbb{P}(\mathbf{f}) \\ \Longrightarrow\ \partial_t \mathbf{u} + \mathbb{P}\left(\mathbf{u}.\nabla\mathbf{u}\right) &= \frac{1}{\rho}\mathbb{P}(\mathbf{f}). \end{aligned}$$

Note that $\mathbb{P}\left(\mathbf{u}.\nabla\mathbf{u}\right) = \mathbf{u}.\nabla\mathbf{u} - \nabla(-p/\rho)$.

*Note* 1. Since $\partial_t \mathbf{u}$ is equal to a divergence-free vector field, if $\mathbf{u}$ starts off divergence-free, it will stay divergence-free for all time.

Consider the operator $\mathbb{P}$. This is a linear projection operator, since $\mathbb{P}\{\mathbf{v_1} + \mathbf{v_2}\} = \mathbb{P}\{\mathbf{v_1}\} + \mathbb{P}\{\mathbf{v_2}\}$ and $\mathbb{P}\{\mathbb{P}\{\mathbf{v}\}\} = \mathbb{P}\{\mathbf{v}\}$. So we may write

$$\mathbf{v} = \mathbb{P}\{\mathbf{v}\} + (\mathbf{v} - \mathbb{P}\{\mathbf{v}\})$$

where, in our previous notation,

$$\begin{aligned} \mathbb{P}\{\mathbf{v}\} &= \mathbf{u}, \\ \mathbf{v} - \mathbb{P}\{\mathbf{v}\} &= \nabla q. \end{aligned}$$

We claim this is an orthogonal projection, with the inner product of two vector fields $\mathbf{v}_1, \mathbf{v}_2$ being

$$\int_\Omega \mathbf{v}_1(\mathbf{x}).\mathbf{v}_2(\mathbf{x})\, d\mathbf{x}.$$

Indeed,

$$\begin{aligned} \int_\Omega \mathbf{u}.\nabla q\, d\mathbf{x} &= \int_\Omega \nabla.(q\mathbf{u})\, d\mathbf{x} \\ &= \int_{\partial\Omega} q\mathbf{u}.\hat{\mathbf{n}} da \\ &= 0. \end{aligned}$$

This will prove useful.

## 2 Energy Evolution

Consider fluid in 3-space.

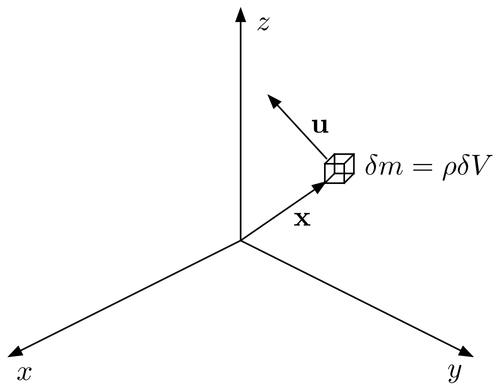



Figure 2.1: Fluid element.

The kinetic energy of a fluid element is $\frac{1}{2}\delta m\|u\|^2$, so the total kinetic energy in some region $\Omega$ is

$$\text{Kinetic Energy} = \frac{1}{2}\rho\int_\Omega \|\mathbf{u}(\mathbf{x},t)\|^2\,d\mathbf{x} = \frac{1}{2}\rho\|\mathbf{u}(\cdot,t)\|_2^2,$$

where the $L^2$-norm is, as usual,

$$\|\mathbf{v}\|_2 = \sqrt{\int_\Omega \mathbf{v}(\mathbf{x})^2\,d\mathbf{x}}.$$

How does the energy evolve in time? Suppose $\mathbf{u}$ satisfies the incompressible Euler equations with body force:

$$\begin{aligned} \partial_t\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla p &= \frac{1}{\rho}\mathbf{f}, \\ \nabla.\mathbf{u} &= 0, \\ \hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned}$$

W.l.o.g., we may take $\mathbf{f}$ to be divergence-free.[1] The kinetic energy evolves in time by

$$\begin{aligned} \frac{d}{dt}\text{KE} &= \frac{d}{dt}\int_\Omega \frac{1}{2}\rho\|\mathbf{u}(\mathbf{x},t)\|^2\,d\mathbf{x} \\ &= \frac{1}{2}\rho\int_\Omega \frac{\partial}{\partial t}\|\mathbf{u}(\mathbf{x},t)\|^2\,d\mathbf{x} \\ &= \rho\int_\Omega \mathbf{u}.\partial_t\mathbf{u}\,d\mathbf{x} \\ &= \rho\int_\Omega \mathbf{u}.\left(-\mathbf{u}.\nabla\mathbf{u} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{f}\right)\,d\mathbf{x} \\ &= -\rho\int_\Omega \mathbf{u}.\left(\mathbf{u}.\nabla\mathbf{u}\right)\,d\mathbf{x} - \int_\Omega \mathbf{u}.\nabla p\,d\mathbf{x} + \int_\Omega \mathbf{u}.\mathbf{f}\,d\mathbf{x}. \end{aligned}$$

[1] Write $\mathbf{f} = \mathbb{P}\{\mathbf{f}\} - \nabla\phi$, and lump $\nabla\phi$ into the pressure term.

Immediately, we see that the second integral is zero, for $\mathbf{u}.\nabla p = \nabla.(\mathbf{u}p)$ and since $\hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} = 0$. Physically, this says that the internal pressures do no work. Now consider the first term. The integrand is

$$\begin{aligned} \mathbf{u}.\nabla p &= u_i(u_j\partial_{x_j}u_i) \\ &= u_ju_i\partial_ju_i \\ &= u_j\partial_j\left(\frac{1}{2}u_iu_i\right) \\ &= \mathbf{u}.\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2\right) \\ &= \nabla.\left(\frac{1}{2}\mathbf{u}\|\mathbf{u}\|^2\right), \end{aligned}$$

so

$$\rho\int_\Omega \mathbf{u}.\left(\mathbf{u}.\nabla\mathbf{u}\right)\,d\mathbf{x} = \rho\int_\Omega \nabla.\left(\frac{1}{2}\mathbf{u}\|\mathbf{u}\|^2\right)\,d\mathbf{x} = 0$$

by the divergence-theorem. Hence,

$$\frac{d}{dt}\mathrm{KE} = \int_\Omega \mathbf{u}.\mathbf{f}\,d\mathbf{x}.$$

*Note* 2. The work done on $\Omega$ is due only to the divergence-free part of $\mathbf{f}$. Consider $\mathbf{f} = -\mathbf{k}\rho g$. We can absorb this into the pressure term by defining the "hydrostatic pressure," $p = -\rho g z + \text{constant}$.

## 3 Vorticity

**Definition.** The *vorticity* of a vector field $\mathbf{u}(\mathbf{x},t)$ is the vector field defined by $\omega(\mathbf{x},t) = \nabla\times\mathbf{u}(\mathbf{x},t)$.

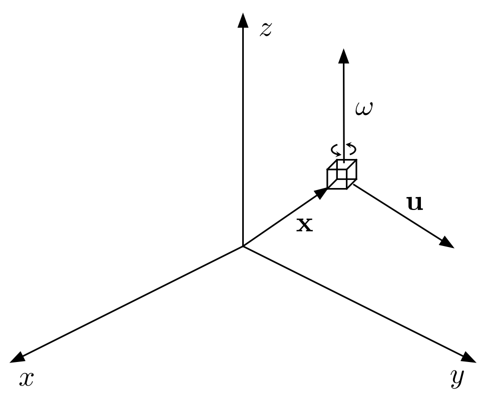



Figure 3.1: Vorticity of a fluid element.

**Proposition.** *Vorticity is twice the angular velocity of a fluid element.*

*Proof.* This is a heuristic derivation. Consider a fluid element with velocity $(u(x,y,z), v(x,y,z), w(x,y,z))$.

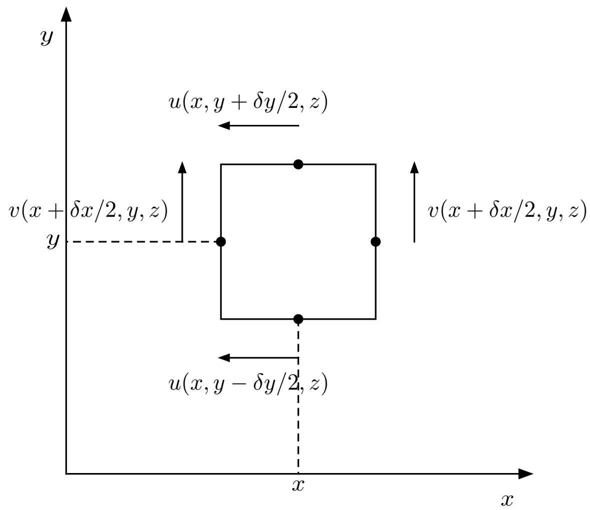



Figure 3.2: Computing angular velocity.

The $z$-component of the average angular velocity (of the boundaries) is

$$\begin{aligned}
\Omega_z &= \frac{1}{4}\left(\frac{v(x+\delta x/2,y,z)}{\delta x/2} - \frac{u(x,y+\delta y/2,z)}{\delta y/2} - \frac{v(x-\delta x/2,y,z)}{\delta x/2} + \frac{u(x,y-\delta y/2,z)}{\delta y/2}\right)\\
&= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}$$

to leading order. Hence the claim. □

Recall that

$$\mathbf{u}.\nabla\mathbf{u} = \omega \times \mathbf{u} + \nabla\left(\frac{1}{2}\|u\|^2\right),$$

for

$$\begin{aligned}
(\mathbf{u}.\nabla\mathbf{u})_i &= u_{i,j}u_j\\
&= \delta_{ik}\delta_{jl}u_{k,l}u_j\\
&= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{il}\delta_{jk})u_{k,l}u_j\\
&= (\epsilon_{ijm}\epsilon_{mkl} + \delta_{il}\delta_{jk})u_{k,l}u_j\\
&= \epsilon_{ijm}(\epsilon_{mkl}u_{k,l})u_j + u_{j,i}u_j\\
&= \epsilon_{imj}\omega_m u_j + \frac{1}{2}\partial_i\left(u_ju_j\right)\\
&= (\omega \times \mathbf{u})_i + \left(\frac{1}{2}\nabla\left(\|\mathbf{u}\|^2\right)\right)_i.
\end{aligned}$$

So we have another representation of the Euler equations, as

$$\begin{aligned}
\partial_t \mathbf{u} + \omega \times \mathbf{u} + \nabla\left(\frac{1}{\rho}p + \frac{1}{2}\|\mathbf{u}\|^2\right) &= \frac{1}{\rho}\mathbf{f}, \\
\nabla.\mathbf{u} &= 0.
\end{aligned}$$

This decomposes the evolution of the natural kinematics into a spin part and a no-spin part. This has immediate implications for the solutions, which we'll see next time.

## Lecture 5: Two More Views of the Euler Equations

Recall our setup, fluid in a domain $\Omega$ with boundary $\partial\Omega$ and outward pointing normal $\hat{\mathbf{n}}$. We saw two views of the incompressible Euler equations:

1. View 1:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla\rho &= \frac{1}{\rho}\mathbf{f}, \\ \nabla.\mathbf{u} &= 0, \\ \hat{\mathbf{n}}.\left.\mathbf{u}\right|_{\partial\Omega} &= 0. \end{aligned}$$

2. View 2:

$$\mathbb{P}\left\{\frac{d}{dt}\mathbf{u}\right\} = \frac{1}{\rho}\mathbb{P}\left\{\mathbf{f}\right\}.$$

Now we'll see a third view.

## 1 A Third View of the Euler Equations

Recall that

$$\mathbf{u}.\nabla\mathbf{u} = \omega \times \mathbf{u} + \nabla\left(\frac{1}{2}\|\mathbf{u}\|^2\right)$$

where $\omega = \nabla \times \mathbf{u}$. This gives a third view of the dynamics:

$$\begin{aligned} \partial_t \mathbf{u} + \omega \times \mathbf{u} + \nabla\left(\frac{1}{\rho}p + \frac{1}{2}\|u\|^2\right) &= \frac{1}{\rho}\mathbf{f}, \\ \nabla.\mathbf{u} &= 0, \\ \omega &= \nabla \times \mathbf{u}. \end{aligned} \tag{1.1}$$

The second implicitly determines pressure, and the third closes the system.

*Note.* In principle, given the vorticity, we could recover the velocity field, for

$$\nabla \times \omega = \nabla \times \nabla \times \mathbf{u} = \nabla\left(\nabla\mathbf{u}\right) - \triangle\mathbf{u} = -\triangle\mathbf{u}.$$

So we would solve the Poisson equation with boundary conditions:

$$\begin{aligned} -\triangle\mathbf{u} &= \nabla \times \omega, \\ \hat{\mathbf{n}}.\left.\mathbf{u}\right|_{\partial\Omega} &= 0. \end{aligned}$$

This has solution by the Biot-Savaart Law, which uses a Green's function.

### 1.1 Local Evolution of Kinetic Energy

Recall that the kinetic energy is $\int_\Omega \frac{1}{2}\rho\|\mathbf{u}\|^2\,d\mathbf{x}$. Dotting $\mathbf{u}$ into (1.1), we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\|\mathbf{u}\|^2\right)+\frac{1}{\rho}\mathbf{u}.\nabla\left(p+\frac{1}{2}\rho\|\mathbf{u}\|^2\right)=\frac{1}{\rho}\mathbf{u}.\mathbf{f},$$

or equivalently,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\|\mathbf{u}\|^2\right)+\nabla.\left[\left(p+\frac{1}{2}\rho\|\mathbf{u}\|^2\right)\mathbf{u}\right]=\frac{1}{\rho}\mathbf{u}.\mathbf{f}.$$

This says that the local evolution of kinetic energy is determined by fluxes and work done by an external force. This has immediate consequences.

### 1.2 Bernoulli's Theorem

**Definition 1.** A flow is said to be *stationary* if the time derivative of any flow property is zero.

For stationary flow with no external force,

$$\mathbf{u}.\nabla\left(p+\frac{1}{2}\rho\|\mathbf{u}\|^2\right)=0,$$

and also $\mathbf{u}.\nabla(\cdot)\equiv\frac{d}{dt}(\cdot)$, hence

$$\frac{d}{dt}\left(p+\frac{1}{2}\rho\|\mathbf{u}\|^2\right)=0.$$

So we've proved a theorem.

**Theorem.** *(Bernoulli) For stationary ideal flows, the quantity $p+\frac{1}{2}\rho\|\mathbf{u}\|^2$ for any fliud element is constant.*

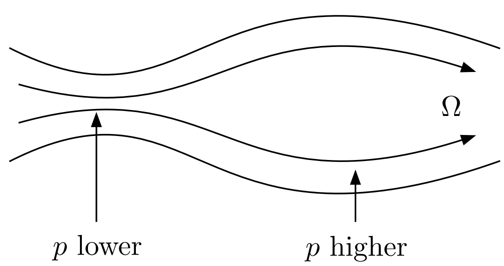



Figure 1.1: Bernoulli's principle.

## 2 A Fourth View of the Euler Equations

Take the curl of (1.1) to get

$$\partial_t\omega+\nabla\times(\omega\times\mathbf{u})=\frac{1}{\rho}\nabla\times\mathbf{f}.$$

A quick calculation shows

$$\begin{aligned}(\nabla \times (\omega \times \mathbf{u}))_i &= \epsilon_{ijk}\partial_j \epsilon_{klm}\omega_l u_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(\omega_{l,j}u_m + \omega_l u_{m,j}) \\ &= \omega_{i,j}u_j + \omega_i u_{j,j} - u_i\omega_{j,j} - u_{i,j}\omega_j \\ &= \omega_{i,j}u_j - u_{i,j}\omega_j .\end{aligned}$$

Putting it together, we arrive at the vorticity evolution equation:

$$\begin{aligned}\partial_t \omega + \mathbf{u}.\nabla\omega - \omega.\nabla\mathbf{u} &= \frac{1}{\rho}\nabla \times \mathbf{f}, \\ \omega &= \nabla \times \mathbf{u}.\end{aligned}$$

There are some interesting consequences.

For example, suppose there is no external force. Then

$$\frac{d}{dt}\omega = \omega.\nabla\mathbf{u}.$$

But note that not every component of the velocity gradient tensor contributes. We can write

$$(\nabla\mathbf{u})_{ij} = u_{j,i} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{j,i} - u_{i,j}) = S_{ij} + A_{ij},$$

and the second term is the anti-symmetric tensor

$$A = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

We claim that

$$\omega.(\nabla\mathbf{u}) = \omega.(S + A) = \omega.S,$$

i.e.,

$$\omega.A \equiv 0.$$

This is a straightforward matrix calculation. So in fact, when there is no external force,

$$\frac{d}{dt}\omega = \omega.\nabla\mathbf{u} = \omega.S,$$

where $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, the "rate-of-strain tensor."

As a consequence, we have the following

**Proposition.** *Suppose we have a smooth solution to the Euler equations, and $\omega$ of a particular fluid element is zero at $t = 0$. Then, it will remain zero for all time. In particular, if $\omega(\mathbf{x}, 0) \equiv 0$, then $\omega \equiv 0$ forever.*

**Definition 2.** Flows with no vorticity are said to be *irrotational*.

What happen if the vorticity is not zero? In $\mathbb{R}^3$, $\omega.\nabla\mathbf{u}$ is called the "vortex-stretching term," for the following reason. Suppose $\omega$ of some fluid element is non-zero, and suppose $\mathbf{u}$ changes across the fluid element in such a way that $\omega.\nabla\mathbf{u} \neq 0$. Then, the ensuing changes in the fluid element's moment of inertia lead to changes in $\omega$, via conservation of angular momentum. Thus, vortex-stretching – see the figure below.

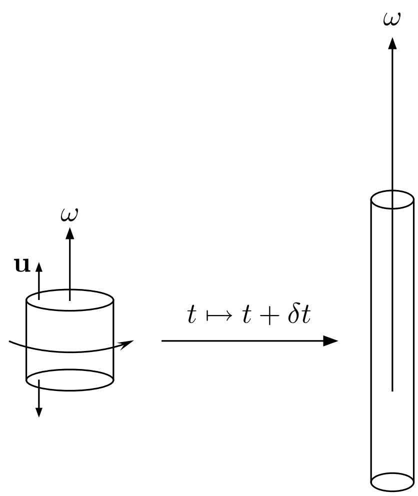



Figure 2.1: Vortex-stretching in a fluid element.

Nobody really knows whether vortex-stretching can lead to finite-time blowup, even given regular initial conditions. Later on, we'll see why vorticity plays a big role in regularity of solutions.

But in $\mathbb{R}^2$, things are more nicely behaved. Indeed, we have $\partial_z \equiv 0$ and $\hat{\mathbf{k}}.\mathbf{u} \equiv 0$. So the dynamics are now

$$\partial_t\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \frac{1}{\rho}\mathbf{f}(x, y, t),$$

with $\hat{\mathbf{k}}.\mathbf{f} = 0$. In components,[1]

$$\begin{aligned} u_t + uu_x + vu_y + \frac{1}{\rho}p_x &= \frac{1}{\rho}f_1(x, y, t), \\ v_t + uv_x + vv_y + \frac{1}{\rho}p_y &= \frac{1}{\rho}f_2(x, y, t). \end{aligned}$$

Vorticity satisfies

$$\begin{aligned} \omega &= \hat{\mathbf{i}}(w_y - v_z) + \hat{\mathbf{j}}(u_z - w_z) + \hat{\mathbf{k}}(v_x - u_y) \\ &= \hat{\mathbf{k}}(v_x - u_y). \end{aligned}$$

[1]And note that

$$\partial_t w + uw_x + vw_y + \frac{1}{\rho}p_z = \frac{1}{\rho}f_3,$$

so $\partial_t w = 0$.

Hence,

$$\omega.\nabla\mathbf{u} = \left(\hat{\mathbf{k}}\omega_3\right).\nabla\mathbf{u} = \omega_3\partial_z\mathbf{u} \equiv 0.$$

So in $\mathbb{R}^2$, there is no vortex-stretching. In this case, we call $\omega_3 = \hat{\mathbf{k}}.\left(\nabla\times\mathbf{u}\right) \equiv \omega$ the "scalar vorticity," which satisfies

$$\frac{d}{dt}\omega = 0.$$

Basically, this says that the moment of inertia of fluid elements does not change for flows in $\mathbb{R}^2$. Note that if there was an external force,

$$\frac{d}{dt}\omega = \frac{1}{\rho}\hat{\mathbf{k}}.\left(\nabla\times\mathbf{f}\right) = \frac{1}{\rho}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

## 3 Enstrophy

Recall that the kinetic energy is $\int_\Omega \frac{1}{2}\rho\|\mathbf{u}\|^2\,d\mathbf{x}$. Similarly, we define another quantity.

**Definition 3.** The *enstrophy* is given by

$$\int_\Omega \|\omega\|^2\,d\mathbf{x} = \|\omega\|_2^2.$$

We saw that in the absence of external forces, energy is conserved. In a way similar to that calculation, one can show that

$$\frac{d}{dt}\left(\text{enstrophy}\right) = \int_\Omega \omega.S.\omega\,d\mathbf{x}.$$

In $\mathbb{R}^2$, $\frac{d}{dt}\left(\text{enstrophy}\right) = 0$.

**Exercise.** In $\mathbb{R}^2$,

$$\begin{aligned} \frac{d}{dt}\omega &= 0, \\ \partial_t\omega + uw_x + vw_y &= 0. \end{aligned}$$

Show that

$$\frac{d}{dt}\int_\Omega \left(\omega(x,y,t)\right)^n\,dxdy = 0.$$

## Lecture 6: Introduction to Potential Flow Theory

We have developed various ways of looking at the Euler equations:

$$\begin{aligned} \partial_t u + u.\nabla u + \frac{1}{\rho}\nabla p &= \frac{1}{\rho} f \\ \nabla . u &= 0, \end{aligned}$$

or

$$\begin{aligned} \partial_t + \omega \times u + \nabla \left( \frac{1}{\rho} p + \frac{1}{2} \|u\|^2 \right) &= \frac{1}{\rho} f \\ \omega &= \nabla \times u, \end{aligned}$$

or

$$\partial_t \omega + u.\nabla \omega = \omega.\nabla u + \frac{1}{\rho} \nabla \times f.$$

Note that these are all nonlocal equations for the dynamics. This is a consequence of the incompressibility condition.

## 1 Irrotational (Potential) Flows

Consider a periodic channel, i.e., consider solutions which are periodic in the $y$-direction with period $L_y$.

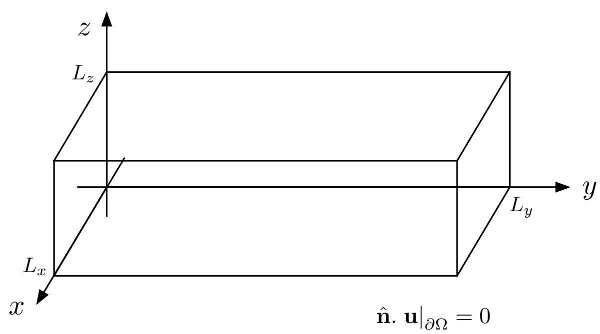



Figure 1.1: Flow in a periodic channel.

One steady solution is

$$\begin{aligned} \mathbf{u} &= \hat{\mathbf{j}} v(x, z), \\ p &= \text{constant}. \end{aligned}$$

This is a solution, for

$$\begin{aligned} \nabla . \mathbf{u} &= \frac{\partial v}{\partial y} \equiv 0, \\ \nabla p &= 0, \text{ and} \\ \mathbf{u}.\nabla &= v(x, y) \frac{\partial}{\partial y}, \text{ so} \\ \mathbf{u}.\nabla \mathbf{u} &= 0. \end{aligned}$$

There is a horrible non-uniqueness here – we have produced an entire class of solutions!

Are there any special solutions? Consider the vorticity,

$$\begin{aligned} \omega &= \nabla \times \mathbf{u} \\ &= -\hat{\mathbf{i}}\frac{\partial v}{\partial z} + \hat{\mathbf{k}}\frac{\partial v}{\partial x}. \end{aligned}$$

A distinguished solution is one with no vorticity. Then, $\partial_z v = 0 = \partial_x v$ we obtain "solid body" flow. Flows of no vorticity are generally distinguished.

**Definition 1.** Flows with no vorticity are said to be *irrotational.*

For the rest of this section, we'll assume that $\omega \equiv 0$, i.e., $\nabla \times \mathbf{u} \equiv 0$. Note that in simply connected domains, $\mathbf{u} = \nabla\phi$ for a unique (up to constant) function $\phi$. So we make a definition.

**Definition 2.** If $\mathbf{u} = \nabla\phi$, we call $\phi$ the *velocity potential* for $\mathbf{u}$.

Why is $\phi$ unique? Consider a simple closed curve in an irrotational flow.

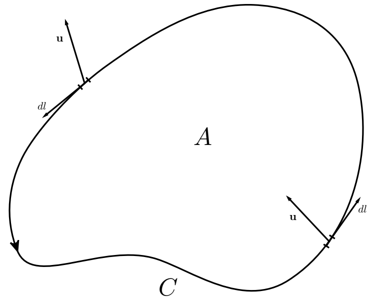



Figure 1.2: A simple closed curve $C$ in flow $\mathbf{u}$.

By Stoke's theorem,

$$\int_C \mathbf{u}.d\mathbf{l} = \int_A (\nabla \times \mathbf{u}) .d\mathbf{a} = 0$$

as the flow has no vorticity. Thus, we may define

$$\phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{u}.d\mathbf{l}$$

along any curve connecting $\mathbf{x}_0$ and $\mathbf{x}$. This is well-defined, by the line above.

**Definition 3.** The *circulation* of a flow about a simple closed curve $C$ is

$$\Gamma_C = \int_C \mathbf{u}.d\mathbf{l}.$$

So irrotational flows have no circulation.

### 1.1 Pressure in a Potential Flow

What do the Euler equations look like for potential flows? Barring the presence of external forces,

$$\begin{aligned} \partial_t \mathbf{u} + \nabla\left(\frac{1}{\rho}p + \frac{1}{2}\|\mathbf{u}\|^2\right) &= 0, \\ \nabla . \mathbf{u} &= 0. \end{aligned}$$

For stationary ($\partial_t \equiv 0$) flows,

$$\nabla\left(p + \frac{1}{2}\rho\|\mathbf{u}\|^2\right) = 0,$$

and thus

$$p(\mathbf{x}) + \frac{1}{2}\rho\|\mathbf{u}(\mathbf{x})\|^2 = \text{constant}.$$

This is much stronger than Bernoulli's theorem.

For time-dependent irrotational flows,

$$\begin{aligned} \mathbf{u} &= \nabla\phi \\ \Longrightarrow \partial_t \mathbf{u} &= \nabla\left(\partial_t \phi\right). \end{aligned}$$

Hence,

$$\begin{aligned} \nabla\left(\partial_t\phi\right) + \nabla\left(\frac{1}{\rho}p + \frac{1}{2}\|\nabla\phi\|^2\right) &= 0 \\ \Longrightarrow \nabla\left(\partial_t\phi + \frac{1}{\rho}p + \frac{1}{2}\|\nabla\phi\|^2\right) &= 0, \end{aligned}$$

and hence

$$\rho\partial_t\phi + p + \frac{1}{2}\rho\|\nabla\phi\|^2 = C(t),$$

a time-dependent, spatially-invariant constant. This is a diagnostic equation that gives the pressure field:

$$p(\mathbf{x}, t) = C(t) - \rho\partial_t\phi - \frac{1}{2}\rho\|\nabla\phi\|^2.$$

We have thus eliminated the need to solve a pde for the pressure. Instead, given a divergence-free velocity field, we can integrate on curves to get the velocity potential and plug it into the equation on the line above. Now, the pressure is not a thermodynamic pressure; it is determined by the need to keep $C(t)$ constant in space.

### 1.2 The Velocity Potential

We now derive an important relation for $\phi$. Since $\mathbf{u} = \nabla\phi$ and $\nabla.\mathbf{u} = 0$, we have

$$\triangle\phi = 0.$$

So the velocity potential satisfies Laplace's equation! We'd like to think of this as a bvp for $\phi$. Recall the boundary conditions in our setup are $\hat{\mathbf{n}}.\, \mathbf{u}|_{\partial\Omega} = 0$, so

$$\hat{\mathbf{n}}.\, \nabla\phi|_{\partial\Omega} = 0.$$

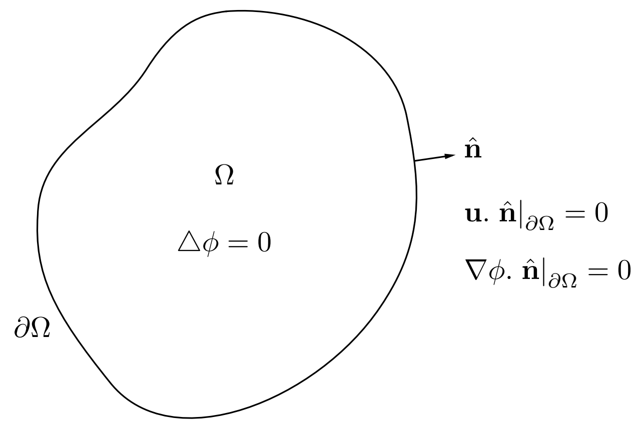



Figure 1.3: Laplace's equation with Neumann boundary condition.

Our problem is now linear – so long as we have $\phi$ which solves Laplace's equation with Neumann boundary conditions, we have generated a velocity field which satisfies the Euler equations (along with the pressure field determined above).

Laplace's equation with homogenous Neumann boundary conditions have a unique solution up to a constant. (See the homework.) But the additive constant does not change the velocity field. In fact,

$$0 = \int_\Omega \phi \triangle \phi \, d\mathbf{x} = -\int_\Omega \|\nabla \phi\|^2 \, d\mathbf{x},$$

after integrating by parts and using the Neumann boundary condition. So the only incompressible, irrotational flows existing on simply-connected domains are the flows with no flow at all (c.f., channel flow, topologically a torus).

A more general bvp comes by using the inhomogenous boundary condition

$$\hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} = u_n(\mathbf{x}, t).$$

Note that

$$\int_{\partial\Omega} u_n(\mathbf{x}, t) \, da = 0$$

must hold for compatibility. So then we'd look for solutions of Laplace's equation

$$\triangle \phi = 0$$

subject to

$$\hat{\mathbf{n}}.\ \nabla\phi|_{\partial\Omega} = u_n.$$

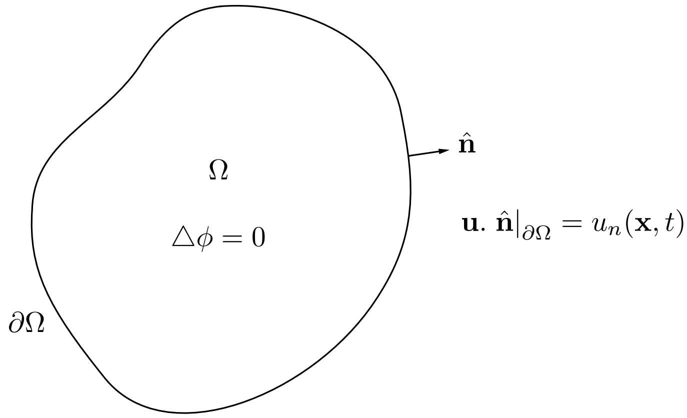



Figure 1.4: Laplace's equation with generic boundary condition.

**Proposition 1.** *There exists a unique (up to time-depedent constant) solution to the bvp posed above.*

So we can find a unique $\phi(\mathbf{x}, t)$ satisfying Laplace's equation with specified flux at the boundary. Given such $\phi$, we can then compute $\mathbf{u} = \nabla\phi$ and $p = -\rho\partial_t\phi - \frac{1}{2}\rho\|\nabla\phi\|^2$. By allowing non-zero flux across the boundaries, we allow for nontrivial flows to exist.

## 2 2D Incompressible, Irrotational, Ideal Fluid Flows

Since we work in $\mathbb{R}^2$, we have $\partial_z \equiv 0$, $w \equiv 0$, and thus $\mathbf{u} = \hat{\mathbf{i}}u(x, y, t) + \hat{\mathbf{j}}v(x, y, t)$. Incompressibility means

$$\nabla.\mathbf{u} = 0,$$

and irrotationality means

$$\nabla \times \mathbf{u} = 0.$$

In coordinates,

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0. \end{aligned}$$

These are the Cauchy-Riemann equations for the real and imaginary parts of a complex analytic function. Indeed, let $z = x + iy \in \mathbb{C}$, and write

$$F(z) = u(x, y) - iv(x, y).$$

Note the minus sign.

**Definition 4.** If

$$\lim_{\|\delta z\| \to 0} \frac{F(z + \delta z) - F(z)}{\delta z}$$

exists, then $F : \mathbb{C} \to \mathbb{C}$ is called *analytic*.

**Proposition 2.** *(Cauchy-Riemann Equations) If $F$ is analytic, then*

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}.\end{aligned}$$

*Proof.* Enforce the limit in the definition of analyticity in two ways. The first relation follows when $\delta z = \delta x$, and the second when $\delta z = i\delta y$. □

We've demonstrated thus that the real and imaginary parts of any complex analytic function comprise the components of a potential flow in $\mathbb{R}^2$.

Recall that for analytic $F$, we may write

$$F(z) = \frac{dW}{dz},$$

allowing for multi-valued $W$ (e.g., $F(z) = \log z$).

**Definition 5.** In the notation above, $W$ is called the *complex potential.*

Now, write $W(z) = \phi(x, y) + i\psi(x, y)$. Then,

$$\begin{aligned}F(z) &= u(x, y) - iv(x, y)\\ &= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\\ &= -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}\end{aligned}$$

by the Cauchy-Riemann equations. Notice that

$$u = \frac{\partial \phi}{\partial x},\ v = \frac{\partial \phi}{\partial y},$$

so $\phi$ is the velocity potential. Also notice that

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x}.$$

**Definition 6.** In the notation above, $\psi$ is called a *stream function.*

Note that

$$\begin{aligned}\mathbf{u}.\nabla\psi &= u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y}\\ &= u(-v) + vu\\ &= 0,\end{aligned}$$

and hence

$$0 = \nabla\phi.\nabla\psi.$$

Thus, potential flows in $\mathbb{R}^2$ have velocity tangent to level sets of $\psi$, and perpendicular to level sets of $\phi$. And if the flow is stationary ($\partial_t \equiv 0$), then the level sets of $\psi$ are the particle paths.

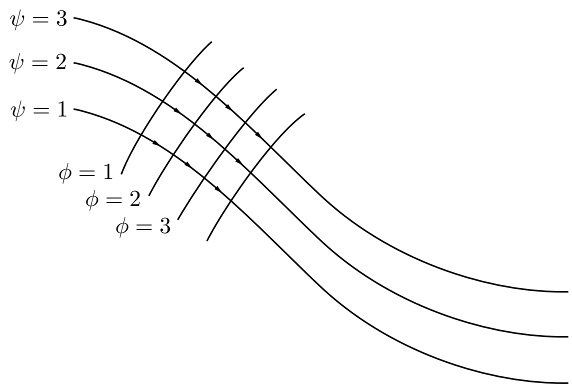



Figure 2.1: Level sets of $\phi$ and $\psi$ in potential flow.

A final calculation:

$$\begin{aligned} \|\mathbf{u}\|^2 &= \|\nabla\phi\|^2 \\ &= u^2 + v^2 \\ &= \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2 \\ &= \|\nabla\psi\|^2. \end{aligned}$$

So the fluid speeds up when the gradient of the stream function is large (when the level sets of $\psi$ bunch together), and slows down when the gradient is small (when the level sets of $\psi$ spread apart). This can be understood via conservation of mass.

## Lecture 7: Potential Flow Theory and Vortex Dynamics

We left off talking about ideal, incompressible, irrotational flow. Taking $\rho \equiv 1$, the Euler equations are

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p &= 0, \\ \nabla.\mathbf{u} &= 0. \end{aligned}$$

Setting $\omega = \nabla \times \mathbf{u}$, we can rewrite as

$$\partial_t \mathbf{u} + \omega \times \mathbf{u} + \nabla \left( p + \frac{1}{2}\|\mathbf{u}\|^2 \right) = 0.$$

Since the flow is irrotational, $\omega = \nabla \times \mathbf{u} = 0$, so on simply-connected domains,

$$\begin{aligned} \mathbf{u} &= \nabla\phi, \\ \partial_t \phi + p + \frac{1}{2}\|\mathbf{u}\|^2 &= C(t). \end{aligned}$$

To find $\mathbf{u}$, we examine solutions of

$$\triangle\phi = 0$$

subject to various boundary conditions.

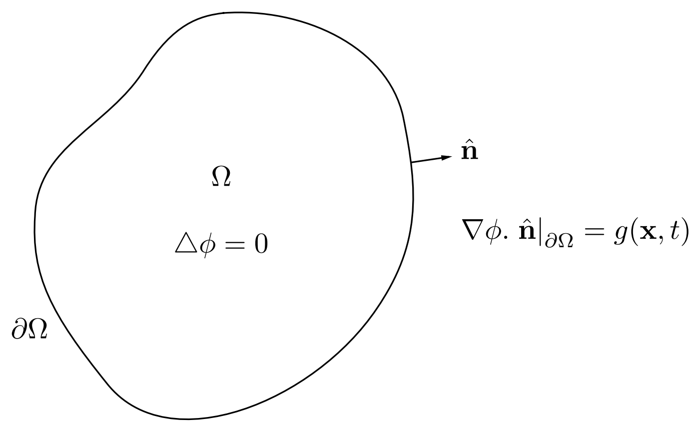



Figure 0.1: Setup for potential flow theory.

This amount to harmonic analysis.

## 1 Potential Flow in $\mathbb{R}^2$

Recall that in $\mathbb{R}^2$, $\omega = \hat{\mathbf{k}}\omega$, and

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0, \end{aligned}$$

the Cauchy-Riemann equations for an analytic function

$$F(z) = u - iv.$$

We saw that we can write the (possibly multivalued) complex potential $W$ so that

$$F(z) = \frac{dW}{dz}.$$

When we write

$$W(z) = \phi(z) + i\psi(z),$$

we recover the velocity potential, $\phi$, and introduce the stream function, $\psi$.

### 1.1 Some Elementary Flows

**Example 1.** Perhaps the simplest analytic function (beside a constant function) is $W_1(z) = (U - iV)\,z$. In this case, $W_1(z) = Ux + Vy + i\,(Uy - Vx)$, so $\mathbf{u} = \nabla\phi = \hat{\mathbf{i}}U + \hat{\mathbf{j}}V$. The level sets of $\psi$ are given by

$$Uy - Vx = \text{const.},$$

which are straight lines along the flow.

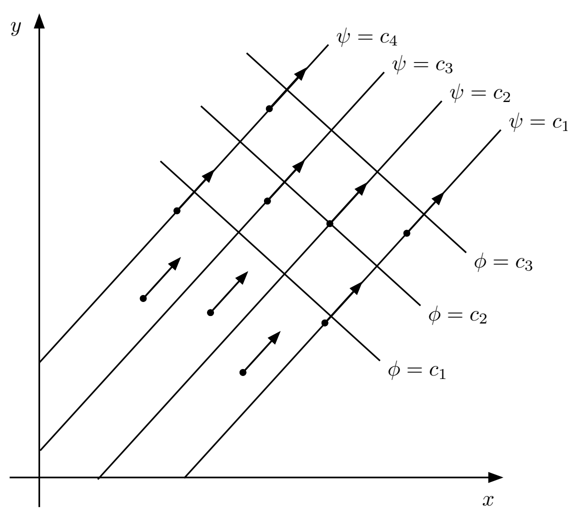



Figure 1.1: Elementary flow I.

**Example 2.** The next simplest would be $W_2(z) = \frac{1}{2}\gamma z^2$. So $W_1(z) = \frac{1}{2}\gamma\left(x^2 - y^2 + 2ixy\right)$, and hence $\phi = \frac{1}{2}\gamma\left(x^2 - y^2\right)$ and $\psi = \gamma xy$. The flow has $\mathbf{u} = \nabla\phi = \hat{\mathbf{i}}\gamma x - \hat{\mathbf{j}}\gamma y$.

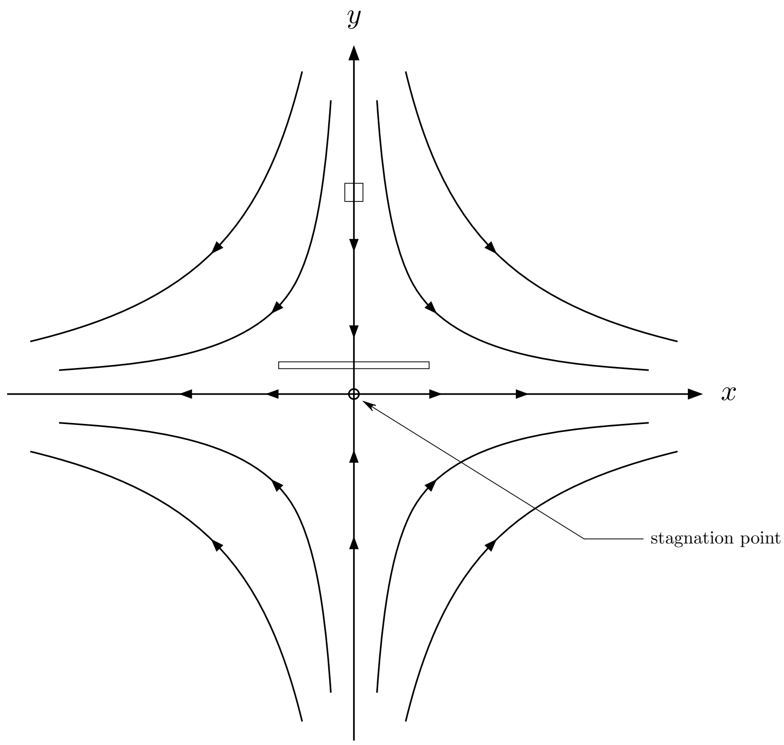



Figure 1.2: Elementary flow II, uniform strain.

Note that the sum of any two analytic functions is also analytic. E.g., we could examine the complex potential $W_1 + W_2$ to get a new flow.

**Example 3.** $W_3(z) = U\left(z + a^2/z\right)$ for $U, a \in \mathbb{R}$. Then

$$\begin{aligned} W_3(z) &= U\left(x + iy + \frac{a^2}{x+iy}\cdot\frac{x-iy}{x-iy}\right) \\ &= U\left(x + \frac{a^2 x}{x^2+y^2} + i\left[y - \frac{a^2 y}{x^2+y^2}\right]\right). \end{aligned}$$

So

$$\begin{aligned} \phi &= Ux\left(1 + \frac{a^2}{r^2}\right), \\ \psi &= Uy\left(1 - \frac{a^2}{r^2}\right), \end{aligned}$$

with $r^2 = x^2 + y^2$.

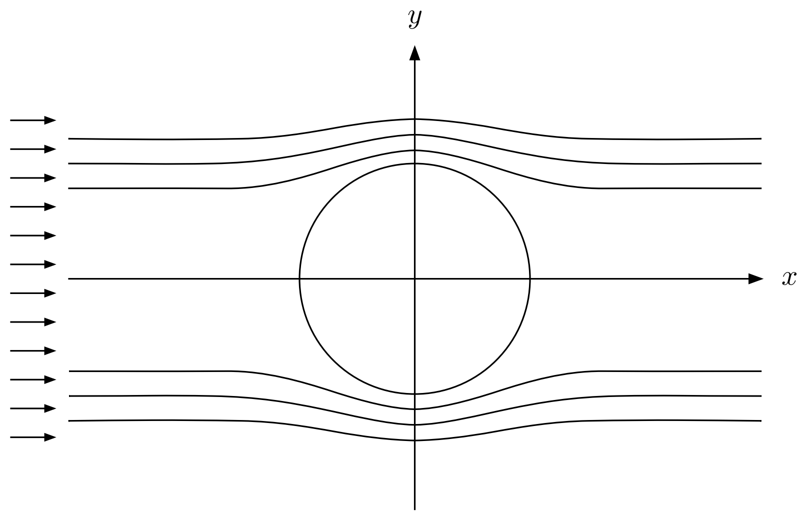



Figure 1.3: Elementary flow III, flow past a cylinder.

**Example 4.** $W_3(z) = \frac{\Gamma}{2\pi i}\log\left(\frac{z}{a}\right)$ for $\Gamma, a \in \mathbb{R}$. Recall that $\log z = i\theta + \log r$, so the complex logarithm is multivalued. But that's okay for our analysis.

$$\begin{aligned} F(z) &= \frac{dW_3}{dz} \\ &= \frac{\Gamma}{2\pi i}\frac{1}{z} \\ &= \frac{\Gamma}{2\pi i}\frac{x-iy}{x^2+y^2}, \end{aligned}$$

so the velocity field is

$$\mathbf{u} = \hat{\mathbf{i}}\left(-\frac{\Gamma}{2\pi}\frac{y}{r^2}\right) + \hat{\mathbf{j}}\left(\frac{\Gamma}{2\pi}\frac{x}{r^2}\right).$$

The complex potential and stream function satisfy

$$\begin{aligned} \phi &= \frac{\Gamma}{2\pi}\theta, \\ \psi &= -\frac{\Gamma}{2\pi}\log\left(\frac{r}{a}\right). \end{aligned}$$

Note that the speed satisfies

$$\|u\|^2 = \frac{\Gamma^2}{4\pi^2}\frac{1}{r^2}.$$

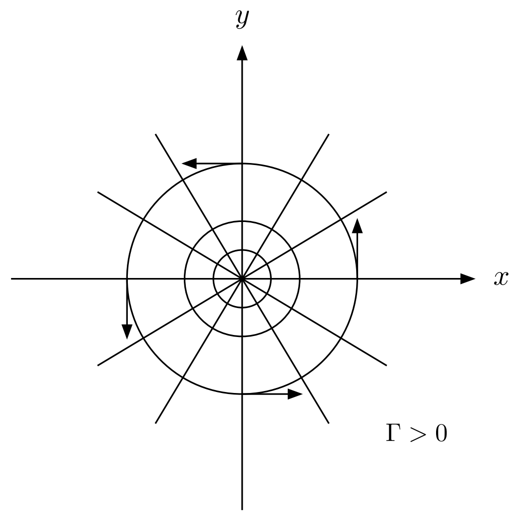



Figure 1.4: Elementary flow IV, elementary vortex.

A quick calculation. For a simple closed curve $C$,

$$\Gamma_C = \oint_C \mathbf{u}.d\mathbf{l} = \int_A \omega \, da$$

by Stoke's theorem. If $C$ does not enclose the origin, $\Gamma_C = 0$. If $C$ encloses the origin, deform it to a circle to get

$$\Gamma_C = \frac{\Gamma}{2\pi r} \cdot 2\pi r = \Gamma \neq 0.$$

So if we expect Stoke's theorem to hold, we could write

$$\nabla \times \mathbf{u} = \Gamma \delta(\mathbf{x}).$$

At the origin, there is a point vortex of infinite strength. We have found a Green's function for the curl operator.

### 1.2 Point Vortex Dynamics

Consider a pair of elementary vortices.

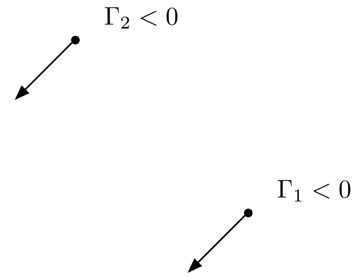



Figure 1.5: A pair of elementary vortices.

Now consider $N$ elementary vortices which interact. We demand that each of the vortices moves in the flow induced by the others, as to satisfy the Euler equations[1]. So write

$$W(z,t) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi i} \log\left(\frac{z-\zeta_k(t)}{a}\right),$$

with

$$\zeta_k(t) = x_k(t) + i y_k(t).$$

Recalling that $F = \frac{dW}{dz} = u - iv$, we define (CHECK THE SIGNS)

$$\begin{aligned} \frac{d}{dt}\bar{\zeta}_k &= \frac{d}{dz} \sum_{j=1, j\neq k}^{N} \frac{\Gamma_j}{2\pi i} \log\left(\frac{z-\zeta_j}{a}\right)\Bigg|_{z=\zeta_k} \\ &= \sum_{j=1, j\neq k}^{N} \frac{\Gamma_j}{2\pi i} \frac{\bar{\zeta}_k - \bar{\zeta}_j}{|\zeta_k - \zeta_j|^2}. \end{aligned}$$

Then, the dynamics are given by

$$\begin{aligned} \dot{x}_k &= -\sum_{j\neq k} \frac{\Gamma_j}{2\pi} \frac{y_k - y_j}{(x_k - x_j)^2 + (y_k - y_j)^2}, \\ \dot{y}_k &= \sum_{j\neq k} \frac{\Gamma_j}{2\pi} \frac{x_k - x_j}{(x_k - x_j)^2 + (y_k - y_j)^2}. \end{aligned}$$

This is a $2N$-dimensional system of ordinary differential equations. But in fact, there is additional structure. This is a Hamiltonian system:

$$\begin{aligned} \Gamma_k \dot{x}_k &= \frac{\partial \mathcal{H}}{\partial y_k}, \\ \Gamma_k \dot{y}_k &= -\frac{\partial \mathcal{H}}{\partial x_k}, \end{aligned}$$

[1] To be specific, this description of vortex dynamics satisfies the weak formulation of the Euler equations.

with

$$\mathcal{H} = \sum_j \sum_{k \neq j} \frac{\Gamma_k \Gamma_j}{2\pi} \log\left(\frac{\|\mathbf{x}_j - \mathbf{x}_k\|}{a}\right).$$

Here, we have $\mathbf{x}_j = (x_j, y_j)$. In the sense of Hamiltonian dynamics, $x_k$ is a canonical position and $y_k$ is a canonical momentum. This is used as a computational tool – e.g., we can model a shear layer as many, many point vortices in a line.

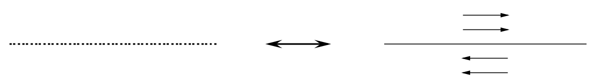


Figure 1.6: Vortex sheet modeling an interface.

## Lecture 8: Vortex Dynamics and D'Alembert's Paradox

The equations:

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} &= \frac{1}{\rho}\mathbf{f},\\ \nabla.\mathbf{u} &= 0.\end{aligned}$$

The boundary condition is $\hat{\mathbf{n}}.\ \mathbf{u}|_{\partial\Omega} = 0$. With the definition $\omega = \nabla \times \mathbf{u}$, we have

$$\begin{aligned}\partial_t \mathbf{u} + \omega \times \mathbf{u} + \nabla\left(\frac{1}{\rho}p + \frac{1}{2}\|\mathbf{u}\|^2\right) &= \frac{1}{\rho}\mathbf{f},\\ \partial_t \omega + \mathbf{u}.\nabla\omega &= \omega.\nabla\mathbf{u} + \frac{1}{\rho}\nabla \times \mathbf{f}.\end{aligned}$$

Recall that irrotationality, $\omega = 0$, means we can write $\mathbf{u} = \nabla\phi$. (The potential is unique up to constant in simply-connected domains.) Incompressiblity implies $\nabla.\mathbf{u} = 0$, so we arrive at

$$\triangle\phi = 0.$$

The pressure is determined by

$$\partial_t\phi + \frac{1}{\rho}p + \frac{1}{2}\|\nabla\phi\|^2 = C(t).$$

In $\mathbb{R}^2$ there is even more structure. The vorticity becomes a scalar,

$$\omega = \hat{\mathbf{k}}\omega = \hat{\mathbf{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

and irrotationality and incompressibility now read

$$\begin{aligned}\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y},\\ \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y}.\end{aligned}$$

These are the Cauchy-Riemann equations for $F = u - iv$. Setting $z = x + iy$, as usual, we can write $F(z) = W'(z)$ with $W(z) = \phi + i\psi$, allowing for multivalued antiderivatives (e.g., $\log(\cdot)$).

## 1 Point Vortex Dynamics (ctd.)

The following is a review.

**Example 1.** Point vortex:

$$W(z) = \frac{\Gamma}{2\pi i}\log\left(\frac{z}{a}\right).$$

Note that there is circulation about the origin, but the flow is everywhere irrotational (except the origin, where it is not defined).

If we set $N$ point vortices in the plane at locations $\zeta_k = x_k + iy_k$, $k = 1, \ldots, N$, we can ensure the flow satisfies the Euler equations by requiring that each point vortex movies in the velocity induced by the others. Then we arrive at

$$\dot{x}_k - i\dot{y}_k = \dot{\bar{\zeta}} = \frac{d}{dz} \sum_{j \neq k} \frac{\Gamma_j}{2\pi i} \log\left(\frac{z - \zeta_j}{a}\right)\Bigg|_{z=\zeta_k}.$$

Then the equations of motion are

$$\begin{aligned} \dot{x}_k &= -\sum_{j \neq k} \frac{\Gamma_j}{2\pi} \frac{y_k - y_j}{(x_k - x_j)^2 + (y_k - y_j)^2}, \\ \dot{y}_k &= \sum_{j \neq k} \frac{\Gamma_j}{2\pi} \frac{x_k - x_j}{(x_k - x_j)^2 + (y_k - y_j)^2}. \end{aligned}$$

This system has a Hamiltonian structure:

$$\begin{aligned} \Gamma_k \dot{x}_k &= \frac{\partial \mathcal{H}}{\partial y_k}, \\ \Gamma_k \dot{y}_k &= \frac{\partial \mathcal{H}}{\partial x_k}, \end{aligned}$$

with

$$\mathcal{H} = -\sum_{j \neq k} \frac{\Gamma_j \Gamma_k}{4\pi} \log\left(\frac{\|\mathbf{x}_j - \mathbf{x}_k\|}{a}\right).$$

What does the Hamiltonian structure tell us? Look for symmetries. The hamiltonian has both translational and rotational invariance, so we arrive at three other conserved quantities

$$\begin{aligned} \mathsf{X} &= \sum_{j=1}^{N} \Gamma_j x_j, \\ \mathsf{Y} &= \sum_{j=1}^{N} \Gamma_j y_j, \\ \mathsf{L} &= \sum_{j=1}^{N} \Gamma_j \left(x_j^2 + y_j^2\right). \end{aligned}$$

The three quantities of interest here are thus $\mathcal{H}$, $\mathsf{L}$, and $\mathsf{X}^2 + \mathsf{Y}^2$.

## 2 D'Alembert's Paradox

Consider a body $B$ in a potential flow with far-field velocity given by the requirement that

$$\mathbf{u} \to \hat{\mathbf{i}} U + \hat{\mathbf{j}} V \text{ as } \|\mathbf{x}\| = |z| \to \infty.$$

Thus,

$$\begin{aligned} F(z) &\to U - iV, \\ W(z) &\to (U - iV)\, z \end{aligned}$$

as $|z| \to \infty$. We ask: what is the magnitude and direction of the force felt by the object?

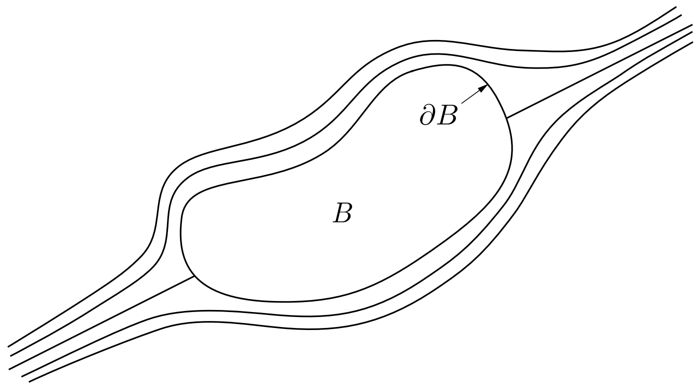



Figure 2.1: Flow over a body.

Let $\mathbf{f} = \hat{\mathbf{i}} f_x + \hat{\mathbf{j}} f_y$ be the force[1] felt by the object. Then we define a "complex force," $\mathcal{F} = f_x - i f_y$. We have the following theorem.

**Theorem 1.** *(Blasius) The force on an object in a stationary, incompressible, irrotational flow with complex velocity $F(z)$ outside the body is*

$$\mathcal{F} = \frac{i\rho}{2} \oint_{\partial B} (F(z))^2 \, dz.$$

*Proof.* Observe that $\mathbf{f}$ is due to pressure differences around $\partial B$, in that

$$\mathbf{f} = -\int_{\partial B} \hat{\mathbf{n}} p \, ds.$$

In components,

$$\begin{aligned} f_x &= \hat{\mathbf{i}}.\hat{\mathbf{f}} = -\int_{\partial B} p \left(\hat{\mathbf{i}}.\hat{\mathbf{n}}\right) ds, \\ f_y &= \hat{\mathbf{j}}.\hat{\mathbf{f}} = -\int_{\partial B} p \left(\hat{\mathbf{j}}.\hat{\mathbf{n}}\right) ds. \end{aligned}$$

We can write an infinitesimal length element as $d\mathbf{l} = \hat{\mathbf{i}} dx + \hat{\mathbf{j}} dy$ and the normal component at the boundary is $\hat{\mathbf{n}} = \hat{\mathbf{i}} \sin\theta + \hat{\mathbf{j}} \cos\theta$, so that

$$d\mathbf{l} = \hat{\mathbf{i}} \left(-\cos\theta\right) ds + \hat{\mathbf{j}} \sin\theta ds.$$

Since $\hat{\mathbf{i}}.\hat{\mathbf{n}} \, ds = \sin\theta \, ds = dy$ and $\hat{\mathbf{j}}.\hat{\mathbf{n}} \, ds = \cos\theta \, ds = -dx$, we have

$$d\mathbf{l} = \hat{\mathbf{i}} dy - \hat{\mathbf{j}} dx.$$

[1] As we're in $\mathbb{R}^2$, really this is force/length. But recall that the object is infinite in and out of the page.

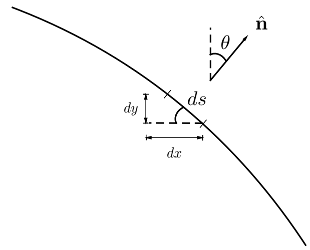



Figure 2.2: Infinitesimal length elements on $\partial B$.

Hence,

$$\begin{aligned} f_x &= -\int_{\partial B} p\,dy, \\ f_y &= \int_{\partial B} p\,dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{\mathcal{F}} &= f_x + i f_y \\ &= -\int_{\partial B} p\,(dy - i\,dx) \\ &= i\int_{\partial B} p\,dz, \end{aligned}$$

as $i\,dz = i\,(dx + i\,dy)$. Since

$$p = -\frac{\rho}{2}\left(u^2+v^2\right) + C(t),$$

we arrive at

$$\overline{\mathcal{F}} = \frac{-i\rho}{2}\int_{\partial B}\left(u^2+v^2\right)\,dz.$$

To finish, consider that

$$\begin{aligned} F^2\,dz &= (u-iv)^2\,(dx+i\,dy) \\ &= \left[\left(u^2-v^2\right) - 2iuv\right](dx + i\,dy) \\ &= \left(u^2-v^2\right)\,dx + 2uv\,dy + i\left[\left(u^2-v^2\right)\,dy - 2uv\,dx\right]. \end{aligned}$$

Recall that flow is tangent to the boundary. The tangent to the boundary is $d\mathbf{l} = \hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy$. Since $\mathbf{u} \parallel d\mathbf{l}$,

$$0 = \mathbf{u}\times d\mathbf{l} = \mathbf{k}\,(udy - vdx)\,,$$

and hence $u\,dy = v\,dx$ on $\partial B$. So we can write

$$\begin{aligned} F^2\,dz &= \left(u^2+v^2\right)\,dx - i\left(u^2+v^2\right)\,dy \\ &= \left(u^2+v^2\right)\,d\overline{z}. \end{aligned}$$

Hence,

$$\overline{\mathcal{F}} = \frac{-i\rho}{2} \int_{\partial B} \overline{(F(z))^2 \; dz},$$

and hence

$$\mathcal{F} = f_x - i f_y = \frac{i\rho}{2} \int_{\partial B} (F(z))^2 \; dz.$$

This proves the claim. □

**Theorem 2.** *(Kutta-Joukowski Theorem) Given the setup above,*

$$\mathbf{f} = \rho \Gamma_C \left( \mathbf{U} \times \hat{\mathbf{k}} \right),$$

*where*

$$\Gamma_C = \oint_C \mathbf{u}.d\mathbf{l}$$

*with $C$ being any closed curve containing the object.*

*Remark.* Thus D'Alembert's paradox is that $\mathbf{f} \perp \mathbf{u}$, so potential flow exerts no drag.

*Proof.* In our setup, $F(z)$ is complex analytic in the domain outside of the object $B$. W.l.o.g., we can place the origin inside $B$. So $F(z)$ has a convergent Laurent expansion

$$F(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots.$$

Note that as $|z| \to \infty$, $F(z) \to a_0$, hence $a_0 = U - iV$ and there are no positive powers in the expansion above. Cauchy's theorem states that

$$\oint_C F(z)\, dz = 2\pi i a_1.$$

We can take $C = \partial B$ to get

$$\begin{aligned} a_1 &= \frac{1}{2\pi i} \oint_{\partial B} F(z)\, dz \\ &= \frac{1}{2\pi i} \int_{\partial B} (u - iv)\,(dx + i\, dy) \\ &= \frac{1}{2\pi i} \int_{\partial B} (u\, dx + v\, dy) + \frac{1}{2\pi} \int_{\partial B} (u\, dy - v\, dx) \\ &= \frac{1}{2\pi i} \int_{\partial B} (u\, dx + v\, dy)\,, \end{aligned}$$

as $u\, dy = v\, dx$ on the boundary. Thus,

$$a_1 = \frac{1}{2\pi i} \int_{\partial B} \mathbf{u}.d\mathbf{l} = \frac{\Gamma_C}{2\pi i}.$$

So we can write

$$\begin{aligned} F(z) &= U - iV + \frac{\Gamma_C}{2\pi i} \frac{1}{z} + \frac{a_2}{z^2} + \ldots \\ \implies (F(z))^2 &= (U - iV)^2 + \frac{2\Gamma_C\,(U - iV)}{2\pi i} \frac{1}{z} + O\left(\frac{1}{z^2}\right), \end{aligned}$$

and by the residue theorem,

$$\oint_{\partial B} (F(z))^2 \, dz = 2\Gamma_C (U - iV) .$$

Blasius' theorem tells us that

$$\mathcal{F} = f_x - i f_y = \frac{i\rho}{2} \cdot 2\Gamma_C (U - iV) ,$$

and hence

$$\begin{aligned} f_x &= \rho\Gamma_C V, \\ f_y &= -\rho\Gamma_C U. \end{aligned}$$

But this is exactly $\mathbf{f} = \rho\Gamma_C (\mathbf{U} \times \mathbf{k})$, so we're done. □

Kutta-Joukowski tells us that there is lift exerted on the object, but there is no drag. This is a non-physical state of affairs. Note that there is a three-dimensional version of this theorem, but still there is no drag. So what are we missing? Viscosity! We'll start this next time.

# Lecture 9: Introduction to Viscous Fluids

In real fluids viscosity plays a role. This lecture we'll try to understand the physics of viscosity. Later we'll see how to incorporate viscosity into Navier-Stokes.

## 1 Viscosity and the Stress Tensor

As a first example, consider two fluids.

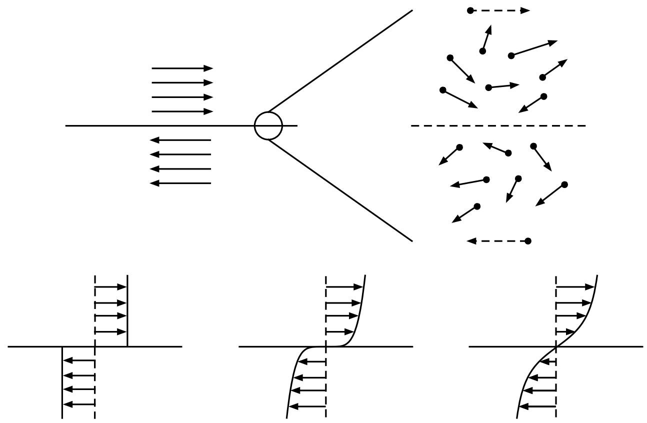


Figure 1.1: Two fluids at an interface.

Shear stress comes about in this example from the diffusion of momentum across the fluid boundary. To incorporate this on a macroscopic level, we consider a fluid element of volume $\delta V = \delta x \delta y \delta z$.

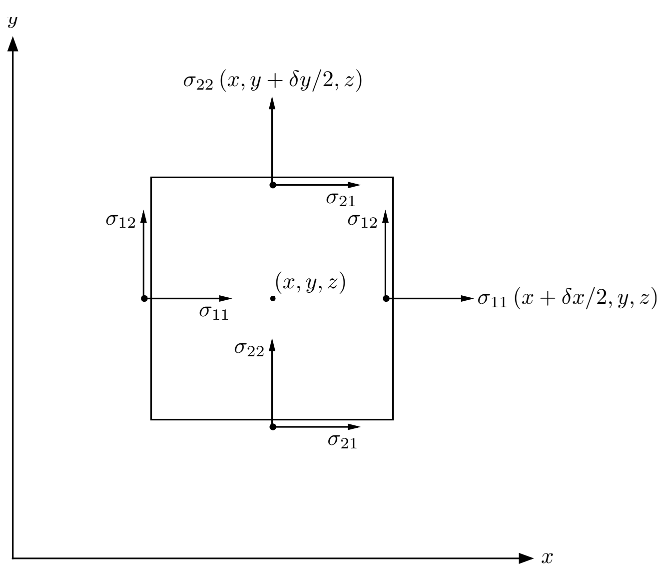



Figure 1.2: Stress on a fluid element.

In our setup, $\sigma_{ij}$ is the force per unit area in the $j$th direction across an area element with normal in the $i$th direction. We'll adopt the following sign convention: $\sigma_{ij}$ is the force on the $(-)$ side due to the fluid on the $(+)$ side. So the net force on the element in the 1-direction is (to first order)

$$\begin{aligned}
\delta F_1 &= \sigma_{11}\left(x+\frac{\delta x}{2},y,z\right)\cdot\delta y\delta z+\sigma_{21}\left(x,y+\frac{\delta y}{2},z\right)\cdot\delta x\delta z-\sigma_{11}\left(x-\frac{\delta x}{2},y,z\right)\cdot\delta y\delta z \\
&\quad -\sigma_{21}\left(x,y-\frac{\delta y}{2},z\right)\cdot\delta x\delta z+\sigma_{31}\left(x,y,z+\frac{\delta z}{2}\right)\cdot\delta x\delta y-\sigma_{31}\left(x,y,z-\frac{\delta z}{2}\right)\cdot\delta x\delta y \\
&= \left(\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{21}}{\partial y}+\frac{\partial\sigma_{31}}{\partial z}\right)\cdot\delta x\delta y\delta z \\
&= (\nabla.\sigma)_1\,\delta V.
\end{aligned}$$

Carrying out this analysis in each direction, we arrive at

$$\delta\mathbf{F}=\nabla.\sigma\delta V,$$

or just

$$\delta F_i=\frac{\partial\sigma_{ji}}{\partial x_j}\delta V.$$

**Definition 1.** In the above notation, we call $\sigma$ the *stress tensor* of the fluid.

**Proposition.** *The stress tensor is symmetric.*

One way to see this is to consider the fluid as having a velocity distribution $\mathbf{u}(\mathbf{x},t)+\mathbf{u}'$. Then $\sigma_{ij}=\rho\left\langle u_i'u_j'\right\rangle$ which is obviously symmetric. Another way is to consider the macroscopic dynamics: we'll see that the stress tensor must be symmetric to prevent infinite angular accelerations. E.g., in the 3-direction, the torque on the fluid element is (to first order)

$$\begin{aligned}
N_3 &= \hat{\mathbf{k}}.\sum_{\text{faces}}\mathbf{r}\times\delta\mathbf{F} \\
&= \frac{\delta x}{2}\cdot\sigma_{12}\left(x+\frac{\delta x}{2},y,z\right)\cdot\delta y\delta z+\frac{\delta x}{2}\cdot\sigma_{12}\left(x-\frac{\delta x}{2},y,z\right)\cdot\delta y\delta z \\
&\quad -\frac{\delta y}{2}\cdot\sigma_{21}\left(x,y+\frac{\delta y}{2},z\right)\cdot\delta x\delta z-\frac{\delta y}{2}\cdot\sigma_{21}\left(x,y-\frac{\delta y}{2},z\right)\cdot\delta x\delta z \\
&= \delta V\cdot(\sigma_{12}-\sigma_{21}).
\end{aligned}$$

We claim this must be zero to prevent infinite angular accelerations. To conclude this, we first compute the moment of inertia. Since $\rho$ is locally constant, we have

$$\begin{aligned}
I_{33} &= \iiint\left(x^2+y^2\right)\rho\,dxdydz \\
&= \frac{1}{12}\left((\delta x)^2+(\delta y)^2\right)\rho\cdot\delta V,
\end{aligned}$$

and thus

$$\text{angular acceleration}=\frac{N_3}{I_{33}}\sim\frac{\sigma_{12}-\sigma_{21}}{(\delta x)^2+(\delta y)^2}.$$

So if $\sigma_{12}\neq\sigma_{21}$, then as $\delta x,\delta y\to 0$, the angular acceleration blows up.

## 2 Rate of Strain Tensor for Newtonian Fluids

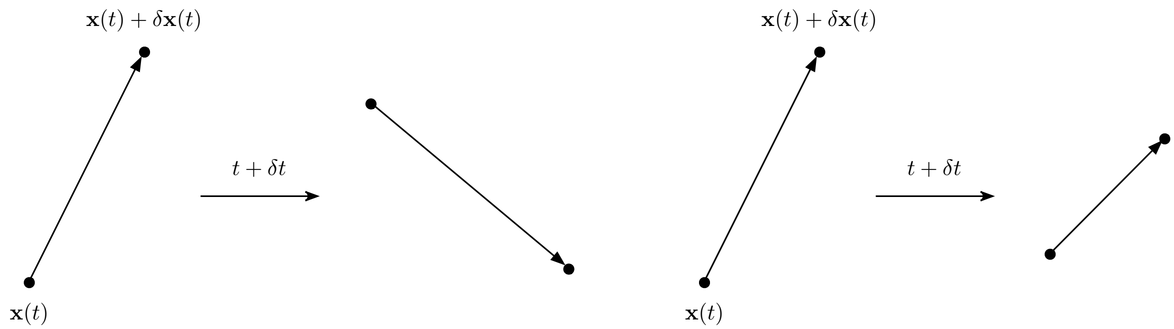
Now we'll seperate the stresses into reversible and irreversible parts. W.l.o.g., write

$$\sigma_{ij} = -\delta_{ij}p + \tau_{ij}$$

as to isolate the normal stresses (due to $p$), which are reversible. The "deviatoric stress tensor" $\tau$ will represent the irreversible (frictional) stresses. Now we have a definition.

**Definition 2.** A *Newtonian fluid* is one in which the frictional stresses are proportional to the "rate of strain" tensor.

The "rate of strain" tensor describes the deviation of the element's motion from rigid body motion. To account for these motions, consider $\delta\mathbf{x}$ which is a small displacement of two points in the fluid, $\mathbf{x}(t)$ and $\mathbf{x}(t) + \delta\mathbf{x}(t)$.




(a) Rigid motion. (b) Non-rigid motion.

Figure 2.1: Rigid vs. non-rigid body motion.

In time $\delta t$, rigid body motion will preserve the length of $\delta\mathbf{x}$. But non-rigid body motion leads to changes in the length of $\delta\mathbf{x}$. The rate of strain tensor will account for the property of the fluid motion which gives rise to the latter motions. So we have a calculation. On the one hand,

$$\frac{d}{dt}\|\mathbf{x}\|^2 = 2\|\mathbf{x}\|\frac{d}{dt}\|\mathbf{x}\|,$$

and on the other hand,

$$\begin{aligned}
\frac{d}{dt}\|\delta\mathbf{x}\|^2 &= 2\delta\mathbf{x}.\left(\frac{d}{dt}\delta\mathbf{x}\right)\\
&= 2\delta\mathbf{x}.\left(\mathbf{u}\left(\mathbf{x}+\delta\mathbf{x},t\right) - \mathbf{u}\left(\mathbf{x},t\right)\right)\\
&= 2\delta x_i\frac{\partial u_i}{\partial x_j}\delta x_j\\
&= \delta x_i\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j.
\end{aligned}$$

Thus,

$$\frac{1}{\|\delta\mathbf{x}\|}\left(\frac{d}{dt}\|\mathbf{x}\|\right)=\frac{\delta x_i}{\|\delta\mathbf{x}\|}\cdot\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\cdot\frac{\delta x_j}{\|\delta\mathbf{x}\|}.$$

So we have a definition.

**Definition 3.** The *rate of strain* tensor is the symmetric part of the velocity gradient tensor,

$$r_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right).$$

Now recall we have written

$$\sigma=-pI+\tau,$$

and we saw that a Newtonian fluid is one for which $\tau\sim r$, with $r=(\nabla\mathbf{u})_{\text{symm.}}$. The most general linear, isotropic relation between two tensors is

$$\tau_{ij}=A_{ijkl}\cdot r_{kl}$$

with

$$A_{ijkl}=a\delta_{ij}\delta_{kl}+b\delta_{ik}\delta_{jl}+c\delta_{il}\delta_{jk}.$$

So

$$\begin{aligned}\tau_{ij}&=a\delta_{ij}\mathrm{tr}\,(r)+br_{ij}+cr_{ji}\\&=a\delta_{ij}\mathrm{tr}\,(r)+(b+c)\,r_{ij},\end{aligned}$$

since $r$ is symmetric, and thus

$$\tau_{ij}=c_1r_{ij}+c_2\mathrm{tr}\,(r)\,\delta_{ij}.$$

Recalling the definition of $r_{ij}$, we find that

$$\mathrm{tr}\,(r)=r_{kk}=u_{k,k}=\nabla.\mathbf{u},$$

and hence that

$$\tau_{ij}=\frac{c_1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+c_2\left(\nabla.\mathbf{u}\right)\delta_{ij}.$$

Adding and subtracting, we may write this as

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}u_{k,k}\delta_{ij}\right)+\lambda u_{k,k}\delta_{ij}$$

where $\mu,\lambda$ are constants determined by $c_1,c_2$. Why would we write this as such? One reason is that the first term is trace-free, so it represents the volume-preserving deformations of fluid elements. The second term accounts for deformations which change the volume of fluid elements.

**Definition 4.** In the notation above, $\mu$ is the *coefficient of dynamic (shear) velocity*, and $\lambda$ is the *coefficient of bulk viscosity.*

So we have arrived at

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\left(r_{ij} - \frac{1}{3}r_{kk}\delta_{ij}\right) + \lambda r_{kk}\delta_{ij}.$$

The rate at which work is being done on the fluid element is given by $\sigma_{ij}r_{ij}$. So for Newtonian fluids,

$$\text{rate at which work is being done on fluid element} = -pr_{kk} + \left(2\mu r_{ij}r_{ij} - \frac{2}{3}\mu\left(r_{kk}\right)^2 + \lambda\left(r_{kk}\right)^2\right).$$

The first term is the rate of $p\,dv$ work; the second is the rate of viscous energy dissapation,

$$\Phi = 2\mu r_{ij}r_{ij} - \frac{2}{3}\mu\left(r_{kk}\right)^2 + \lambda\left(r_{kk}\right)^2.$$

For thermodynamic reasons, we expect that $\Phi \geq 0$. This is the same as asking that $\lambda, \mu \geq 0$.

Next time, we'll see how to put stresses into Navier-Stokes, mainly by

$$\rho\frac{d}{dt}\mathbf{u} = \text{total force per volume}.$$

# Lecture 10: Navier-Stokes Equation

A quick review of what we had last time.

## 1 Stress/Rate-of-Strain Tensors (review)

Recall we had the stress tensor, $\sigma$, with

$$\sigma_{ij} = \text{force/area in } j\text{-direction acting across area element with normal in } i\text{-direction.}$$

More generally,

$$\hat{\mathbf{n}} \cdot \sigma \cdot \hat{\mathbf{m}} = \text{force/area in } \hat{\mathbf{m}}\text{-direction acting across area element with normal in } \hat{\mathbf{n}}\text{-direction.}$$

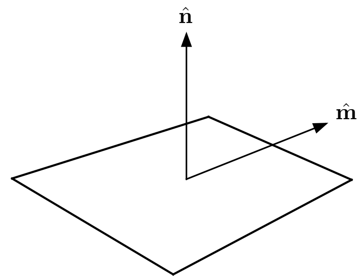



Figure 1.1: Area element.

The force on a fluid element of volume $\delta V$ is

$$\begin{aligned} \delta F &= \nabla.\sigma \delta V, \\ \delta F_i &= \frac{\partial}{\partial x_i} \sigma_{ji} \delta V. \end{aligned}$$

Note that $\sigma_{ij} = \sigma_{ji}$. We also developed the rate of strain tensor, $r$, has

$$r_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right).$$

At each point in the flow, $r$ is symmetric, so we can decompose the dynamics w.r.t. an orthonormal basis of eigenvectors of $r$. Then a rod oriented along $\hat{e}_i$ instantaneously stretches at rate $\lambda_i$.

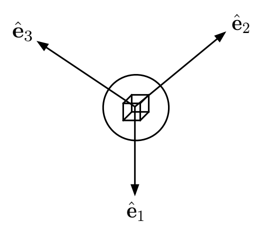



Figure 1.2: Eigenvectors of $r$.

Motion in $\mathbb{R}^3$ has the following components:

| | |
|---|---|
| Translation | $\mathbf{u}$ |
| Rotation | $\omega = \nabla \times \mathbf{u}$ |
| Compression/extension | $\nabla . \mathbf{u}$ |
| Deformation ("pure strain") | $r_{ij} - \frac{1}{3} r_{kk} \delta_{ij}$ |

Note that $\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr} r = \nabla . \mathbf{u}$, so compression/extension is captured by $r$. And so is deformation, which is the trace-free part of $r$. Have we lost information or put more in somehow? If we take the velocity vector field and all of its derivatives, we have $3 + 9 = 12$ pieces of information. In the four types of motion listed above, we have $3 + 3 + 1 + 5 = 12$ pieces of information. So it seems that indeed all of the kinematic properties needed to propagate a fluid element forward one step in time is included in the velocity field (and its derivatives).

Now observe that the rate at which work done on the fluid element is given by

$$\underbrace{\text{stress} \times \text{rate}}_{\text{force} \times \text{velocity}} = \sigma : r = \sigma_{ij} r_{ij} = \operatorname{tr}(\sigma \cdot r).$$

W.l.o.g., we wrote

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij},$$

with $\tau$ being the deviatoric stress tensor. And for Newtonian fluids, we saw that $\tau \sim r$ along with isotropy forces

$$\begin{aligned} \tau_{ij} &= A_{ijkl} r_{kl} \\ &= \left(c_1 \delta_{ij}\delta_{kl} + c_2 \delta_{ik}\delta_{jl} + c_3 \delta_{il}\delta_{jk}\right) r_{kl} \\ &= c_1 \delta_{ij} r_{kk} + (c_2 + c_3)\, r_{ij} \\ &= \underbrace{(c_2 + c_3)}_{2\mu} \left( r_{ij} - \frac{1}{3}\delta_{ij} r_{kk} \right) + \underbrace{\left( c_1 + \frac{c_2 + c_3}{3} \right)}_{\lambda} \delta_{ij} r_{kk}. \end{aligned}$$

Note that $\mu, \lambda$ are physical properties of the fluid in consideration. So for Newtonian fluids,

$$\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \tau_{ij} \\ &= -p\delta_{ij} + 2\mu \left( r_{ij} - \frac{1}{3}\delta_{ij} r_{kk} \right) + \lambda \delta_{ij} r_{kk}. \end{aligned}$$

What should the signs be on $\mu, \lambda$? First, note that for any tensor $M$ we define

$$\|M\|^2 = \operatorname{tr}(M \cdot M) = M_{ij} M_{ij}.$$

Then the power/unit volume is

$$\begin{aligned} \sigma_{ij} r_{ij} &= -p r_{ii} + \left[ 2\mu \left( r_{ij} - \frac{1}{3}\delta_{ij} r_{kk} \right) + \lambda \delta_{ij} r_{kk} \right] \left( r_{ij} - \frac{1}{3}\delta_{ij} r_{ll} + \frac{1}{3}\delta_{ij} r_{ll} \right) \\ &= -p\,(\nabla . \mathbf{u}) + \left[ 2\mu \| r - \frac{1}{3} \operatorname{tr}(r) \cdot T \|^2 + \lambda \,(\operatorname{tr} r)^2 \right]. \end{aligned}$$

So as long as $\mu, \lambda > 0$, then the work done by the deviatoric stresses will be positive. So it is like a friction. The first term is the $p\,dv$ work, and the second term is the dissapative work. Let's give the second term a name:

$$\Phi = 2\mu \|r - \frac{1}{3}\mathrm{tr}\,(r)\cdot T\|^2 + \lambda\,(\mathrm{tr}\,r)^2\,,$$

which is the rate of dissapation in the fluid.

## 2 Viscous Navier-Stokes

Consider that the total force/volume acting on a fluid element is given by

$$\nabla.\sigma + \mathbf{f}^{\mathrm{ext}}.$$

For Newtonian fluids, we can make the following calculation:

$$\begin{aligned}
\frac{\delta F_i}{\delta V} &= \frac{\partial}{\partial x_j}\sigma_{ji} \\
&= \frac{\partial}{\partial x_j}\left(-p\delta_{ji} + 2\mu\left(r_{ij} - \frac{1}{3}\delta_{ji}r_{kk}\right) + \lambda\delta_{ij}r_{kk}\right) + f_i^{\mathrm{ext}} \\
&= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(u_{i,i} + u_{j,i}\right) - \frac{2}{3}\mu\frac{\partial}{\partial x_i}\left(\nabla.\mathbf{u}\right) + \lambda\frac{\partial}{\partial x_i}\left(\nabla.\mathbf{u}\right) + f_i^{\mathrm{ext}} \\
&= -\frac{\partial p}{\partial x_i} + \mu\triangle u_i + \left(\lambda + \frac{1}{3}\mu\right)\frac{\partial}{\partial x_i}\left(\nabla.\mathbf{u}\right) + f_i^{\mathrm{ext}}.
\end{aligned}$$

Now we're ready to present full Navier-Stokes.

Fluid properties we're interested in are $\rho, p, \mathbf{u}$, and the internal energy/mass $E$. The dynamics are thus:

- Conservation of mass:

$$\begin{aligned}
\frac{d}{dt}\rho + \rho\left(\nabla.\mathbf{u}\right) &= 0 \\
\iff \frac{\partial\rho}{\partial t} + \nabla.\left(\rho\mathbf{u}\right) &= 0.
\end{aligned}$$

- Conservation of momentum:

$$\rho\frac{d}{dt}\mathbf{u} = \nabla.\sigma + \mathbf{f}^{\mathrm{ext}}.$$

  For Newtonian fluids,

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right)\mathbf{u} = -\nabla p + \mu\triangle\mathbf{u} + \left(\lambda + \frac{\mu}{3}\right)\nabla\left(\nabla.\mathbf{u}\right) + \mathbf{f}^{\mathrm{ext}}.$$

  This is know as the **Navier-Stokes** equation.

- Conservation of (internal) energy:

$$\rho\frac{d}{dt}E = -p\left(\nabla.\mathbf{u}\right) + \Phi + \nabla.\left(k\nabla T\right),$$

where $T$ is the temperature at a point and $k$ is a material coefficient called the "thermal conductivity" coefficient. The third term comes from Fourier's law. We have

$$E = cT,$$

with $c$ being a material parameter called the "specific heat."

- Thermodynamically determined variables:

$$\text{pressure: } p = p(\rho, T),$$
$$\text{material coefficients: } \mu, \lambda, k, c.$$

Note that the material coefficients generally are all functions of $\rho, T$.

We have five unknowns $(\rho, p, \mathbf{u}, E, T)$ and five equations. Throughout the rest of the course, we'll assume that all the material parameters are constants. This will greatly simplify the calculations, but we'll still find a wealth of information in the equations. We'd also like to only consider incompressible flows, i.e., we'd like $\rho =$ constant and $\nabla.\mathbf{u} = 0$. But consider a special class of fluids – flows driven by bouancy due to a gravatational field. Certiainly we can't demand incompressibility everywhere for these flows!

For many fluids, the "coefficient of expansion"

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p > 0$$

is a relatively small number. E.g., $p = \rho k_B T$ for an ideal gas and

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial}{\partial T}\frac{p}{k_B T}\right)_p = \frac{1}{T} \sim 10^{-2}\text{-}10^{-3}\ \frac{1}{\mathrm{K}}.$$

So then Taylor expanding $\rho(T)$ to get

$$\begin{aligned} \rho(T) &= \rho_0 - \alpha(T_0)\,\rho(T_0)\,(T - T_0) + \ldots \\ \Longrightarrow \frac{\rho(T)}{\rho_0} &= 1 - \alpha(T - T_0) + \cdots \approx 1 \pm \%. \end{aligned}$$

Sometimes those %'s will matter, and in particular for bouyancy-driven flows. Thus we take the "Oberbeck-Boussinesq" approximation – we say the flow is incompressible <u>except</u> for bouancy forces. In particular, we have the following approximations:

1. We model density as
$$\rho = \rho_0 + \rho_0 \alpha (T - T_0),$$
and so for external forces due to gravity we write
$$\mathbf{f}^{\text{ext}} = -\hat{\mathbf{k}}\rho g = -\hat{\mathbf{k}}\rho_0 g - \hat{\mathbf{k}}\rho_0 \alpha g (T - T_0).$$

2. We neglect viscous heating, i.e., we neglect $\Phi$.

To justify the second assumption, recall the internal energy equation

$$\rho c\left(\frac{\partial T}{\partial t}+\mathbf{u}.\nabla T\right)=\rho\frac{d}{dt}E=-p\left(\nabla.\mathbf{u}\right)+\Phi+\nabla.\left(k\nabla T\right).$$

We'll try to identify the (temperature-driven) situations which might be characterized as having $\Phi$ negligible compared to the convective term, $\rho c\mathbf{u}.\nabla T$. Neglecting $\Phi$ should be OK if

$$\mu\frac{U^2}{l^2}=\mu\|\nabla\mathbf{u}\|^2\ll\rho_0 c_0 U\frac{\delta T}{l}.$$

Note that as we already assume flows to be incompressible, the $\lambda$ term in $\Phi$ doesn't enter into this calculation (as $\nabla.\mathbf{u}=0$ from the start). If the flow is driven by bouancy, the momentum equation tells us that

$$\rho_0\frac{U}{t}\sim\rho_0 g\alpha\delta T,$$

or just

$$\frac{U}{\delta T}\sim g\alpha t.$$

Hence,

$$\begin{aligned}\frac{\mu\frac{U^2}{l^2}}{\rho_0 c_0 U\frac{\delta T}{l}} &= \frac{\mu_0}{\rho_0 c_0}\frac{U}{l\delta T}\\ &= \frac{\mu_0}{\rho_0 c_0}\frac{g\alpha t}{l}.\end{aligned}$$

For a typical ideal gas, the first term $\sim 10^{-8}$ sK. Typically, $g=10$ m/s and $\alpha=1/300$ K. So,

$$\frac{\mu\frac{U^2}{l^2}}{\rho_0 c_0 U\frac{\delta T}{l}}\approx\left(10^{-10}\ \frac{s}{m}\right)\cdot\frac{t}{l}.$$

Thus for dynamics with reasonable $t/l$, the second approximation is justified.

# Lecture 11: Boussinesq Approximation, Boundary Conditions, and Rayleigh-Benard Convection

We developed the continuum equations of motion, including Navier-Stokes. The dynamics are given by

$$\begin{aligned}\partial_t\rho + \nabla.(\mathbf{u}\rho) &= 0 \iff \frac{d}{dt}\rho + \rho\nabla.\mathbf{u} \;=\; 0,\\ \rho\left(\partial_t\mathbf{u} + \mathbf{u}.\nabla\mathbf{u}\right) &= \nabla.\sigma + \mathbf{f}^{\text{ext}}.\end{aligned}$$

For Newtonian fluids,

$$\sigma_{ij} = -p\delta_{ij} + \underbrace{\left(\mu\left(u_{i,j} + u_{j,i} - \frac{2}{3}\delta_{ij}u_{k,k}\right) + \lambda\delta_{ij}u_{k,k}\right)}_{\sigma_{ij}^{\text{diss}}}.$$

A usual example of external forces is $\mathbf{f}^{\text{ext}} = -\hat{\mathbf{k}}\rho g$. And we had

$$\rho\frac{d}{dt}E = -\rho\nabla.\mathbf{u} + \Phi + \nabla.(k\nabla T),$$

where $E = cT$ and $\Phi = \sigma_{ij}^{\text{diss}}r_{ij}$. To close it, we need an equation of state

$$p = p(\rho, T).$$

Note that all material parameters $\mu, \lambda, c, k$ are generally functions of $T, \rho$.

## 1 Boussinesq Approximation (ctd.)

We'd like to study a subset of the equations which capture most of the phenomena we might see elsewhere. So we make the approximation:

$$\begin{aligned}\nabla.\mathbf{u} &= 0,\\ \rho &= \text{constant},\\ \mu, \lambda, c, k &= \text{constant}.\end{aligned}$$

Also, we assume that the thermal expansion coefficient

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p > 0$$

is approximately constant:

$$\rho \approx \rho_0 - \alpha\rho_0(T - T_0).$$

Note that once we make these assumptions, the external force (supposing gravity only) can be written as

$$\mathbf{f}^{\text{ext}} = -\hat{\mathbf{k}}\rho g \approx \mathbf{k}\alpha\rho_0 gT - \nabla\left[\rho_0(1 + \alpha T_0)z\right].$$

The second term is the gradient of a hydrostatic pressure. The first is the buoyancy force. Last time, we argued that if the flow is driven by buouancy and if the $l, t$-scales of the flow satisfy $\frac{l}{t} \ll 10^{10}$ m/s, then we can neglect the viscous heating terms in the energy equation.

All of the above constitute the so-called Boussinesq approximation. In this setup, the evolution equations become

$$\begin{aligned}\nabla.\mathbf{u} &= 0,\\ \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla p &= \frac{\mu}{\rho}\triangle\mathbf{u} + \hat{\mathbf{k}}g\alpha T,\\ \partial_t T + \mathbf{u}.\nabla T &= \frac{k}{\rho c}\triangle T.\end{aligned}$$

Note that compressional heating can make a difference in gaseous flow! So there is a "ghost" term here, coming from the term $\rho\nabla.\mathbf{u}$ in the original energy equation. But if we define

$$c = \begin{cases} c_v & \text{liquids} \\ c_p & \text{gases} \end{cases},$$

this we can capture this piece of the dynamics.

What has changed from incompressible Navier-Stokes? The only new term showing up in either the continuity or momentum equations is the term

$$\frac{\mu}{\rho}\triangle\mathbf{u}.$$

And there is a similar term in the energy equation:

$$\frac{k}{\rho c}\triangle T.$$

So we make two new definitions.

**Definition 1.** The *kinematic viscosity* is

$$\nu = \frac{\mu}{\rho} \quad \left[\tfrac{\mathrm{L}^2}{T}\right].$$

The *thermal diffusion coefficient* is

$$\kappa = \frac{k}{\rho c}.$$

Note that there are some inconsistancy in the Boussinesq approximation. For example, the approximation does not strictly satisfy mass or energy conservation. But in these approximations we can (to first order) capture the dynamics of bouancy-driven flows.

There are other applications that take Boussinesq-like approximations. One such application is oceanography. In this setting, we are interested in salinity throughout the fluid,

$$S(\mathbf{x}, t) = \text{concentration of salt}.$$

Then one might take the approximation

$$\begin{aligned}\rho &= \rho_0 + \beta(S - S_0),\\ \mathbf{f}^{\text{ext}} &= -\hat{\mathbf{k}}g\beta S,\end{aligned}$$

and write the evolution equation

$$\frac{d}{dt}S = \kappa_S \triangle S.$$

Here,

$$\frac{\kappa_T}{\kappa_S} = \text{Lewis number} \gg 1$$

in practice.

## 2 Boundary Conditions

As usual, we require

$$\hat{\mathbf{n}}.\,\mathbf{u}|_{\partial\Omega} = 0,$$

i.e., we expect that no flow occurs across the walls.

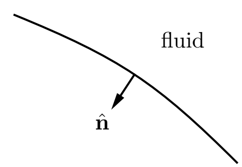



Figure 2.1: Outward pointing normal at $\partial\Omega$.

But now that we have the Laplacian operator to deal with, we need more boundary conditions. A basic modelling assumption is that the fluid near a wall remains stuck to the wall. This is called the "no-slip" boundary condition:

$$\mathbf{u}|_{\partial\Omega} = 0.$$

Note that in many situations, this is empirically valid. But in some applications, the boundary applies negligible shear stress (e.g., the ocean/air boundary, or the liquid-core/mantle boundary). In these cases, one would use the "free-slip" condition, as follows. Consider a boundary with normal component in the negative $z$-direction.

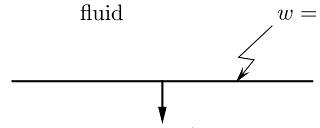



Figure 2.2: Boundary with downward pointing normal.

Write

$$\mathbf{u} = \hat{\mathbf{i}}u + \hat{\mathbf{j}}v + \hat{\mathbf{k}}w$$

and note $w = 0$ at the wall. Free-slip means

$$
\begin{aligned}
0 &= (\hat{\mathbf{n}}.\tau) \times \hat{\mathbf{n}} \\
&= \left[ \begin{pmatrix} 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \partial_x u & \frac{1}{2}(\partial_y u + \partial_x v) & \frac{1}{2}(\partial_z u + \partial_x w) \\ & \partial_y v & \frac{1}{2}(\partial_z v + \partial_y w) \\ & & \partial_z w \end{pmatrix} \right] \times \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2}(\partial_z v + \partial_y w) \\ -\frac{1}{2}(\partial_z u + \partial_x w) \\ 0 \end{pmatrix} \\
&= (\partial_z v, \partial_z u, 0)
\end{aligned}
$$

at the rigid surface, in our setup. In general, free-slip conditions are

$$
\begin{aligned}
\hat{\mathbf{n}}.\, \mathbf{u}|_{\partial U} &= 0, \\
\hat{\mathbf{n}}.\, \nabla \mathbf{u}_{\text{tang}}|_{\partial U} &= 0.
\end{aligned}
$$

So the velocity follows no-slip or stress-free (free-slip) boundary conditions. Temperature is specified on the boundary, or heat flux

$$\hat{\mathbf{n}}.\, \nabla T|_{\partial U}$$

is specified through the boundary.

## 3 Rayleigh-Benard Convection

Fluid sits between two flat plates which are held at different temperatures. Both no-slip and stress-free boundary conditions are interesting to study (or some combination). Everything is assumed to be periodic in $x$ and $y$ with periods $L_x$ and $L_y$.

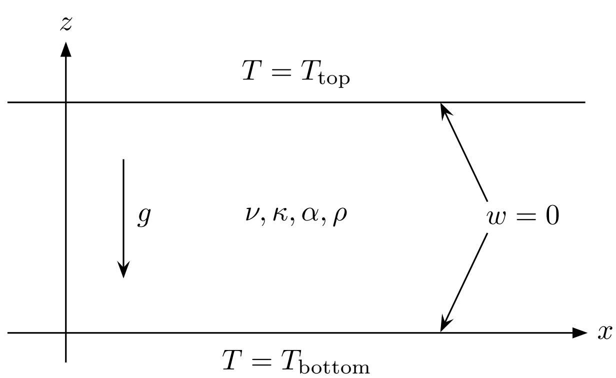



Figure 3.1: Setup for Rayleigh-Benard convection.

For all parameter values, there exists an exact solution:

$$\begin{aligned}
\mathbf{u} &\equiv 0, \\
T &= T_{\text{bot}} + (T_{\text{top}} - T_{\text{bot}}) \cdot \frac{z}{h}, \\
p &= \rho g \alpha \cdot (T_{\text{top}} - T_{\text{bot}}) \cdot \left( z - \frac{1}{2}\frac{z^2}{h} \right).
\end{aligned}$$

This is know as the "conduction" solution. Note that for some choices of parameter values, this can be proven to be the unique long-time solution. And then it is globally stable.

Let's count the parameters in the problem:

$$\rho, \nu, \kappa, \alpha, g, h, L_x, L_y, T_{\text{top}}, T_{\text{bot}}$$

make up 10 parameters. But note that these parameters are not really all indepedent, at least from a mathematical point of view. We can pass to non-dimensional variables. We can rescale time and space (and consequently velocity), temperature, and pressure. In so doing, we can non-dimensionalize the dynamics. So first, define a new time variable by

$$t' = \frac{\kappa t}{h^2},$$

motivated by the physical fact that $\frac{h^2}{\kappa} \approx$ time for heat to diffuse across layer. Also, define

$$\begin{aligned}
(x', y', z') &= \frac{1}{h}(x, y, z, ), \\
T' &= \frac{T}{T_{\text{bot}} - T_{\text{top}}} - \frac{T_{\text{top}}}{T_{\text{bot}} - T_{\text{top}}}, \\
\mathbf{u}' &= \frac{h}{\kappa}\mathbf{u}, \text{ and} \\
p' &= \left(\frac{h}{\kappa}\right)^2 \frac{p}{\rho}.
\end{aligned}$$

Then the problem now looks as follows:

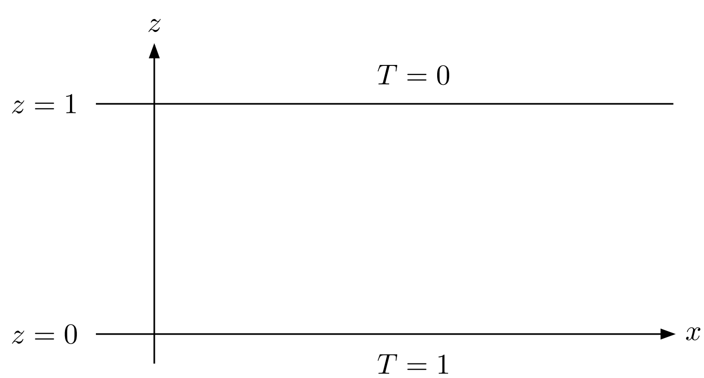



Figure 3.2: Non-dimensionalized Rayleigh-Benard convection.

The equations of motion are

$$\begin{aligned}
\nabla.\mathbf{u} &= 0, \\
(\partial_t + \mathbf{u}.\nabla)\,\mathbf{u} + \nabla p &= \mathrm{Pr}\triangle\mathbf{u} + \hat{\mathbf{k}}\mathrm{Pr}\cdot\mathrm{Ra}\cdot T, \\
(\partial_t + \mathbf{u}.\nabla)\,T &= \triangle T,
\end{aligned}$$

in the new variables. The dimensionless parameters are

$$\frac{L_x}{h}, \frac{L_y}{h}, \mathrm{Pr}, \mathrm{Ra}$$

with the following definitions.

**Definition 2.** The *Prandtl number* is

$$\mathrm{Pr} = \frac{\nu}{\kappa}.$$

The *Rayleigh number* is

$$\mathrm{Ra} = \frac{g\alpha\,(T_{\mathrm{top}} - T_{\mathrm{bot}})\,h^3}{\nu\kappa}.$$

In our new variables, the conductive solution is

$$\begin{aligned}
\mathbf{u} &= 0, \\
T &= 1 - z, \\
p &= \left(z - \frac{1}{2}z^2\right)\cdot\mathrm{Pr}\cdot\mathrm{Ra}.
\end{aligned}$$

Note that for given $\frac{L_x}{h}, \frac{L_y}{h}$, there is a critical $\mathrm{Ra}_c$ so that if $\mathrm{Ra} < \mathrm{Ra}_c$, the conduction solution is absolutely stable. If $\mathrm{Ra} > \mathrm{Ra}_c$, then the conduction solution is unstable. But the Prandtl number does not play a role in the stability, only in the ensuing fluid dynamics.

## Lecture 12: Incompressible Navier-Stokes and Plane Coutte Flow

We added more physics (viscosity) to our fluid model. Then we started to boil down the equations to capture a large variety of dynamics we're interested in. The equations are now:

$$\begin{aligned} \nabla.\mathbf{u} &= 0, \\ \rho\left(\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u}\right) + \nabla p &= \mu\triangle\mathbf{u} + \mathbf{f}^{\text{ext}}. \end{aligned}$$

The dynamics happen in some domain $\Omega$ with a boundary $\partial\Omega$. We wrote down boundary conditions last time – beyond $\hat{\mathbf{u}}.\ \mathbf{u}|_{\partial\Omega} = 0$,

- no-slip: $\mathbf{u}$ specified on $\partial\Omega$;
- stress-free: no stress at the walls.

Note that the incompressibility condition gives

$$0 = \int_\Omega \left(\nabla.\mathbf{u}\right)\, d\mathbf{x} = \int_{\partial\Omega} \left(\mathbf{u}.\hat{\mathbf{n}}\right)\, da,$$

an implicit condition any boundary conditions must satisfy.

## 1 Pressure

As with Euler, the pressure is determiend instantaneously everywhere by $\mathbf{u}$. Taking the divergence of the equations, we arrive at

$$\triangle p = -\mathbf{u}.\left(\mathbf{u}.\nabla\mathbf{u}\right) + \nabla.\mathbf{f}^{\text{ext}}.$$

And the boundary conditions look like

$$\hat{\mathbf{n}}.\ \nabla p|_{\partial\Omega} = \hat{\mathbf{n}}.\ (\text{everything else})|_{\partial\Omega}\,.$$

So in principle, if we're given at time $t = 0$ the velocity

$$\mathbf{u}\left(\mathbf{x}, 0\right) = \mathbf{u}_0\left(\mathbf{x}\right),$$

then (numerically) we'd solve for the pressure and update forward in time.

## 2 Case Study: Plane Couette Flow

Consider fluid between two plates, periodic in the directions tangent to the plates. An exact solution is

$$\begin{aligned} \mathbf{u} &= \hat{\mathbf{i}}U\frac{y}{h}, \\ p &= \text{constant}. \end{aligned}$$

This is known as **plane Couette flow**.

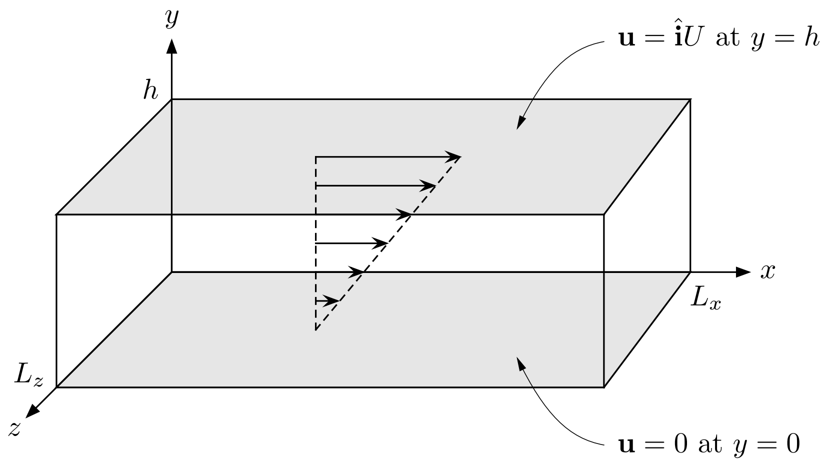



Figure 2.1: Plane Couette flow.

In fact this is how a "Couette viscometer" measures viscosity of a fluid: the instrument measures how much force/torque it takes to maintain steady flow in the given setup. Even though the viscosity does not show up in the above solution, it is intrinsic to the equations. Now we'll consider various properties of this solution.

### 2.1 Vorticity

Recall that $\omega = \nabla \times \mathbf{u}$, so here

$$\omega = -\hat{\mathbf{k}}\frac{U}{h}.$$

The fluid elements are rolling and slipping over each other. (D2)

### 2.2 Shear Stress

By definition,

$$\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \mu\left(u_{i,j} - u_{j,i}\right) \\ \implies \sigma &= -pI + \mu\left(\hat{\mathbf{i}}\hat{\mathbf{j}} + \hat{\mathbf{j}}\hat{\mathbf{i}}\right)\frac{U}{h}, \end{aligned}$$

so we have

$$\sigma = \begin{pmatrix} -p & \mu\frac{U}{h} & 0 \\ \mu\frac{U}{h} & -p & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

So here

$$\mu\frac{U}{h} = \text{vertical flux of horizontal momentum}.$$

What is the force felt by the top plate? The shear stress at the top plate is

$$-\sigma_{12}\big|_{y=h} = -\mu\frac{U}{h}.$$

Consider that the force opposes the velocity of the top plate, which makes sense. So we see that

$$\text{work/time done on fluid} = \left(\mu\frac{U}{h}\right)\cdot A\cdot U = \mu\frac{U^2}{h}L_xL_z,$$

and also

$$\text{force applied} = F = \mu\frac{U}{h}L_xL_z.$$

So given such a flow, the applied force is proportional to the velocity of the top plate. This gives us a way to measure viscosity. But what geometric/material conditions assure that (steady) plane Couette flow appears? Under what conditions is plane Couette flow stable? Before we can answer this, we need to discuss energy.

### 2.3 Energy

Recall that the kinetic energy present in the fluid is

$$K(t) = \int_\Omega \frac{1}{2}\rho\|\mathbf{u}\|^2\,d\mathbf{x}.$$

Supposing the plane Couette geometry and $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$, we have

$$\begin{aligned}\frac{d}{dt}K(t) &= \int_\Omega \rho\mathbf{u}.\frac{\partial}{\partial t}\mathbf{u}\,d\mathbf{x}\\ &= \int_\Omega \rho\mathbf{u}.\left(-\mathbf{u}.\nabla\mathbf{u} - \nabla p + \nu\triangle\mathbf{u}\right)\,d\mathbf{x}.\end{aligned}$$

Let's consider each term here individually. Since the flow is divergence free,

$$\begin{aligned}\int_\Omega \mathbf{u}.\nabla p\,d\mathbf{x} &= \int_\Omega \nabla\left(\mathbf{u}.p\right)\,d\mathbf{x}\\ &= \int_{\partial\Omega} p\mathbf{u}.\hat{\mathbf{n}}\,da\\ &= 0\end{aligned}$$

due to the periodicity. Now the advection term:

$$\begin{aligned}\int_\Omega u_i\left(u_j\partial_j u_i\right)\,d\mathbf{x} &= \int_\Omega u_i\left(\mathbf{u}.\nabla u_i\right)\,d\mathbf{x}\\ &= \int_\Omega \mathbf{u}.\nabla\left(\frac{1}{2}u_iu_i\right)\,d\mathbf{x}\\ &= \int_\Omega \nabla.\left(\mathbf{u}\frac{1}{2}\|\mathbf{u}\|^2\right)\,d\mathbf{x}\\ &= \int_{\partial\Omega}\frac{1}{2}\|\mathbf{u}\|^2\mathbf{u}.\hat{\mathbf{n}}\,da\\ &= 0\end{aligned}$$

for the same reasons. What about the dissapative term? Consider that

$$\begin{aligned}
\int_\Omega \mathbf{u}.\triangle\mathbf{u}\,d\mathbf{x} &= \int_\Omega u_i\triangle u_i\,d\mathbf{x}\\
&= \int_\Omega u_i\nabla.\left(\nabla u_i\right)\,d\mathbf{x}\\
&= \int_\Omega \left[\nabla.\left(u_i\nabla u_i\right)-\left(\nabla u_i\right).\left(\nabla u_i\right)\right]\,d\mathbf{x}\\
&= \int_{\partial\Omega} u_i\hat{\mathbf{n}}.\nabla u_i\,da-\int_\Omega u_{i,j}u_{i,j}\,d\mathbf{x}\\
&= \int_{\text{top}} U\frac{\partial}{\partial y}u\,dxdz-\int_\Omega \|\nabla\mathbf{u}\|^2\,d\mathbf{x}.
\end{aligned}$$

So the rate of change of total kinetic energy is

$$\frac{d}{dt}K\left(t\right)=\rho\nu U\int_{\text{top}}\frac{\partial u}{\partial y}\,dxdz-\rho\nu\int_\Omega\|\nabla\mathbf{u}\|^2\,dxdydz.$$

The first term here is

$$U\int_{\text{top}}\mu\frac{\partial u}{\partial y}\,dxdz=U\int_{\text{top}}\sigma_{12}\,dxdz=U\cdot F,$$

so the first term is the power being put into the flow. The second term is

$$-\int_\Omega \mu\|\nabla\mathbf{u}\|^2\,dxdydz$$

which is negative definite, and is the power dissapated in the flow (by heat).[1]

Let's check this power balance for plane Couette flow:

$$\frac{dK}{dt}=U\cdot\left(\mu\frac{U}{h}\right)L_xL_z-\mu\frac{U^2}{h^2}hL_xL_z=0.$$

Now we can ask if plane Couette flow is stable, and if so, how quickly will arbitrary flows converge to it?

### 2.4 Stability of Plane Couette Flow

Consider starting in the plane Couette geometry with a general initial condition, i.e., $\mathbf{u}\left(\mathbf{x},0\right)=\mathbf{u}_0\left(\mathbf{x}\right)$. Suppose there exists a unique solution $\mathbf{u}\left(\mathbf{x},t\right)$. Let's compare this with the plane Couette solution

$$\mathbf{u}^{\text{PC}}\left(\mathbf{x}\right)=\hat{\mathbf{i}}U\frac{y}{h}.$$

Define

$$\mathbf{v}\left(\mathbf{x},t\right)=\mathbf{u}\left(\mathbf{x},t\right)-\mathbf{u}^{\text{PC}}\left(\mathbf{x}\right).$$

Note that

$$\mathbf{v}|_{y=0}=0=\mathbf{v}|_{y=h}$$

[1] So an experimentalist would worry about removing the extra heat through the plates.

as the solutions match the same boundary conditions. If we can conclude that $\mathbf{v} \to 0$ in some sense, then we can conclude that $\mathbf{u}^{\mathrm{PC}}$ is absolutely stable.

Since $\mathbf{u} = \mathbf{u}^{\mathrm{PC}} + \mathbf{v}$, Navier-stokes tells us that

$$\rho\left(\partial_t\left(\mathbf{u}^{\mathrm{PC}} + \mathbf{v}\right) + \left(\mathbf{u}^{\mathrm{PC}} + \mathbf{v}\right).\nabla\left(\mathbf{u}^{\mathrm{PC}} + \mathbf{v}\right)\right) + \nabla p = \mu\triangle\left(\mathbf{u}^{\mathrm{PC}} + \mathbf{v}\right).$$

Carrying this out, we have

$$\rho\left(\partial_t\mathbf{v} + U\frac{y}{h}\partial_x\mathbf{v} + \mathbf{v}.\left(\hat{\mathbf{j}}\hat{\mathbf{i}}\frac{U}{h}\right) + \mathbf{v}.\nabla\mathbf{v}\right) + \nabla p = \mu\triangle\mathbf{v}.$$

Then dividing by $\rho$ yields

$$\partial_t\mathbf{v} + \mathbf{v}.\nabla\mathbf{v} + \frac{U}{h}\partial_x\mathbf{v} + \hat{\mathbf{i}}v_y\frac{U}{h} + \nabla p = \nu\triangle\mathbf{v},$$

where we've written $\mathbf{v} = \hat{\mathbf{i}}v_x + \hat{\mathbf{j}}v_y + \hat{\mathbf{k}}v_z$. And we have the boundary conditions from above.

Now we make a definition:

**Definition 1.** The *energy in the perturbation* is

$$E(t) = \frac{1}{2}\rho\int_\Omega \|\mathbf{v}(\mathbf{x},t)\|^2\, dxdydz.$$

Note that $E(t) \geq 0$ at all time. If $\mathrm{E}(t) \to 0$ as $t \to 0$, then $\mathbf{v} \to 0$ with respect to $L^2(\Omega)$ and $\mathbf{u} \to \mathbf{u}^{\mathrm{PC}}$ with respect to $L^2(\Omega)$. So we look for conditions yielding this result.

We have

$$\begin{aligned}\frac{d}{dt}E(t) &= \rho\int_\Omega \mathbf{v}.\partial_t\mathbf{v}\, d\mathbf{x} \\ &= \rho\int_\Omega \mathbf{v}.\left(-\mathbf{v}.\nabla\mathbf{v} - \frac{U}{h}\partial_x\mathbf{v} - \hat{\mathbf{i}}v_y\frac{U}{h} - \nabla p + \nu\triangle\mathbf{v}\right) dxdydz.\end{aligned}$$

After distributing (in the order above), the first term and the fourth term are zero exactly by the same reasons as before. What about the second term? Observe that

$$\int \mathbf{v}.\partial_x\mathbf{v} = \int \partial_x\left(\frac{1}{2}\|v\|^2\right) = 0$$

by the fundamental theorem of calculus and the periodic boundary conditions. So we have

$$\begin{aligned}\frac{d}{dt}E(t) &= -\rho\int_\Omega \frac{U}{h}v_xv_y\, dxdydz + \rho\int_\Omega \mathbf{v}.(\nu\triangle\mathbf{v})\, dxdydz \\ &= -\rho\int_\Omega \frac{U}{h}v_xv_y\, dxdydz - \rho\nu\int_\Omega \underbrace{\|\nabla\mathbf{v}\|^2}_{v_{i,j}v_{i,j}} dxdydz.\end{aligned}$$

The second term is $\leq 0$. But the first term is indefinite. So perhaps if the viscosity is large enough, then the second term here dominates the first. Next time we'll derive conditions on $\nu$ to ensure that the energy decreases.

## Lecture 13: Plane Couette Flow (ctd.)

The equations are now

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla\frac{p}{\rho} &= \frac{\mu}{\rho}\triangle\mathbf{u}, \\ \nabla.\mathbf{u} &= 0.\end{aligned}$$

## 1 Plane Couette Flow

Let's recall what we did last time. We were considering the following setup:

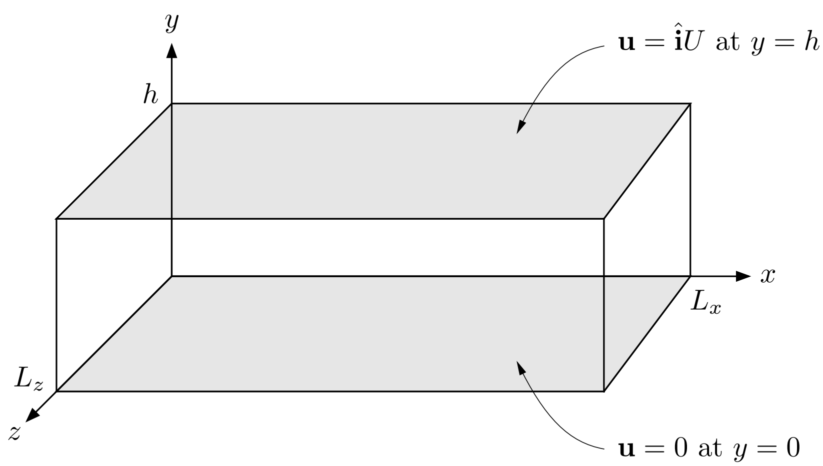



Figure 1.1: Setup for plane Couette flow.

We exhibited a steady solution,

$$\begin{aligned}\mathbf{u}^{\mathrm{PC}} &= \hat{\mathbf{i}}U\frac{y}{h}, \\ p^{\mathrm{PC}} &= \text{constant},\end{aligned}$$

called "plane Couette flow." The force required on the top plate to maintain steady flow is

$$F = L_x L_z \sigma_{21} = \frac{\mu U A}{h}$$

and so requires a power

$$F \cdot U = \frac{\mu U^2 A}{h}.$$

The energy dissipation per unit mass is

$$\epsilon = \frac{\mu U^2 A}{h} \cdot \frac{1}{\rho A h} = \nu\frac{U^2}{h^2}.$$

### 1.1 Stability of Plane Couette

A major question we studied last time concerned the stablity of this solution. So we considered an arbitrary initial condition $\mathbf{u}_0(\mathbf{x})$ and subesquent solution $\mathbf{u}(\mathbf{x},t)$. We defined

$$\mathbf{v}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \mathbf{u}^{\mathrm{PC}}.$$

Setting $\mathbf{u} = \mathbf{u}^{\mathrm{PC}} + \mathbf{v}$ into the equations left us with

$$\begin{aligned} \partial_t \mathbf{v} + \mathbf{v}.\nabla\mathbf{v} + U\frac{y}{h}\partial_x \mathbf{v} + \hat{\mathbf{i}}\frac{U}{h}v_y + \frac{1}{\rho}\nabla p &= \nu\triangle\mathbf{v}, \\ \nabla.\mathbf{v} &= 0. \end{aligned}$$

Similarly we obtained boundary conditions

$$\mathbf{v}|_{y=0} = 0 \text{ and } \mathbf{v}|_{y=h} = 0.$$

To check stability, we considered the evolution of

$$E(t) = \frac{1}{2}\rho\int \|\mathbf{v}(\mathbf{x},t)\|^2\, dxdydz = \frac{1}{2}\rho\|\mathbf{v}(\cdot,t)\|_2^2.$$

We arrived at

$$\frac{d}{dt}E(t) = -\rho\int\left[\nu\|\nabla\mathbf{v}\|^2 + \frac{U}{h}v_x v_y\right] dxdydz.$$

So we ask: what explicit conditions will lead to the integrand being positive? These conditions will yield stability of the plane Couette solution. But first we need a mathematical result.

### 1.2 Poincare's Inequality

**Theorem 1.** *(Poincare) Suppose $f(y)$ is a (smooth enough) function on $y \in [0,h]$ and $f(0) = 0 = f(h)$. Then*

$$\|f\|_2^2 = \int_0^h (f(y))^2\, dy \le \frac{\pi^2}{h^2}\int_0^h (f'(y))^2\, dy = \frac{\pi^2}{h^2}\|f'\|_2^2.$$

*Proof.* Write

$$\begin{aligned} f(y) &= \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi y}{h}\right) \text{ with} \\ f_n &= \frac{2}{h}\int_0^h f(y)\sin\left(\frac{n\pi y}{h}\right) dy. \end{aligned}$$

Recall Parseval's theorem states

$$\|f\|_2^2 = \frac{h}{2}\sum_{n=1}^{\infty} f_n^2.$$

Assuming $f'$ is nice enough to have its own Fourier representation,

$$f'(y) = \sum_{n=1}^{\infty} f_n \frac{n\pi}{h}\cos\left(\frac{n\pi y}{h}\right).$$

Then calculate

$$\begin{aligned}
\|f'\|_2^2 &= \int_0^h \left(f'(y)\right)^2\, dy \\
&= \frac{h}{2}\sum_{n=1}^{\infty}\frac{n^2\pi^2}{h^2}f_n^2 \\
&\geq \frac{h}{2}\sum_{n=1}^{\infty}\frac{\pi^2}{h^2}f_n^2 \\
&= \frac{\pi^2}{h^2}\left(\frac{h}{2}\sum_{n=1}^{\infty}f_n^2\right) \\
&= \frac{\pi^2}{h^2}\|f\|_2^2,
\end{aligned}$$

which is what we wanted to show. □

This can be generalized.

**Theorem 2.** *Suppose $f(\mathbf{x})$ is a function on a domain $\Omega$ satisfying $f|_{\partial\Omega} = 0$. Then*

$$\|f\|_2^2 \leq \lambda_0 \|\nabla f\|_2^2$$

*where $\lambda_0$ is the smallest eigenvalue of $-\triangle$ on $\Omega$ with homogenous Dirichlet boundary conditions.*

In our case we have functions $f(x, y, z)$ which are periodic in $x, z$ but not $y$. So we write

$$f(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{n,m,l}\sin\left(\frac{n\pi y}{h}\right)e^{i2\pi\frac{mx}{L_x}}e^{i2\pi\frac{lz}{L_z}}$$

with $f_{n,m,l}$ being complex in general. Explicitly,

$$f_{n,m,l} = \frac{2}{hL_xL_z}\int_0^{L_x} dx\int_0^h dy\int_0^{L_z} dz\, \sin\left(\frac{n\pi y}{h}\right)e^{-\frac{2\pi imx}{L_x}}e^{-\frac{2\pi ilz}{L_z}}f(x, y, z).$$

By Parseval we have

$$\|f\|_2^2 = \int_0^{L_x} dx\int_0^h dy\int_0^{L_z} dz = \frac{h}{2}L_xL_z\sum_{n=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\left|f_{n,m,l}\right|^2.$$

Here, Poincare's inequality becomes

$$\|\frac{\partial f}{\partial y}\|_2^2 \geq \frac{\pi^2}{h^2}\|f\|_2^2.$$

This analysis will apply to $v_x(x, y, z, t), v_y(x, y, z, t),$ and $v_z(x, y, z, t)$.

### 1.3 Energy of Perturbation

Now we continue tracking the evolution of the perturbation. So write

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) = -\int\left[\nu\|\nabla\mathbf{v}\|^2 + \frac{U}{h}v_x v_y\right].$$

Then by Poincare's inequality,

$$\begin{aligned}
\|\nabla\mathbf{v}\|_2^2 &= \|\frac{\partial v_x}{\partial x}\|_2^2 + \|\frac{\partial v_y}{\partial x}\|_2^2 + \|\frac{\partial v_z}{\partial x}\|_2^2 \\
&\quad + \|\frac{\partial v_x}{\partial y}\|_2^2 + \|\frac{\partial v_y}{\partial y}\|_2^2 + \|\frac{\partial v_z}{\partial y}\|_2^2 \\
&\quad + \|\frac{\partial v_x}{\partial z}\|_2^2 + \|\frac{\partial v_y}{\partial z}\|_2^2 + \|\frac{\partial v_z}{\partial z}\|_2^2 \\
&\geq \|\frac{\partial v_x}{\partial y}\|_2^2 + \|\frac{\partial v_y}{\partial y}\|_2^2 \\
&\geq \frac{\pi^2}{h^2}\|v_x\|_2^2 + \frac{\pi^2}{h^2}\|v_y\|_2^2.
\end{aligned}$$

It's a fact that

$$ab \leq \frac{1}{2c}a^2 + \frac{c}{2}b^2$$

for any $0 < c < \infty$.[1] Therefore,

$$\left|\int v_x v_y\,dxdydz\right| \leq \int\left(\frac{1}{2}v_x^2 + \frac{1}{2}v_y^2\right)\,dxdydz = \frac{1}{2}\int\left(v_x^2 + v_y^2\right).$$

So,

$$\nu\|\nabla\mathbf{v}\|_2^2 \geq \nu\frac{\pi^2}{h^2}\left(\|v_x\|_2^2 + \|v_y\|_2^2\right),$$

and we have arrived at

$$\nu\|\nabla\mathbf{v}\|_2^2 \geq \frac{2\nu\pi^2}{h^2}\left|\int v_x v_y\,dxdydz\right|.$$

Setting this back into our energy estimate, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) &= -\nu\|\nabla\mathbf{v}\|_2^2 - \frac{U}{h}\int v_x v_y \\
&\leq -\nu\|\nabla\mathbf{v}\|_2^2 + \frac{U}{h}\left|\int v_x v_y\right| \\
&\leq -\nu\|\nabla\mathbf{v}\|_2^2 + \frac{U}{h}\cdot\frac{h^2}{2\pi^2}\|\nabla\mathbf{v}\|_2^2
\end{aligned}$$

and thus

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) \leq -\left(\nu - \frac{Uh}{2\pi^2}\right)\|\nabla\mathbf{v}\|_2^2.$$

[1] This follows from noting that $0 \leq \left(\frac{a}{\sqrt{c}} - \sqrt{c}b\right)^2$.

So we have the following result:

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) \le 0$$

if

$$\nu > \frac{Uh}{2\pi^2},$$

i.e., if

$$\frac{Uh}{\nu} < 2\pi^2 \approx 20.$$

**Definition.** The nondimensional parameter

$$\mathrm{Re} = \frac{Uh}{\nu}$$

is called the *Reynolds number.*

*Note.* The Reynolds number is often reffered to as "(inertial forces)/(damping forces)."

We've shown that the energy cannot increase if the Reynolds number is less than $2\pi^2$. Now let's show that this is sufficient to ensure stability. Rearranging things and reapplying Poincare's inequality, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) &\le -\nu\|\nabla\mathbf{v}\|_2^2 + \frac{U}{h}\left|\int v_x v_y\right| \\
&\le -\nu\cdot\frac{\pi^2}{h^2}\|\mathbf{v}\|_2^2 + \frac{U}{h}\frac{1}{2}\left(\|v_x\|_2^2 + \|v_y\|_2^2\right) \\
&\le -\left(\nu\frac{\pi^2}{h^2} - \frac{U}{2h}\right)\|\mathbf{v}\|_2^2.
\end{aligned}$$

Multiplying through by two, we arrive at

$$\frac{d}{dt}\|\mathbf{v}\|_2^2 \le -\underbrace{\left(\frac{2\nu\pi^2}{h^2} - \frac{U}{h}\right)}_{r}\|\mathbf{v}\|_2^2.$$

Note that $r > 0$ when $\mathrm{Re} < 2\pi^2$. To show exponential decay of perturbations, we need one more result – Gronwall's inequality.

**Theorem 3.** *(Gronwall) Given $f(t)$ with $|f(0)| < \infty$ and*

$$\frac{df}{dt} \le -rf,$$

*then*

$$f(t) \le f(0)\,e^{-rt}.$$

*Proof.* Write

$$\begin{aligned}
\frac{df}{dt} &\le -rf \\
\iff \left(\frac{df}{dt} + rf\right) &\le 0 \\
\iff e^{-rt}\frac{d}{dt}\left(e^{rt}f(t)\right) &\le 0
\end{aligned}$$

and conclude

$$e^{rt}f(t) \le f(0),$$

which is what we wanted to show. □

Applying this here, we get

$$\|\mathbf{v}(\cdot,t)\|_2^2 \le \|\mathbf{v}(\cdot,0)\|_2^2 e^{-rt}.$$

So we've proved a theorem.

**Theorem 4.** *Plane Couette flow is absolutely stable (in $L^2$ sense) for*

$$Re = \frac{Uh}{\nu} < 2\pi^2.$$

*Remark.* It's a fact that plane Couette flow is absolutely stable if $\mathrm{Re} < 88$.

How would we obtain the sharper result? Write

$$\begin{aligned}\frac{d}{dt}\|\mathbf{v}\|_2^2 &= -\left(\frac{-2\int\left(\nu\|\nabla\mathbf{v}\|^2 + \frac{U}{h}v_x v_y\right)}{\|\mathbf{v}\|_2^2}\right)\|\mathbf{v}\|_2^2 \\ &\le -\left[\min_{\substack{\nabla.\mathbf{w}=0\\ \mathbf{w}|_{y=0}=0=\mathbf{w}|_{y=h}}} \frac{-2\int\left(\nu\|\nabla\mathbf{w}\|^2 + \frac{U}{h}w_x w_y\right)}{\|\mathbf{w}\|_2^2}\right]\|\mathbf{v}\|_2^2, \qquad (1.1)\end{aligned}$$

then proceed by variational calculus to calculate this minimum (which is just the norm of an operator). The number that results is exactly 88.

### 1.4 Energy Dissipation

We wrote

$$\epsilon = \frac{\text{energy dissipation rate}}{\text{unit mass}}$$

and we had

$$\epsilon^{\mathrm{PC}} = \nu\frac{U^2}{h^2}.$$

As per convention, consider[2]

$$\beta(\mathrm{Re}) = \frac{\bar{\epsilon}h}{U^3} = \text{function of Re}.$$

This quantity is called the "dissipation factor" or "friction factor." (In general, this would be a function of the aspect ratio of the box, but we're suppressing that here.) For plane Couette flow,

$$\beta^{\mathrm{PC}} = \frac{\epsilon^{\mathrm{PC}}h}{U^3} = \frac{\nu U^2}{h^2}\cdot\frac{h}{U^3} = \frac{1}{\mathrm{Re}}.$$

[2] Here, $\overline{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\,dt$.

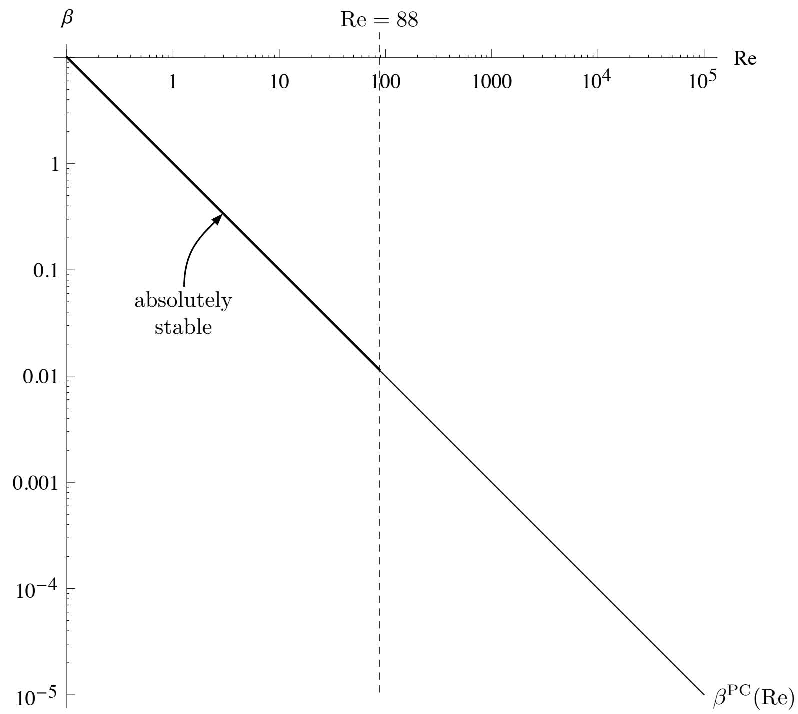



Figure 1.2: $\beta^{\mathrm{PC}}$ vs. Re.

We showed today that if $\mathrm{Re} < 88$, then $\beta \to \beta^{\mathrm{PC}}$ for arbitrary initial conditions. Thus plane Couette flow is absolutely stable for $\mathrm{Re} < 88$. Next time we'll investigate $\mathrm{Re} \geq 88$.

## Lecture 14: Plane Couette Flow (ctd.)

Recall we considered stability of plane Couette flow,

$$\begin{aligned} \mathbf{u}^{\mathrm{PC}} &= \hat{\mathbf{i}} U \frac{y}{h}, \\ p &= \text{constant}. \end{aligned}$$

Taking a perturbation, we considered the energy density of the difference,

$$\epsilon = \frac{\nu}{|\Omega|} \|\nabla \mathbf{v}\|_2^2.$$

We introduced the Reynold's number,

$$\mathrm{Re} = \frac{Uh}{\nu},$$

and wrote

$$\epsilon^{\mathrm{PC}} = \nu \frac{U^2}{h^2} \text{ and } \beta^{\mathrm{PC}} = \frac{1}{\mathrm{Re}}.$$

In our analysis we found plane Couette flow to be absolutely stable for Re < 88. All this is depicted below.

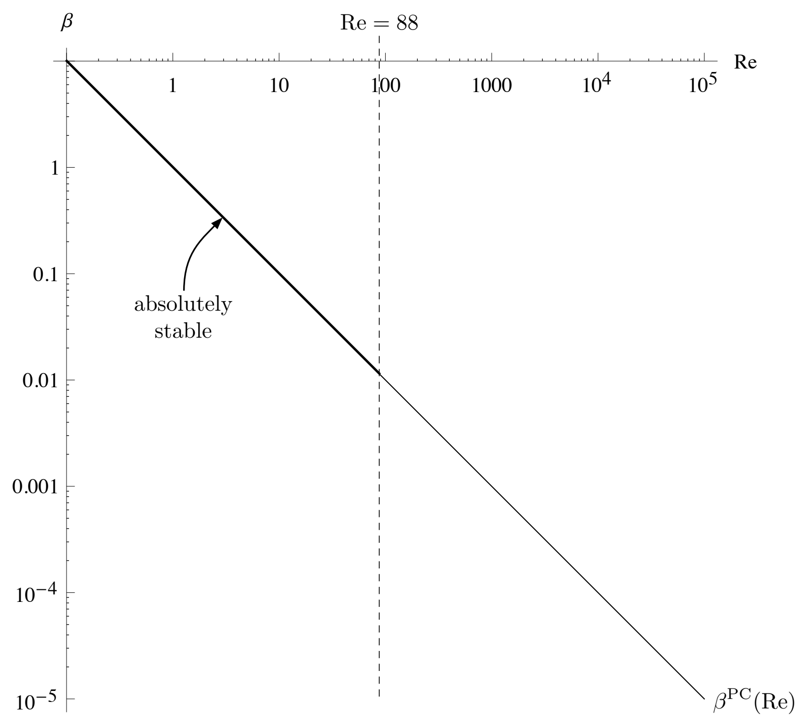



Figure 0.1: $\beta^{\mathrm{PC}}$ vs. Re.

## 1 Energy in the Large

Consider a general solution $\mathbf{u}(\mathbf{x},t)$ starting from $\mathbf{u}_0(\mathbf{x})$. Then consider the energy

$$\epsilon(t) = \frac{\nu}{hL_xL_z}\int \|\nabla\mathbf{u}(x,y,z,t)\|^2\,dxdydz.$$

Let

$$\mathbf{u} = \hat{\mathbf{i}}U\frac{y}{h} + \mathbf{v}(x,y,z,t)$$

be a perturbation to plane Couette flow, where

$$\mathbf{v}|_{y=0} = 0 = \mathbf{v}|_{y=h}\,.$$

Then

$$\begin{aligned} \nabla\mathbf{u} &= \hat{\mathbf{j}}\hat{\mathbf{i}}\frac{U}{h} + \nabla\mathbf{v} \\ \implies \|\nabla\mathbf{u}\|^2 &= \left(\frac{\partial v_x}{x}\right)^2 + \left(\frac{\partial v_y}{x}\right)^2 + \left(\frac{\partial v_z}{x}\right)^2 + \left(\frac{U}{h} + \frac{\partial v_x}{y}\right)^2 + \left(\frac{\partial v_y}{y}\right)^2 + \left(\frac{\partial v_z}{z}\right)^2 + \dots \\ &= \|\nabla\mathbf{v}\|^2 + \frac{U^2}{h^2} + 2\frac{U}{h}\frac{\partial v_x}{\partial y}. \end{aligned}$$

Integrating yields

$$\begin{aligned} \|\nabla\mathbf{v}\|_2^2 &= \|\nabla\mathbf{v}\|_2^2 + hL_xL_z\frac{U^2}{h^2} + 2\frac{U}{h}\int_0^{L_x}dx\int_0^h dy\int_0^{L_z}dz\,\frac{\partial v_x}{y} \\ &= \|\nabla\mathbf{v}\|_2^2 + hL_xL_z\frac{U^2}{h^2}. \end{aligned}$$

So with a perturbation,

$$\epsilon(t) = \nu\frac{U^2}{h^2} + \frac{\nu}{hL_xL_z}\|\nabla\mathbf{v}\|_2^2 \geq \nu\frac{U^2}{h^2},$$

and for long-time averages,

$$\bar{\epsilon} \geq \nu\frac{U^2}{h^2} \implies \beta \geq \nu\frac{U^2}{h^2}\cdot\frac{h}{U^3} = \frac{1}{\mathrm{Re}}.$$

Thus for high Re, $\beta$ is at least as large as $\beta^{\mathrm{PC}} = \mathrm{Re}^{-1}$.

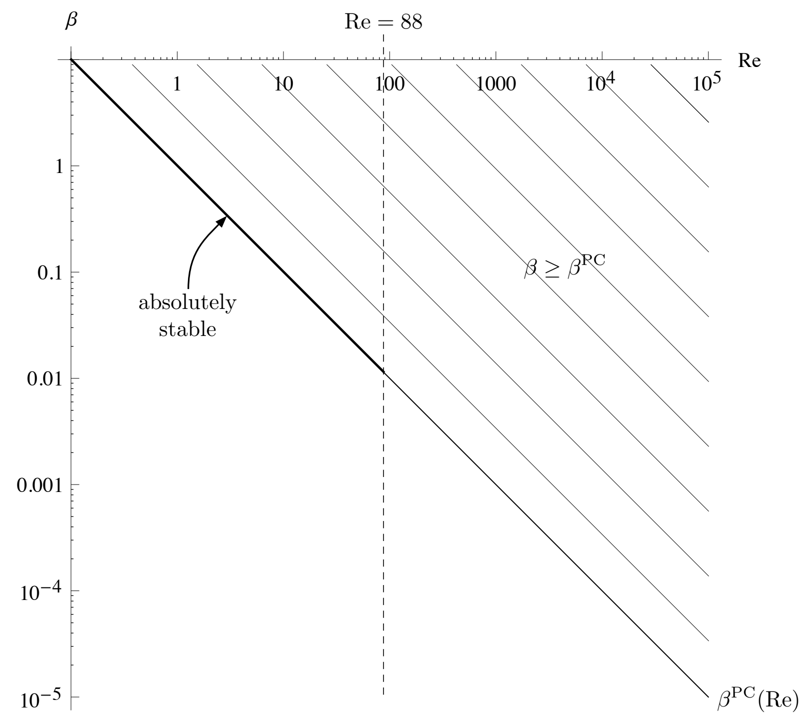



Figure 1.1: $\beta^{\mathrm{PC}}$ vs. Re.

## 2 Stability Analysis

Suppose $\mathbf{u}_0(\mathbf{x})$ is an exact steady solution of Navier-Stokes (or any dynamical system in general).

**Definition 1.** We say that $\mathbf{u}_0(\mathbf{x})$ is *absolutely stable (globally stable)* if starting from any initial condition,

$$\mathbf{u}(\mathbf{x}, t) \to \mathbf{u}_0(\mathbf{x})$$

as $t \to \infty$.

**Definition 2.** We say $\mathbf{u}_0(\mathbf{x})$ is *nonlinearly stable* if there exists $\delta > 0$ so that

$$\|\mathbf{u}(\mathbf{x}, 0) - \mathbf{u}_0(\mathbf{x})\| < \delta \implies \mathbf{u}(\mathbf{x}, t) \to \mathbf{u}_0(\mathbf{x})$$

as $t \to \infty$.

**Definition 3.** We say $\mathbf{u}_0(\mathbf{x})$ is *linearly stable* if infinitesimal perturbations decay.

*Remark.* This last notion of stability is quite weak. So we use it in general to test unstability.

**Definition 4.** We say $\mathbf{u}_0(\mathbf{x})$ is *unstable* if it is not linearly stable.

### 2.1 Plane Parallel Flows

We want to analyze stability of a more general class of solutions to Navier-Stokes. So we make a definition.

**Definition 5.** A *plane-parallel flow* is one in which

$$\mathbf{u}_0(\mathbf{x}) = \hat{\mathbf{i}}U(y).$$

These flows solve

$$\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = \nu\triangle\mathbf{u} + \hat{\mathbf{i}}f(y).$$

Note that at time $t = 0$, we can write

$$0 = \hat{\mathbf{i}}\nu U''(y) + \hat{\mathbf{i}}f(y),$$

so the initial condition arises via

$$U''(y) = -\frac{1}{\nu}f(y).$$

As usual, write

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}) + \mathbf{v}(\mathbf{x},t)$$

with $\mathbf{u}_0 = \hat{\mathbf{i}}U(y)$ and $\mathbf{v}$ "small." So the dynamics become

$$\begin{aligned}\partial_t\mathbf{v} + \mathbf{v}.\nabla\mathbf{v} + \hat{\mathbf{i}}U'(y)v_y + U(y)\partial_x\mathbf{v} + \nabla p &= \nu\triangle\mathbf{v} + \hat{\mathbf{i}}\nu U''(y) + \hat{\mathbf{i}}f(y)\\ &= \nu\triangle\mathbf{v}.\end{aligned}$$

After linearizing (for small $\mathbf{v}$) we have

$$\partial_t\mathbf{v} + \hat{\mathbf{i}}U'(y)v_y + U(y)\partial_x\mathbf{v} + \nabla p = \nu\triangle\mathbf{v},$$

a linear homogenous equation for the perturbation.

Now we look for solutions of the form $\mathbf{v}(\mathbf{x},t) = e^{-\lambda t}\mathbf{v}_\lambda(\mathbf{x})$.[1] Plugging this in yields

$$\begin{aligned}\lambda\mathbf{v}_\lambda(\mathbf{x}) &= -\nu\triangle\mathbf{v}_\lambda + \hat{\mathbf{i}}U'(y)v_{y\lambda} + U(y)\partial_x\mathbf{v}_\lambda + \nabla p_\lambda,\\ \nabla.\mathbf{v}_\lambda(\mathbf{x}) &= 0,\\ \mathbf{v}_\lambda|_{\partial\Omega} &= 0,\\ p(\mathbf{x},t) &= p_\lambda(\mathbf{x})e^{-\lambda t}.\end{aligned}$$

We seek a solution to this system – if any $\Re\{\lambda\} < 0$, we'll conclude linear instability.

W.l.o.g., say $\mathbf{v}_\lambda(\mathbf{x})$ has $\exp(ik_x x + ik_z z)$ $x, z$ dependance,[2] with

$$k_x = \frac{2\pi}{L_x}n \text{ and } k_z = \frac{2\pi}{L_z}m \qquad -\infty < n, m < \infty.$$

So we write

$$\mathbf{v}_\lambda(\mathbf{x}) = \hat{\mathbf{i}}u(y,\lambda)e^{ik_x x + ik_z z} + \hat{\mathbf{j}}v(y,\lambda)e^{ik_x x + ik_z z} + \hat{\mathbf{k}}w(y,\lambda)e^{ik_x x + ik_z z}.$$

[1] Time-translation invariance allows us to take the Laplace transform in time. (??)
[2] Due to spatial invariance (periodicity) in $x, z$.

The divergence-free condition becomes

$$0 = ik_x u(y) + v'(y) + ik_z w(y),$$

and the pressure satisfies

$$0 = ik_x U'(y) v + U'(y) ik_x v + \left(-k_x^2 + \frac{\partial^2}{\partial y^2} - k_z^2\right) p.$$

The pressure is thus determined by

$$\left(-\frac{d^2}{dy^2} + k_x^2 + k_y^2\right) p = 2ik_x U'(y) v.$$

The components of the perturbation satisfy the linearized equations:

$$\begin{aligned}
0 &= \lambda u + \nu\left(-k_x^2 + \frac{d^2}{dy^2} - k_z^2\right) u - U'(y) v - U(y) ik_x u - ik_x p && (2.1)\\
0 &= \lambda v + \nu\left(-k_x^2 + \frac{d^2}{dy^2} - k_z^2\right) v - U(y) ik_x v - \frac{dp}{dy} && (2.2)\\
0 &= \lambda w + \nu\left(-k_x^2 + \frac{d^2}{dy^2} - k_z^2\right) w - U(y) ik_x w - ik_z p. && (2.3)
\end{aligned}$$

Writing the pressure equation similarly, we have

$$0 = 2ik_x U'(y) v + \left(\frac{d^2}{dy^2} - k_x^2 - k_z^2\right) p. \tag{2.4}$$

Currently we have a three-dimensional system with a divergence-free constraint. Next we aim to turn this into a two-dimensional system with a divergence-free constraint. This will allow us to use a stream-function approach.

## 2.2 Squire's Transformation

Define $k^2 = k_x^2 + k_z^2$, $\tilde{u} = \frac{k_x u + k_z w}{k}$, and $\tilde{p} = \frac{kp}{k_x}$. Here we assume $k_x \neq 0$ (the dynamics are simplified already if $k_x = 0$). Then enforcing $\frac{k_x \cdot (2.1) + k_z \cdot (2.3)}{k}$ yields

$$0 = \lambda \tilde{u} + \nu\left(\frac{d^2}{dy^2} - k^2\right)\tilde{u} - \frac{k_x}{k} U'(y) v - ik_x U(y) \tilde{u} - ik_x \tilde{p}.$$

Multiplying through by $k/k_x$ gives

$$0 = \lambda\left(\frac{k}{k_x}\right)\tilde{u} + \nu\left(\frac{d^2}{dy^2} - k^2\right)\left(\frac{k}{k_x}\right)\tilde{u} - U'(y) v - ikU(y)\tilde{u} - ik\tilde{p}.$$

So now we have equations for $\frac{k}{k_x}\tilde{u}, v, \tilde{p}$.

We want to renormalize our equations. By rearranging the previous equation and by multiplying (2.2) and (2.4) by $k/k_x$, we arrive at the system

$$\begin{aligned}
0 &= \left(\lambda \frac{k}{k_x}\right) \tilde{u}(y) + \left(\nu \frac{k}{k_x}\right) \left(\frac{d^2}{dy^2} - k^2\right) \tilde{u}(y) - U'(y)\, v(y) - ikU(y)\, \tilde{u}(y) - ik\tilde{p}(y) \\
0 &= \left(\lambda \frac{k}{k_x}\right) v(y) + \left(\nu \frac{k}{k_x}\right) \left(\frac{d^2}{dy^2} - k^2\right) v(y) - ikU(y)\, v(y) - \frac{d\tilde{p}}{dy} \\
0 &= 2ikU'(y)\, v + \left(\frac{d^2}{dy^2} - k^2\right) \tilde{p}(y).
\end{aligned}$$

We can relabel as $\tilde{\lambda} = \lambda k/k_x$ and $\tilde{\nu} = \nu k/k_x$. We have now recast the dynamics in terms of $\tilde{u}(y), v(y), \tilde{p}(y)$. We have now a two-dimensional problem with a pressure induced by the divergence-free condition. Note that we can write the divergence-free condition in the new variables:

$$\begin{aligned}
0 &= ik_x u + \frac{dv}{dy} + ik_z w \\
\Longrightarrow 0 &= ik\tilde{u} + \frac{dv}{dy}.
\end{aligned}$$

As we expect, this looks like a two-dimensional condition.

Now suppose there exists $\tilde{\nu}_c$ so that $\tilde{\nu} < \tilde{\nu}_c \implies \Re(\tilde{\nu}) < 0$. Then as $\tilde{\nu} = \nu k/k_x > \nu$, we immediately conclude $\Re(\nu) < 0$. So then there will exist $\nu_c$ so that $\nu < \nu_c$ yields instability. And the three-dimensional instability will be worse than the two-dimensional instability!

Next time we'll eliminate the pressure and take a stream-function approach.

## Lecture 15: Plane Couette Stability – Reduction to 1-D and Non-Normal Growth

Recall what we did last time – we considered stability of plane parallel flows, which satisfy

$$\begin{aligned} \nabla.\mathbf{u} &= 0, \\ \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p &= \nu\triangle\mathbf{u} + \mathbf{f}, \end{aligned}$$

and which arise via

$$\begin{aligned} \mathbf{u}_0 &= \hat{\mathbf{i}}U(y), \\ p &= \text{constant}. \end{aligned}$$

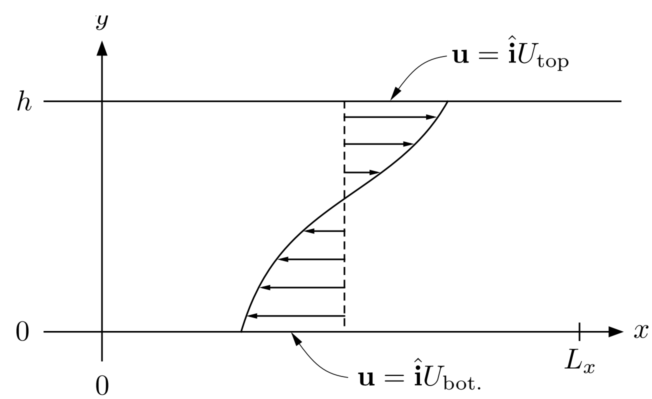



Figure 0.1: Plane parallel flow.

Note that the initial condition is determined by the forcing term, via

$$\nu U''(y) + f(y) = 0$$

subjected to

$$U(0) = U_{\text{bot}} \text{ and } U(h) = U_{\text{top}}.$$

To determine stability (rather, instability) of the plane parallel flows, we look for linear instability. We set

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0 + \mathbf{v}(\mathbf{x},t)$$

and after linearizing the dynamics for $\mathbf{v}$ we get

$$\partial_t \mathbf{v} + \mathbf{v}.\nabla\mathbf{u}_0 + \mathbf{u}_0.\mathbf{v} + \nabla p = \nu\triangle\mathbf{v}.$$

We looked for solutions of the form

$$\mathbf{v} = \left(\hat{\mathbf{i}}u(y) + \hat{\mathbf{j}}v(y) + \hat{\mathbf{k}}w(y)\right)e^{-\lambda t}e^{ik_x x}e^{ik_z z}$$

with

$$k_x = \frac{2\pi n}{L_x} \text{ and } k_z = \frac{2\pi m}{L_z} \qquad -\infty < n, m < \infty.$$

This gave us also

$$p = p(y)\,e^{-\lambda t}e^{ik_x x}e^{ik_z z}.$$

## 1 Linear Stability of Plane Parallel Flows

We continue with the equations

$$
\begin{aligned}
-\lambda u + U'(y)\, v + ik_x U(y)\, u + ik_x p &= \nu\left(-k_x^2 + \frac{d^2}{d^2} - kz^2\right) u && (1.1)\\
-\lambda v + ik_x U(y)\, v + \frac{d}{dy} p &= \nu\left(-k_x^2 + \frac{d^2}{dy^2} - kz^2\right) v && (1.2)\\
-\lambda w + ik_x U(y)\, w + ik_z p &= \nu\left(-k_x^2 + \frac{d^2}{dy^2} - kz^2\right) w && (1.3)\\
ik_x u + \frac{dv}{dy} + ik_z w &= 0
\end{aligned}
$$

subject to

$$
\begin{aligned}
u(0) = {} & 0 = u(h)\\
v(0) = {} & 0 = v(h)\\
w(0) = {} & 0 = w(h).
\end{aligned}
$$

First, consider $k_x = 0 = k_z$. Then the equations become

$$
\begin{aligned}
-\lambda u + U'(y)\, v &= \nu \frac{d^2}{dy^2} u\\
-\lambda v + \frac{d}{dy} p &= \nu \frac{d^2}{dy^2} v\\
-\lambda w &= \nu \frac{d^2}{dy^2} w\\
ik_x u + \frac{dv}{dy} + ik_z w &= 0,
\end{aligned}
$$

and hence the solution is

$$
u(y) = \sin \frac{n\pi y}{h} \qquad n = 1, 2, 3, \ldots.
$$

So here,

$$
\lambda = \frac{n^2 \pi^2 \nu}{h^2} > 0
$$

and there is no instability.

Next consider the case $k_x = 0$ but $k_z \neq 0$. Then the equations become

$$
\begin{aligned}
-\lambda u + U'(y)\, v &= \nu\left(\frac{d^2}{dy^2} - k_z^2\right) u\\
-\lambda v + \frac{dp}{dy} &= \nu\left(\frac{d^2}{dy^2} - k_z^2\right) v\\
-\lambda w + ik_z p &= \nu\left(\frac{d^2}{dy^2} - k_z^2\right) w\\
\frac{dv}{dy} + ik_z w &= 0.
\end{aligned}
$$

If $w = 0$, then $v = \text{constant} = 0$, so

$$\nu \frac{d^2}{dy^2} u = \left(\nu k_z^2 - \lambda\right) u$$

and hence

$$\lambda = \frac{\nu l^2 \pi^2}{h^2} + k_z^2 > 0 \qquad l = 1, 2, \ldots.$$

If $w \neq 0$, take $p = \text{constant} = 0$ w.l.o.g., then

$$\nu \frac{d^2}{dy^2} w = \left(\nu k_z^2 - \lambda\right) w$$

which also gives $\lambda > 0$.

Next, we consider the most general case.

## 2 Reduction of Dynamics to 1-D

Consider $k_x \neq 0$ ($k_z$ arbitrary)? W.l.o.g., take $k_x > 0$. Then we use Squire's tranformation. So we define

$$\begin{aligned} \tilde{u} &= \frac{k_x u + k_z w}{k} \text{ with } k = \sqrt{k_x^2 + k_z^2} \\ \tilde{p} &= \frac{k}{k_x} p, \end{aligned}$$

then we take $\frac{k_x x(1.1) + k_z x(1.3)}{k}$ then multiply the result by $\left(\frac{k}{k_x}\right)$ to get

$$-\lambda \left(\frac{k}{k_x}\right) \tilde{u} + U'(y) v + ikU(y) \tilde{u} + ik\tilde{p} = \left(\nu \frac{k}{k_x}\right) \left(\frac{d^2}{dy^2} - k^2\right) \tilde{u} \tag{2.1}$$

and we multiply (1.3) by $\left(\frac{k}{k_x}\right)$ as well to get

$$-\lambda \left(\frac{k}{k_x}\right) v + ikU(y) v + \frac{d\tilde{p}}{dy} = \left(\nu \frac{k}{k_x}\right) \left(\frac{d^2}{dy^2} - k^2\right) v. \tag{2.2}$$

The incompressibility condition is now

$$0 = ik\tilde{u} + \frac{dv}{dy}.$$

So we've reduced to a two-dimensional system. Let's now introduce a stream-function and reduce to a one-dimensional system.

Define a stream-function $\psi$ by

$$ik\psi(y) = v(y),$$

which implies

$$\tilde{u} = -\frac{d\psi}{dy} = -\psi'(y)$$

by incompressibility. And write now $\tilde{\lambda} = \lambda \frac{k}{k_x}$ and $\tilde{\nu} = \nu \frac{k}{k_x}$. Taking $\frac{d}{dy}(2.1) - ik(2.2)$ yields

$$-\tilde{\lambda}\left(-\psi'' + k^2\psi\right) + ikU''\psi + ikU\left(y\right)\left(-\psi'' + k^2\psi\right) = \tilde{\nu}\left(\frac{d^2}{dy^2} - k^2\right)\left(-\psi'' + k^2\psi\right)$$

subject to the boundary conditions

$$\begin{aligned} \psi\left(0\right) &= 0 = \psi\left(h\right) \text{ and} \\ \psi'\left(0\right) &= 0 = \psi'\left(h\right). \end{aligned}$$

This is known as the **Orr-Sommerfeld equation**.

For plane Couette flows, we can simplify this equation even further. For these flows, we have

$$U\left(y\right) = U\frac{y}{h}$$

so

$$U'\left(y\right) = \frac{U}{h} \text{ and } U''\left(y\right) = 0,$$

and so the equation becomes

$$\begin{aligned} -\tilde{\lambda}f\left(y\right) + \frac{ikU}{h}yf\left(y\right) &= \tilde{\nu}\left(\frac{d^2}{dy^2} - k^2\right)f\left(y\right) \text{ with} \\ f\left(y\right) &= -\psi''\left(y\right) + \psi\left(y\right). \end{aligned}$$

This is a complex version of the **Airy equation**. In 1973, Ramonov proved that $\Re\left(\tilde{\lambda}\right) > 0$ for all $\tilde{\lambda}$. So there is no linear instability for plane Couette flows! But how does this translate back to the non-linear case?

## 3 Non-Normal Growth of Solutions to ODE

**Example 1.** Consider the system

$$\begin{aligned} \dot{X}\left(t\right) &= -X + Y + \ldots \\ \dot{Y}\left(t\right) &= -\alpha Y + \ldots \end{aligned}$$

with $\alpha > 0$. There is a unique fixed point at $(X, Y) = (0, 0)$. Look for a solution $e^{-\lambda t}$ and investigate time dependence. Doing so, we arrive at the eigenvalue/eigenvector problem

$$-\lambda\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}.$$

There is a solution if

$$0 = \det\begin{pmatrix} -1+\lambda & 1 \\ 0 & -\alpha+\lambda \end{pmatrix} = \left(\lambda - 1\right)\left(\lambda - \alpha\right),$$

so

$$\begin{aligned} \lambda &= 1 > 0 \text{ and} \\ \lambda &= \alpha > 0. \end{aligned}$$

The eigenvectors are

$$\lambda = 1 : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and}$$
$$\lambda = \alpha : \begin{pmatrix} 1 \\ 1-\alpha \end{pmatrix}.$$

Supposing $0 < \alpha < 1$, we have the following flow:

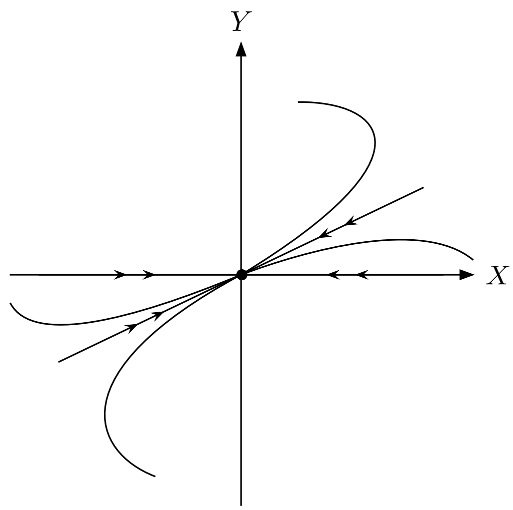



Figure 3.1: Phase portrait with eigendirections.

The general solution is not hard to write down:

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-\alpha t}\begin{pmatrix} 1 \\ 1-\alpha \end{pmatrix}.$$

So if we define an "energy" by $E(t) = \frac{1}{2}\left(X(t)^2 + Y(t)^2\right)$, we see it evolves as

$$\begin{aligned} \frac{d}{dt}E &= X\dot{X} + Y\dot{Y} \\ &= X(-X+Y) + Y(-\alpha Y) \\ &= -X^2 + XY - \alpha Y^2 \\ &= -\left[\frac{2\left(X^2 - XY + \alpha Y^2\right)}{X^2+Y^2}\right]E. \end{aligned}$$

Then we may write

$$\frac{dE}{dt} \leq -2\left[\min_{\tilde{X},\tilde{Y}} \frac{\tilde{X}^2 - \tilde{X}\tilde{Y} + \alpha\tilde{Y}^2}{\tilde{X}^2+\tilde{Y}^2}\right]E(t).$$

If this max is positive, then Grenwall's inequality tells us solutions decay immediately. (??) Define $\mu$ to be this maximum, then we can write

$$\mu = \min_{\tilde{X}^2+\tilde{Y}^2=1} \begin{pmatrix} \tilde{X} & \tilde{Y} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \alpha \end{pmatrix}}_{M} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix},$$

so $\mu$ is the smallest eigenvalue of $M$. The eigenvalues are

$$\mu_{\pm} = \frac{1}{2}\left(1 + \alpha \pm \sqrt{\alpha^2 - 2\alpha + 2}\right),$$

and in particular when $\alpha = \frac{1}{4}$, $\mu = 0$. As $\alpha \to 0$, $\mu_- < 0$, so $\mu$ becomes negative. So for small enough $\alpha$, the energy in the perturbation can grow before it decays.

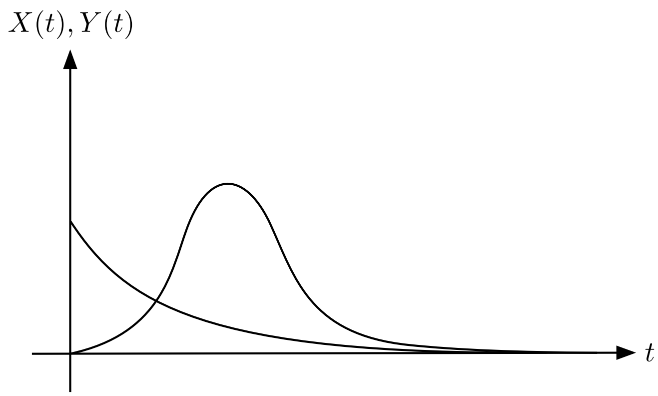



Figure 3.2: Non-normal growth.

If this were a non-linear problem, and if the energy grows too quickly, then the non-linear solution could become unstable (even though the linear problem is stable). This is known as "non-normal growth."

We'll take the same approach for plane Couette flow. Now,

$$\begin{aligned} \partial_t \mathbf{v} + \hat{\mathbf{i}}\frac{U}{h}v_y + U\frac{y}{h}\partial_x \mathbf{v} + \nabla p &= \nu \triangle \mathbf{v}, \\ \nabla . \mathbf{v} &= 0. \end{aligned}$$

Let's look for a solution with $\partial_x \equiv 0$. So write

$$\mathbf{v} = \hat{\mathbf{i}} v_x(y,z,t) + \hat{\mathbf{j}} v_y(y,z,t) + \hat{\mathbf{k}} v_z(y,z,t),$$

and enforce

$$0 = \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

In this case,

$$\begin{aligned}
\partial_t v_x + \frac{U}{h} v_y &= \nu \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_x \\
\partial_t v_y + \frac{\partial p}{\partial y} &= \nu \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_y \\
\partial_t v_z + \frac{\partial p}{\partial z} &= \nu \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_z .
\end{aligned}$$

Now let's look for a solution of the form

$$\begin{aligned}
v_x &= V_x (t) \sin (kz) \sin \left( \frac{\pi y}{h} \right) \\
v_y &= V_y (t) \sin (kz) \sin \left( \frac{\pi y}{h} \right) \\
v_z &= V_z (t) \cos (kz) \cos \left( \frac{\pi y}{h} \right) .
\end{aligned}$$

*Note.* We're cheating a little here! The $z$-equation certainly does not satisfy the correct boundary conditions. This is for illustrative purposes; if we were to require the correct boundary conditions, we'd have sinh's and cosh's instead. So a similar analysis applies.

Continuing on, we write $V_z (t) = \frac{\pi}{hk} V_y (t)$ and then we have the system

$$\begin{aligned}
\dot{V}_x + \frac{U}{h} V_y &= -\nu \left( \frac{\pi^2}{h^2} + k^2 \right) V_x \\
\dot{V}_y &= -\nu \left( \frac{\pi^2}{h^2} + k^2 \right) V_y .
\end{aligned}$$

A particular solution having $V_x (0) = 0$ and $V_y (0) \neq 0$ is

$$\begin{aligned}
V_y (t) &= V_y (0) e^{-s} \\
V_x (t) &= -\gamma V_y (0) s e^{-s} ,
\end{aligned}$$

where

$$\begin{aligned}
s &= \nu \left( \frac{\pi^2}{h^2} + k^2 \right) t, \\
\gamma &= \frac{\mathrm{Re}}{\pi^2 + h^2 k^2}, \text{ and} \\
\mathrm{Re} &= \frac{Uh}{\nu} .
\end{aligned}$$

What is the energy here? We have

$$E (t) = \frac{1}{2} \left( V_x^2 + V_y^2 \right) = \frac{1}{2} \left( \gamma^2 s^2 + 1 \right) e^{-s} \left( V_y (0) \right)^2 .$$

Just as in the example, we have the following diagram:

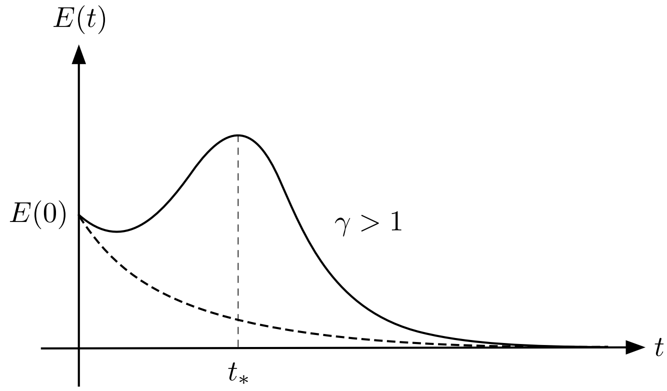



Figure 3.3: Non-normal growth in plane Couette flow.

The critical value $t_*$ comes from

$$s_* = 1 + \sqrt{1 - \frac{1}{\gamma^2}}.$$

And the maximum energy is

$$E_{\max} = E\left(0\right)\gamma^2\left(1 + \sqrt{1 - \frac{1}{\gamma^2}}\right)e^{-1-\sqrt{1-\frac{1}{\gamma^2}}} \to 2E\left(0\right)e^{-2}\gamma^2$$

asymptotically as $\gamma \to \infty$. Observe that $\gamma \sim \mathrm{Re}$, so $E_{\max} \sim \mathrm{Re}^2$. In practice, it's quite easy to obtain very high Reynold's numbers. And small perturbations in those flows can lead to instabilities and turbulence, even though the linear problem is stable.

**Example 2.** As a first step towards pde, and towards understanding the result we just saw, consider the following system:

$$\begin{aligned} \dot{X}_1 &= -X_1 + aX_2 \\ \dot{X}_2 &= -X_2 + aX_3 \\ \dot{X}_3 &= -X_3 + aX_4 \\ \dot{X}_4 &= -X_4 + \dots \\ &\vdots \end{aligned}$$

As a "matrix" equation, this is

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} -1 & a & 0 & 0 & 0 & \cdots \\ 0 & -1 & a & 0 & 0 & \cdots \\ 0 & 0 & -1 & a & 0 & \cdots \\ 0 & 0 & 0 & -1 & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \end{pmatrix}.$$

All eigenvalues are $-1$ (specifically, any truncation has all eigenvalues $-1$). A general solution is

$$X_n(t) = e^{-t} \sum_{m=0}^{\infty} X_{n+m}(0) \frac{(at)^m}{m!}.$$

Suppose we take $X_n(0) = r^m$ with $r < 1$, then

$$X_n(t) = r^n e^{(ra-1)t}$$

if $a > \frac{1}{r}$. This is incredibly degenerate non-normal growth!

## Lecture 16: Plane Couette Flow (ctd.)

Recall we had been studying plane Couette flow,

$$\begin{aligned}\mathbf{u}^{\mathrm{PC}} &= \hat{\mathbf{i}}\frac{Uy}{h},\\ p^{\mathrm{PC}} &= \text{constant},\end{aligned}$$

a solution to incompressible Navier-Stokes in the following setup:

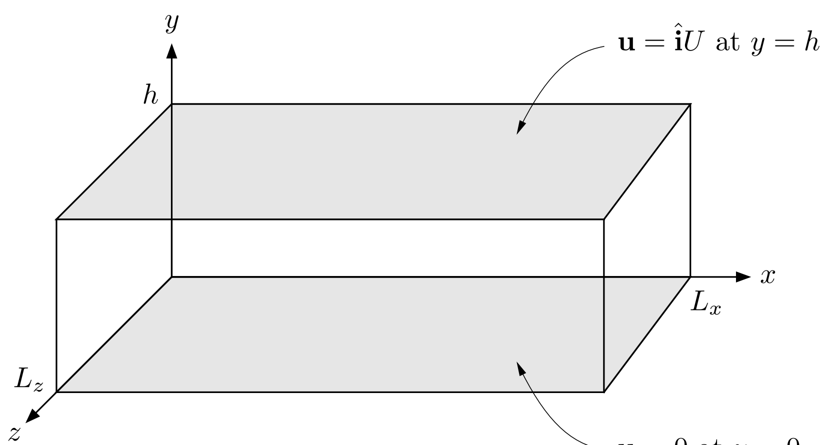



Figure 0.1: Setup for plane Couette flow.

We defined the energy dissipation rate as a space-time average, i.e.,

$$\epsilon = \left\langle \nu \|\nabla \mathbf{u}\|^2 \right\rangle.$$

In the plane Couette case,

$$\begin{aligned}\epsilon^{\mathrm{PC}} &= \gamma\frac{U^2}{h^2} \text{ and}\\ \beta^{\mathrm{PC}} &= \frac{\epsilon^{\mathrm{PC}} h}{U^3} = \frac{\nu}{Uh} = \frac{1}{\mathrm{Re}}.\end{aligned}$$

We found that plane Couette flow is the unique long-time solution in this setup for sufficiently small Re; we also showed $\beta \geq \beta^{\mathrm{PC}}$ in general. All this is indicated on the following $\beta - \mathrm{Re}$ diagram.

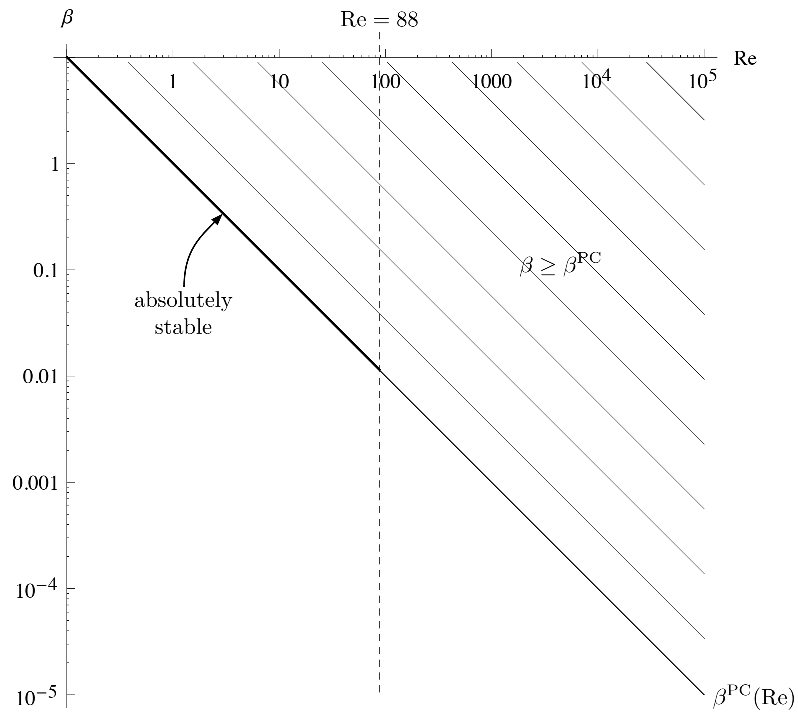



Figure 0.2: $\beta$ vs. Re.

We'll continue to investigate this diagram today.

## 1 An Upper Bound on Dissipation

Again, we have the Navier-Stokes equations with the plane Couette geometry/boundary conditions,

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \nabla p &= \nu \triangle \mathbf{u} \\ \nabla . \mathbf{u} &= 0. \end{aligned}$$

We'll proceed as did E. Hopf in 1941: decompose the flow as

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{i}} \Upsilon(y) + \mathbf{v}(x, t)$$

where $\Upsilon$ is a "background flow" which satisfies the inhomogenous boundary conditions,

$$\Upsilon(0) = 0 \text{ and } \Upsilon(h) = U.$$

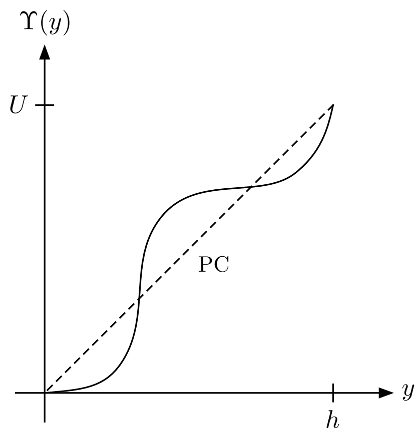



Figure 1.1: E. Hopf's background flow.

So then

$$\mathbf{v}(\mathbf{x},t)|_{y=0} = 0 = \mathbf{v}(\mathbf{x},t)|_{y=h}\,.$$

Having done this, we find

$$\begin{aligned} \nabla.\mathbf{v} &= 0 \\ \partial_t \mathbf{v} + \mathbf{v}.\nabla\mathbf{v} + \underbrace{\mathbf{v}.\nabla\left(\hat{\mathbf{i}}\Upsilon(y)\right)}_{\hat{\mathbf{i}}\Upsilon'(y)v_y} + \underbrace{\hat{\mathbf{i}}\Upsilon(y).\nabla\mathbf{v}}_{\Upsilon(y)\partial_x\mathbf{v}} + \nabla p &= \nu\triangle\mathbf{v} + \hat{\mathbf{i}}\nu\Upsilon''(y)\,. \end{aligned}$$

Consider the kinetic energy in $\mathbf{v}$. So dot the second relation above with $\mathbf{v}$ and integrate over the space, which yields

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) + \underbrace{\int_{\text{box}} \mathbf{v}.(\nabla\mathbf{v}).\mathbf{v}\,d\mathbf{x}}_{=0} + \int_{\text{box}} \Upsilon'(y)\,v_x v_y\,d\mathbf{x} & \\ + \underbrace{\int_{\text{box}} \Upsilon(y)\,\mathbf{v}.(\partial_x\mathbf{v})\;d\mathbf{x}}_{=0} + \underbrace{\int_{\text{box}} \mathbf{v}.\nabla p\,d\mathbf{x}}_{=0} &= -\nu\|\nabla\mathbf{v}\|_2^2 + \nu\int_{\text{box}} v_x\Upsilon''(y)\;d\mathbf{x}. \end{aligned}$$

We saw why the terms which dissapear evaluate to zero in previous lectures. By integrating by parts on the last term we arrive at

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) + \int \Upsilon'(y)\,v_x v_y\,d\mathbf{x} = -\nu\|\nabla\mathbf{v}\|_2^2 - \nu\int \Upsilon'(y)\,\frac{\partial v_x}{\partial y}\,d\mathbf{x}. \tag{1.1}$$

Now observe that

$$\nabla\mathbf{u} = \hat{\mathbf{j}}\hat{\mathbf{i}}\Upsilon'(y) + \nabla\mathbf{v},$$

so then

$$\|\nabla\mathbf{u}\|_2^2 = L_x L_z \int_0^h \left(\Upsilon'(y)\right)^2\,dy + 2\int \Upsilon'(y)\,\frac{\partial v_x}{\partial_y}\,d\mathbf{x} + \|\nabla\mathbf{v}\|_2^2.$$

Multiplying this relation by $\nu/2$, we get

$$\frac{\nu}{2}\|\nabla \mathbf{u}\|_2^2 = \frac{1}{2}L_x L_z \int_0^h \nu\left(\Upsilon'(y)\right)^2\,dy + \frac{\nu}{2}\|\nabla \mathbf{v}\|_2^2 + \nu\int \Upsilon'(y)\,\frac{\partial v_x}{\partial_y}\,d\mathbf{x}. \tag{1.2}$$

Adding (1.1) and (1.2) yields

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) + \frac{\nu}{2}\|\nabla \mathbf{u}\|_2^2 = \frac{1}{2}L_x L_z \nu \int_0^h \left(\Upsilon'(y)\right)^2\,dy - \frac{1}{2}\nu\|\nabla \mathbf{v}\|_2^2 - \int \Upsilon'(y)\,v_x v_y\,d\mathbf{x}.$$

Then we multiply by two and write

$$\frac{d}{dt}\|\mathbf{v}\|_2^2 + \nu\|\nabla \mathbf{u}\|_2^2 = L_x L_z \nu \int_0^h \left(\Upsilon'(y)\right)^2\,dy - \int \left[\nu\|\nabla \mathbf{v}\|_2^2 + 2\Upsilon'(y)\,v_x v_y\right]\,d\mathbf{x}.$$

We have yet to choose $\Upsilon$. Our goal is the following: Choose $\Upsilon(y)$ so that $\exists\ c > 0$ with

$$\int \left[\nu\|\nabla \mathbf{v}\|_2^2 + 2\Upsilon'(y)\,v_x v_y\right]\,d\mathbf{x} \geq c\|\mathbf{v}\|_2^2$$

for all divergence free $\mathbf{v}$ which also satisfy the homogeneous boundary conditions.

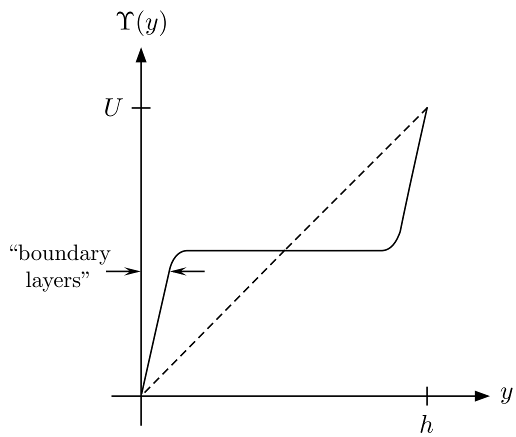



Figure 1.2: A candidate $\Upsilon$.

If we can choose such an $\Upsilon$, then we could write

$$\frac{d}{dt}\|\mathbf{v}\|_2^2 \leq \frac{d}{dt}\|\mathbf{v}\|_2^2 + \nu\|\nabla \mathbf{u}\|_2^2 \leq \underbrace{L_x L_z \int_0^h \left(\Upsilon'(y)\right)^2\,dy}_{A} - c\|\mathbf{v}\|_2^2.$$

Call $f(t) = \|\mathbf{v}(\cdot, t)\|_2^2$, then we have

$$\frac{df}{dt} \leq A - c\cdot f(t)\,.$$

By the general Gronwall's inequality (proved in the homework) we conclude

$$f(t) \leq f(0)\, e^{-ct} + A\frac{\left(1-e^{-ct}\right)}{c}.$$

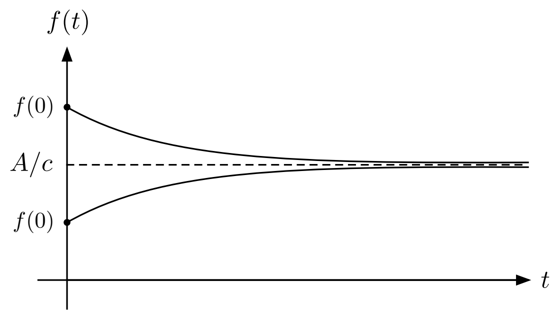



Figure 1.3: Energy bound on $f(t) = \|\mathbf{v}(\cdot,t)\|_2^2$.

So if such an $\Upsilon$ exists, then we'll have found that the energy in the perturbation is uniformly bounded in long time. And then we can do the following. Take the time average ($\frac{1}{T}\int_0^T [\cdot]\, dt$) of the differential inequality

$$\frac{d}{dt}\|\mathbf{v}\|_2^2 + \nu\|\nabla\mathbf{u}\|_2^2 \leq L_x L_z \int_0^h \left(\Upsilon'(y)\right)^2 \, dy - c\|\mathbf{v}\|_2^2 \leq L_x L_z \int_0^h \left(\Upsilon'(y)\right)^2 \, dy$$

to get

$$\frac{1}{T}\|\mathbf{v}(\cdot,T)\|_2^2 - \frac{1}{T}\|\mathbf{v}(\cdot,0)\|_2^2 + \frac{1}{T}\int_0^T \nu\|\nabla\mathbf{u}(\cdot,t)\|_2^2\, dt \leq L_x L_z \int_0^h \left(\Upsilon'(y)\right)^2 \, dy.$$

Sending $T \to \infty$ and dividing the result by $L_x L_z h$ we conclude

$$\epsilon \leq \frac{1}{h}\int_0^h \nu\left(\Upsilon'(y)\right)^2 \, dy,$$

thereby bounding the energy dissipation rate. This all hinges on finding an acceptable $\Upsilon$, which we'll do now.

So we want to see that

$$\int \left(\nu\|\nabla\mathbf{v}\|^2 + 2\Upsilon'(y)\, v_x v_y\right) \, d\mathbf{x} \geq c\|\mathbf{v}\|_2^2$$

for some $c > 0$. Try

$$\Upsilon(y) = \begin{cases} \frac{Uy}{2\delta} & 0 \leq y \leq \delta \\ \frac{U}{2} & \delta \leq y \leq h-\delta \\ \frac{U(y-h)}{2\delta} & h-\delta \leq y \leq h \end{cases}.$$

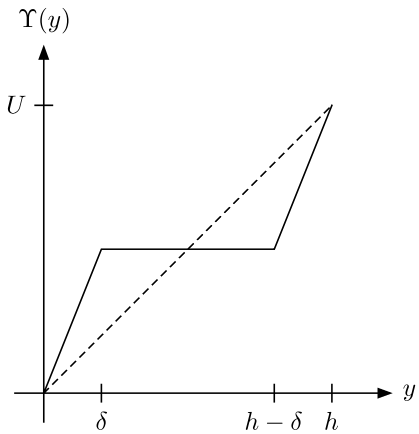



Figure 1.4: One-parameter family of $\Upsilon$'s.

Note that

$$\Upsilon'(y) = \begin{cases} \frac{U}{2\delta} & 0 \le y < \delta \\ 0 & \delta < y < h-\delta \\ \frac{U}{2\delta} & h-\delta < y \le h \end{cases},$$

so

$$\int_0^h \left(\Upsilon'(y)\right)^2\, dy = \frac{U^2}{4\delta^2}\cdot\delta + \frac{U^2}{4\delta^2}\cdot\delta = \frac{U^2}{2\delta}.$$

We'd like to get $\delta$ as small as possible to produce as sharp as possible a bound on $\epsilon$. First we need a lemma.

**Lemma.** *(Howard) Suppose $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(0)=0=f(L)$ and*

$$\|f'\|_2^2 = \int_0^L \left(f'(x)\right)^2\, dx < \infty.$$

*Then,*

$$|f(x)|^2 \le x(L-x)\frac{1}{L}\|f'\|_2^2.$$

*Proof.* By the fundamental theorem of calculus and the Cauchy-Schwarz inequality, we may write

$$\begin{aligned} |f(x)| &= \left|\int_0^x 1\cdot f'(\xi)\, d\xi\right| \\ &\le \underbrace{\left(\int_0^x 1^2\, d\xi\right)^{\frac{1}{2}}}_{\sqrt{x}} \left(\int_0^x \left(f'(\xi)\right)^2\, d\xi\right)^{\frac{1}{2}}, \end{aligned}$$

and hence

$$\frac{|f(x)|^2}{x} \le \int_0^x \left(f'(\xi)\right)^2\, d\xi.$$

On the other hand,

$$\begin{aligned} |f(x)| &= \left| \int_x^L f'(\xi)\, d\xi \right| \\ &\leq (L-x)^{\frac{1}{2}} \left( \int_x^L \left( f'(\xi)^2\, d\xi \right) \right), \end{aligned}$$

and hence

$$\frac{|f(x)|^2}{L-x} \leq \int_x^L (f'(\xi))^2\, d\xi.$$

Adding these up we get

$$|f(x)|^2 \left( \frac{1}{x} + \frac{1}{L-x} \right) \leq \|f'\|_2^2,$$

or equivalently

$$|f(x)|^2 \left( \frac{L}{x(L-x)} \right) \leq \|f'\|_2^2.$$

Hence the result. □

Now we'll use the result. First write (as in the proof)

$$\begin{aligned} v_x(x,y,z) &= \int_0^y \frac{\partial v_x}{\partial y}(x,\eta,z)\, d\eta \\ &\leq \sqrt{y} \left( \int_0^\delta \left( \frac{\partial v_x}{\partial y}(x,\eta,z) \right)^2 d\eta \right)^{\frac{1}{2}} \end{aligned}$$

for $0 \leq y \leq \delta$. A similar expression holds for $v_y$, and hence

$$|v_x v_y(x,y,z)| \leq y \left( \int_0^\delta \left( \frac{\partial v_x}{\partial y} \right)^2 d\eta \right)^{\frac{1}{2}} \left( \int_0^\delta \left( \frac{\partial v_y}{\partial y} \right)^2 d\eta \right)^{\frac{1}{2}}.$$

Now recall that

$$ab \leq \frac{1}{2} \left( \frac{1}{\phi} a^2 + \phi b^2 \right),$$

so

$$|v_x v_y(x,y,z)| \leq \frac{y}{2} \left[ \phi \int_0^\delta \left( \frac{\partial v_x}{\partial y} \right)^2 d\eta + \frac{1}{\phi} \int_0^\delta \left( \frac{\partial v_y}{\partial y} \right)^2 d\eta \right].$$

We'll finish this next time.

## Lecture 17: Plane Couette Flow and RANS

Recall we were in the midst of a calculation which would give an upper bound on the dissipation rate in the flow.

# 1 An Upper Bound on Dissipation (ctd.)

We decomposed the flow into the background flow $\Upsilon(y)$ and the rest:

$$\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{i}}\Upsilon(y) + \mathbf{v}(\mathbf{x},t)$$

with

$$\nabla.\mathbf{v} = 0$$

and subject to boundary conditions

$$\mathbf{v}|_{y=0} = 0 = \mathbf{v}|_{y=h}.$$

At this point, we know that if we can choose $\Upsilon(y)$ so that

$$\int_{\text{box}} \left(\nu\|\nabla\mathbf{v}\|^2 + 2\Upsilon'(y)\, v_x v_y\right)\, d\mathbf{x} \geq c\|\mathbf{v}\|_2^2 \tag{1.1}$$

for some $c > 0$ and for all divergence-free vector fields $\mathbf{v}$ satisfying the homogeneous boundary conditions, then we can conclude that the energy in $\mathbf{v}$ (and the total energy) is bounded for all time, and that

$$\epsilon = \left\langle \nu\|\nabla\mathbf{u}\|^2 \right\rangle \leq \frac{1}{h}\int_0^h \nu\left(\Upsilon'(y)\right)^2\, dy.$$

We proposed trying $\Upsilon$ of the form

$$\Upsilon(y) = \begin{cases} \frac{Uy}{2\delta} & 0 \leq y \leq \delta \\ \frac{U}{2} & \delta \leq y \leq h-\delta \\ \frac{U(y-h)}{2\delta} & h-\delta \leq y \leq h \end{cases}$$

so the background flow looks as follows.

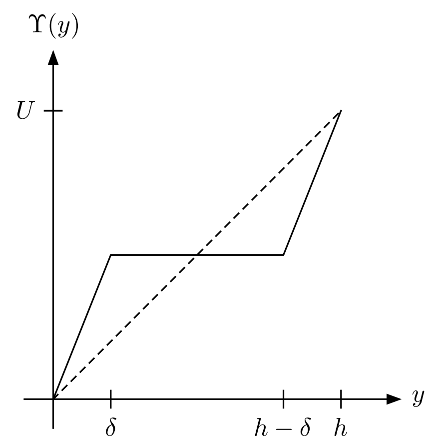



Figure 1.1: The proposed $\Upsilon$.

In this case, if the above requirement holds for some $\delta \in (0, h/2]$, then

$$\epsilon \le \nu \frac{U^2 h}{2\delta}.$$

We'll pick up the analysis here today.

So write

$$\begin{aligned}
|v_x(x,y,z)| &= \left| \int_0^h 1 \cdot \frac{\partial v_x}{\partial y}(x,\eta,z)\, d\eta \right| \\
&\le \left( \int_0^y 1^2\, d\eta \right)^{\frac{1}{2}} \left( \int_0^y \left( \frac{\partial v_x}{\partial y}(x,\eta,z) \right)^2 d\eta \right)^{\frac{1}{2}} \\
&= \sqrt{y} \left( \int_0^y \left( \frac{\partial v_x}{\partial y}(x,\eta,z) \right)^2 d\eta \right)^{\frac{1}{2}}
\end{aligned}$$

and similarly for $v_y$. Then

$$\begin{aligned}
|v_x(x,y,z)\, v_y(x,y,z)| &\le y \left( \int_0^y \left( \frac{\partial v_x}{\partial y} \right)^2 d\eta \right)^{\frac{1}{2}} \left( \int_0^y \left( \frac{\partial v_y}{\partial y} \right)^2 d\eta \right)^{\frac{1}{2}} \\
&\le \frac{y}{2} \left[ \frac{1}{\phi} \int_0^y \left( \frac{\partial v_x}{\partial y} \right)^2 d\eta + \phi \int_0^y \left( \frac{\partial v_y}{\partial y} \right)^2 d\eta \right].
\end{aligned}$$

Now

$$\begin{aligned}
\left| 2 \int\int\int_0^h \Upsilon'(y)\, v_x v_y\, dydzdx \right| &\le 2\left( \frac{U}{2\delta} \right) \left[ \int\int\int_0^\delta |v_x v_y|\, dydzdx + \int\int\int_{h-\delta}^h |v_x v_y|\, dydzdx \right] \\
&\le \frac{U}{\delta} \left[ \frac{\delta^2}{4} \left( \frac{1}{\phi} \int\int\int_0^\delta \left( \frac{\partial v_x}{\partial y} \right)^2 dydzdx + \phi \int\int\int_0^\delta \left( \frac{\partial v_y}{\partial y} \right)^2 dydzdx \right.\right. \\
&\quad \left.\left. + \frac{1}{\phi} \int\int\int_{h-\delta}^h \left( \frac{\partial v_x}{\partial y} \right)^2 dydzdx + \phi \int\int\int_{h-\delta}^h \left( \frac{\partial v_y}{\partial y} \right)^2 dydzdx \right) \right] \\
&\le \frac{U\delta}{4} \left( \frac{1}{\phi} \left\| \frac{\partial v_x}{\partial y} \right\|_2^2 + \phi \left\| \frac{\partial v_y}{\partial y} \right\|_2^2 \right) \\
&\le \frac{U\delta}{4\phi} \left( \left\| \frac{\partial v_x}{\partial y} \right\|_2^2 + \phi^2 \left\| \frac{\partial v_y}{\partial y} \right\|_2^2 \right).
\end{aligned}$$

We showed in homework that if

$$\begin{aligned}
\nabla . \mathbf{v} &= 0 \text{ and} \\
\mathbf{v}|_{y=0} &= 0 = \mathbf{v}|_{y=h}
\end{aligned}$$

then

$$\left\| \frac{\partial v_x}{\partial y} \right\|_2^2 + 2 \left\| \frac{\partial v_y}{\partial y} \right\|_2^2 \le \|\nabla \mathbf{v}\|_2^2.$$

Now we take $\phi = \sqrt{2}$ to acheive

$$\left| \int \int \int_0^h 2\Upsilon'(y)\, v_x v_y \, dydzdx \right| \leq \frac{U\delta}{4\sqrt{2}} \|\nabla \mathbf{v}\|_2^2$$

and thus

$$\begin{aligned} \int_{\text{box}} \left( \nu \|\nabla \mathbf{v}\|^2 + 2\Upsilon'(y)\, v_x v_y \right) \, d\mathbf{x} &\geq \nu \|\nabla \mathbf{v}\|_2^2 - \frac{U\delta}{4\sqrt{2}} \|\nabla \mathbf{v}\|_2^2 \\ &= \left( \nu - \frac{U\delta}{4\sqrt{2}} \right) \|\nabla \mathbf{v}\|_2^2 \\ &\geq \nu \left( 1 - \frac{U\delta}{4\sqrt{2}\nu} \right) \frac{\pi^2}{h^2} \|\mathbf{v}\|_2^2 \end{aligned}$$

by Poincare's inequality. So pick any $\delta$ satisfying

$$\delta \leq \frac{4\sqrt{2}\nu}{U}.$$

We have shown that for any $\delta$ satisfying the above requirement,

$$\begin{aligned} \epsilon &\leq \frac{1}{h} \int_0^h \nu \left( \Upsilon'(y) \right)^2 \, dy \\ &= \nu \left( \frac{U}{2\delta} \right)^2 \frac{2\delta}{h} \\ &= \frac{\nu U^2}{2h\delta}. \end{aligned}$$

Once we take

$$\delta = \frac{4\sqrt{2}\nu}{U},$$

we have acheived

$$\epsilon \leq \frac{\nu U^2 U}{2h4\sqrt{2}\nu} = \frac{1}{8\sqrt{2}} \frac{U^3}{h}$$

and hence

$$\beta = \frac{\epsilon h}{U^3} \leq \frac{1}{8\sqrt{2}} \approx 0.088.$$

*Note.* The above analysis required $\delta \in (0, h/2]$, so the resulting bound only holds if

$$\frac{4\sqrt{2}\nu}{U} \leq \frac{h}{2},$$

i.e., if

$$\mathrm{Re} \geq 8\sqrt{2} \approx 11.$$

Thus the upper bound is consistent with $\beta \geq \mathrm{Re}^{-1}$.

We have proved a general theorem.

**Theorem.** *In the plane Couette flow setup,*

$$\beta \geq \frac{1}{Re}.$$

*If also* $Re \geq 8\sqrt{2}$*, then*

$$\beta \leq \frac{1}{8\sqrt{2}}.$$

*Remark.* Suppose we had chosen $\delta = h/2$ instead, i.e., $\Upsilon(y) = u_y^{\mathrm{PC}}$. Then so long as $h/2 \leq 4\sqrt{2}\nu/U$, i.e., $\mathrm{Re} \leq 8\sqrt{2}$, the above calculations would yield $\beta \leq \mathrm{Re}^{-1}$. Motivated by this, one could compare the problem posed at the start of this lecture (equation (1.1)) with the variational problem posed in Lecture 13 (equation (1.1) in said lecture).

This was a quick calculation with a one parameter family of $\Upsilon$'s which were easy to work with. One could pose a variational problem to optimize the shape of $\Upsilon$ and solve for the associated optimal bound computationally. With this approach it was recently shown that

$$\beta \leq 0.011.$$

The $\beta - \mathrm{Re}$ diagram below summarizes everything we've discussed regarding plane Couette flow.

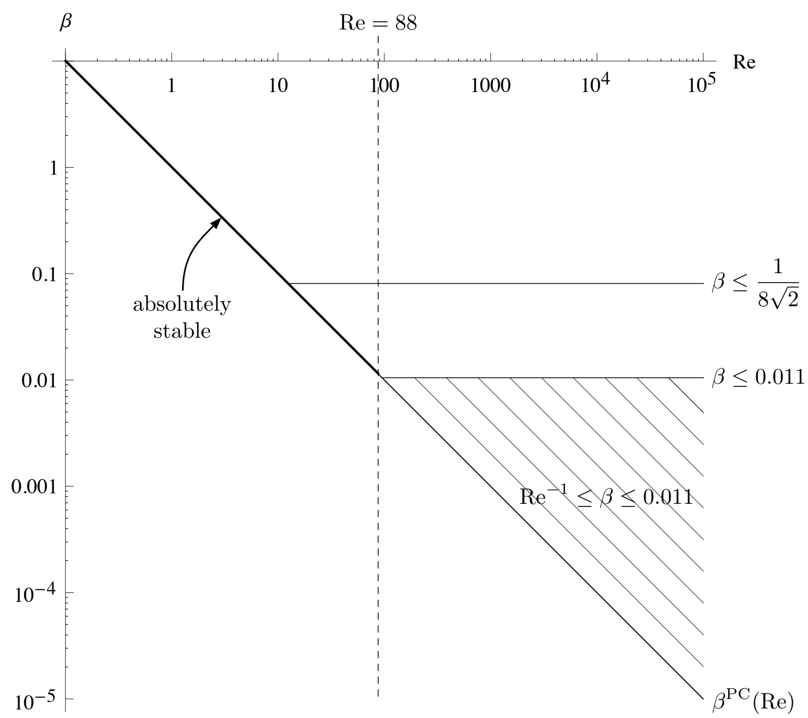



Figure 1.2: $\beta$ vs. Re.

This is as far as we can go with analysis alone. For large enough Re, experimental data fits to a curve

$$\frac{\epsilon h}{U^3} \sim \frac{1}{(\log \mathrm{Re})^2}.$$

Next we'll explore a classical theory of turbulence which is capable of predicting such a law.

## 2 Turbulence: Reynold's Averaged Navier-Stokes (RANS)

At very large Reynolds number, turbulent flows are observed. Tracer particle seem to move about randomly. Even so,

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \nabla p &= \nu \triangle \mathbf{u}, \\ \nabla.\mathbf{u} &= 0. \end{aligned}$$

The idea here is to understand

$$\overline{\mathbf{u}}(x,y,z) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{u}(x,y,z,t)\ dt,$$

a long time average. So we aim to derive differential equations for average quantities.

What do we know so far? In the plane Couette setup,

$$\overline{\mathbf{u}}(x,0,z) = 0 = \overline{\mathbf{u}}(x,h,z).$$

And time-averaging the divergence-free condition yields

$$\nabla.\overline{\mathbf{u}} = 0.$$

We have left the momentum equation; time-averaging yields

$$\overline{\partial_t \mathbf{u}} + \overline{\mathbf{u}.\nabla \mathbf{u}} + \overline{\nabla p} = \overline{\nu \triangle \mathbf{u}}.$$

Let's look at each term.

First, we have

$$\begin{aligned} \overline{\partial_t \mathbf{u}} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\partial u}{\partial t}(x,y,z,t)\ dt \\ &= \lim_{T\to\infty} \frac{1}{T}\left(u(x,y,z,T) - u(x,y,z,t)\right) \\ &= 0 \end{aligned}$$

so long as $u$ is bounded pointwise. This is the first modeling assumption. So we arrive at the **RANS** equations:

$$\begin{aligned} \partial_j\left(\overline{u_i u_j}\right) + \partial_i \overline{p} &= \nu \triangle \overline{u_i}, \\ \partial_i \overline{u_i} &= 0. \end{aligned}$$

Note that we've used the fact that

$$(\mathbf{u}.\nabla\mathbf{u})_i = u_j\partial_j u_i = \partial_j\left(u_j u_i\right)$$

via the divergence-free condition.

Now write

$$\mathbf{u}\left(x,y,z,t\right) = \overline{\mathbf{u}}\left(x,y,z\right) + \mathbf{v}\left(x,y,z,t\right)$$

with $\mathbf{v}$ the fluctuations about the mean flow. Then

$$\overline{\mathbf{v}} = 0,$$

which says that the mean fluctuation about the mean is zero, and

$$\nabla.\mathbf{v} = 0$$

and

$$\left.\mathbf{v}\right|_{y=0} = 0 = \left.\mathbf{v}\right|_{y=h}.$$

If we plug this into the full Navier-Stokes and average, we see

$$\overline{\left(\overline{\mathbf{u}}+\mathbf{v}\right).\nabla\left(\overline{\mathbf{u}}+\mathbf{v}\right)} + \nabla\overline{p} = \nu\overline{\triangle\left(\overline{\mathbf{u}}+\mathbf{v}\right)}$$

and thus

$$\overline{\mathbf{u}}.\nabla\overline{\mathbf{u}} + \underbrace{\overline{\overline{\mathbf{u}}.\nabla\mathbf{v}}}_{=0} + \underbrace{\overline{\mathbf{v}.\nabla\overline{\mathbf{u}}}}_{=0} + \overline{\mathbf{v}.\nabla\mathbf{v}} + \nabla\overline{p} = \nu\triangle\overline{\mathbf{u}} + \underbrace{\nu\triangle\overline{\mathbf{v}}}_{=0}.$$

So we arrive at

$$\begin{aligned}\overline{\mathbf{u}}.\nabla\overline{\mathbf{u}} + \nabla\overline{p} &= \nu\triangle\overline{\mathbf{u}} - \underbrace{\nabla.\left(\overline{\mathbf{v}\mathbf{v}}\right)}_{\partial_j\left(\overline{v_j v_i}\right)}, \\ \nabla.\overline{\mathbf{u}} &= 0.\end{aligned}$$

This is steady Navier-Stokes with an additional term, $-\overline{\mathbf{v}\mathbf{v}}$. So we make a definition.

**Definition.** The *Reynolds stress tensor* is

$$\sigma^{\text{Reynolds}} = -\overline{\mathbf{v}\mathbf{v}}.$$

In components,

$$\sigma_{ij} = -\overline{v_i v_j}.$$

With this we can write

$$\overline{\mathbf{u}}.\nabla\overline{\mathbf{u}} + \nabla\overline{p} = \nu\triangle\overline{\mathbf{u}} + \nabla.\sigma^{\text{Reynolds}}.$$

Recall that

$$\nu\triangle\mathbf{u} = \nabla.\sigma^{\text{viscous}}.$$

So now it looks like we have an additional forcing term, due to the Reynolds stress. Recall that viscous stresses arise due to random motion of molecules across planes. The Reynolds stress arises here due to chaotic motion of fluid elements across planes. Next time we'll examine a model for the Reynold's stress.

## Lecture 18: RANS, Turbulent Fluctuations, and Some Modeling Assumptions

We have been talking about turbulence in the context of plane Couette flow. The following $\beta$-Re diagram depicts three important flow regimes: laminar flow, transition to turbulence, and fully turbulent flow.

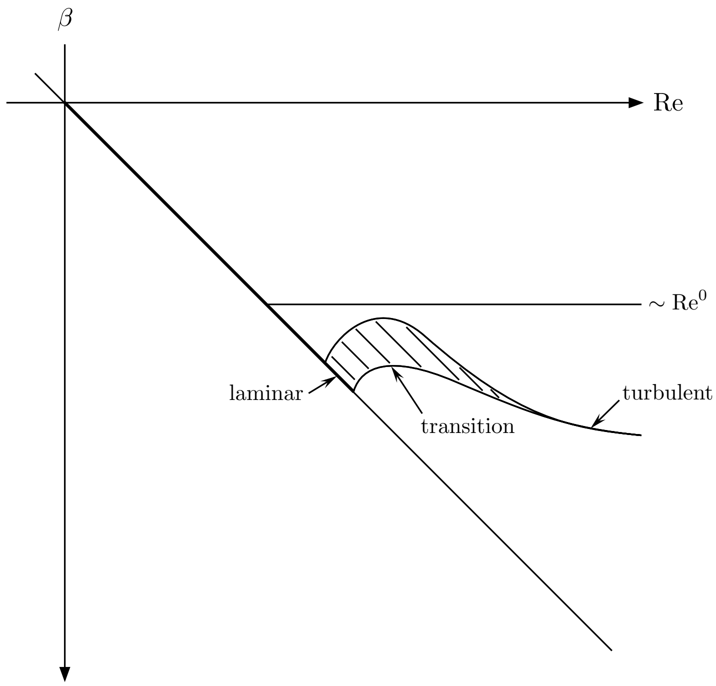



Figure 0.1: $\beta$ vs. Re.

The transition regime is (purposefully) badly defined in the figure. In reality, it depends on many factors and changes shape/size based on these. To understand the turbulent regime, we started discussing RANS.

So we considered

$$\begin{aligned} \overline{\mathbf{u}}(\mathbf{x}) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{u}(\mathbf{x},t)\, dt \\ &= \hat{\mathbf{i}}\overline{u} + \hat{\mathbf{j}}\overline{v} + \hat{\mathbf{k}}\overline{w}. \end{aligned}$$

For plane Couette flow,

$$\rho \int_{\text{top}} \nu \frac{\partial \overline{u}}{\partial y}\, dxdz = \overline{F},$$

and then we have

$$\epsilon = \frac{\overline{F}U}{\rho L_x L_z h}.$$

So the averaging procedure here is quite natural.

As we saw last time, averaging the Navier-Stokes equations yields

$$\overline{\mathbf{u}.\nabla.\mathbf{u}} + \nabla\overline{p} = \nu\triangle\overline{\mathbf{u}}.$$

Then one sets

$$\mathbf{u}(\mathbf{x},t) = \overline{\mathbf{u}}(\mathbf{x}) + \mathbf{v}(\mathbf{x},t)$$

with

$$\begin{aligned}\nabla\mathbf{v} &= 0 \text{ and}\\ \overline{\mathbf{v}} &= 0.\end{aligned}$$

Setting this into the averaged Navier-Stokes equations and averaging again yields the RANS equations:

$$\overline{\mathbf{u}}.\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{p}} = \nu\triangle\overline{\mathbf{u}} + \nabla\sigma^{\text{Rey}},$$

with

$$\sigma^{\text{Rey}} = -\overline{\mathbf{v}\mathbf{v}}$$

being the Reynolds stress. So

$$\sigma_{ij}^{\text{Rey}} = -\overline{v_i v_j}.$$

Note that

$$\begin{aligned}\|\nabla\mathbf{u}\|^2 &= \|\nabla\overline{\mathbf{u}} + \nabla\mathbf{v}\|^2\\ &= \|\nabla\overline{\mathbf{u}}\|^2 + 2\nabla\overline{\mathbf{u}} : \nabla\mathbf{v} + \|\nabla\mathbf{v}\|^2,\end{aligned}$$

and averaging yields

$$\overline{\|\nabla\mathbf{u}\|^2} = \|\nabla\overline{\mathbf{u}}\|^2 + \overline{\|\nabla\mathbf{v}\|^2}.$$

Hence

$$\epsilon = \epsilon_{\text{mean}} + \epsilon_{\text{fluct}}.$$

Now we'll investigate the fluctuations in more depth.

## 1 Fluctuations

Take a change of variables $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{v}$ in the Navier-Stokes equations, which yields

$$\partial_t\mathbf{v} + \mathbf{v}.\nabla\mathbf{v} + \mathbf{v}.\nabla\overline{\mathbf{u}} + \overline{\mathbf{u}}.\nabla\mathbf{v} + \overline{\mathbf{u}}.\nabla\overline{\mathbf{u}} + \nabla p = \nu\triangle\overline{\mathbf{u}} + \nu\triangle\mathbf{v}.$$

And also we have

$$\overline{\mathbf{u}}.\nabla\overline{\mathbf{u}} + \nabla\overline{p} = \nu\triangle\overline{\mathbf{u}} + \nabla.\sigma^{\text{Rey}}.$$

Subtracting the second from the first yields

$$\partial_t\mathbf{v} + \mathbf{v}.\nabla\mathbf{v} + \mathbf{v}.\nabla\overline{\mathbf{u}} + \overline{\mathbf{u}}.\nabla\mathbf{v} + \nabla(p - \overline{p}) = \nu\triangle\mathbf{v} - \nabla.\sigma^{\text{Rey}}.$$

Now we want to look at energy. So dot $\mathbf{v}$ into this relation and integrate. This yields

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) + \int\mathbf{v}.(\nabla\overline{\mathbf{u}}).\mathbf{v} = -\nu\|\nabla\mathbf{v}\|_2^2 - \int\left(\nabla.\sigma^{\text{Rey}}\right).\mathbf{v},$$

and time-averging gives

$$\underbrace{\overline{\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right)}}_{=0} + \int \overline{\mathbf{v}.\left(\nabla\overline{\mathbf{u}}\right).\mathbf{v}} = -\nu\overline{\|\nabla\mathbf{v}\|_2^2} - \underbrace{\int \overline{\left(\nabla.\sigma^{\mathrm{Rey}}\right).\mathbf{v}}}_{=0}.$$

Therefore,

$$\epsilon_{\mathrm{fluct}} = \left\langle \nu\|\nabla\mathbf{v}\|^2\right\rangle = \left\langle -\left(\nabla\overline{\mathbf{u}}\right) : \mathbf{v}\mathbf{v}\right\rangle = \left\langle R_{ij}\sigma_{ij}^{\mathrm{Rey}}\right\rangle,$$

where

$$R_{ij} = \frac{1}{2}\left(\frac{\partial\overline{u_j}}{\partial x_i} + \frac{\partial\overline{u_i}}{\partial x_j}\right).$$

This is as far as we can go without making modeling assumptions. But we'd like to have a way to solve RANS forward in time. We need physical intuition to link $\sigma^{\mathrm{Rey}}$ and $\mathbf{u}$.

## 2 Three Modeling Assumptions

The first big assumption:

$$\sigma^{\mathrm{Rey}} \sim R = \text{mean rate of strain}.$$

This is as in molecular dynamics, but as the scales are much larger here this is indeed a modeling assumption.

**Definition.** Given the assumption above, we call the constant of proportionality between $\sigma^{\mathrm{Rey}}$ and $R$ the *eddy viscosity*, and write

$$\sigma^{\mathrm{Rey}} = \nu_e R.$$

Now we ask: what can $\nu_e$ depend on? We know that

$$[\nu_e] = \frac{\text{length}^2}{\text{time}},$$

and the fluctuations are driven by gradients in the mean velocity, which have

$$[\nabla\overline{\mathbf{u}}] \sim \frac{1}{\text{time}}.$$

It seems, at least in the plane Couette setup, that our length scale should be

$$\text{length} \sim \text{distance to the wall}.$$

This line of reasoning is due to Prandtl and von Karman. As did they, we now make a second modeling assumption:

$$\nu_e \sim \|\nabla\overline{\mathbf{u}}\| \cdot (\text{distance to wall})^2.$$

And finally we have a third modeling assumption: We assume the turbulent statistics reflect the maximal symmetry of the problem.

In the plane Couette setup, this last assumption implies that all mean quantities can only depend on $y$. So we can write[1]

$$\begin{aligned}\overline{\mathbf{u}} &= \hat{\mathbf{i}}\overline{u} + \underbrace{\hat{\mathbf{j}}\overline{v}}_{=0} + \underbrace{\hat{\mathbf{k}}\overline{w}}_{=0}\\ &= \hat{\mathbf{i}}\overline{u}\,(y)\,,\end{aligned}$$

and similarly

$$\begin{aligned}\sigma^{\text{Rey}} &= -\overline{\mathbf{v}\mathbf{v}}\,(y) \ \text{ and}\\ \overline{p} &= \overline{p}\,(y)\,.\end{aligned}$$

Furthermore, by imposing the third assumption on RANS we get

$$0 = \nu\frac{\partial^2}{\partial y^2}\overline{u}\,(y) + \frac{\partial}{\partial y}\sigma_{21}^{\text{Rey}}\,(y)\,, \tag{2.1}$$

$$\frac{\partial\overline{p}}{\partial y} = \frac{\partial}{\partial y}\sigma_{22}^{\text{Rey}}\,(y)\,. \tag{2.2}$$

Now let's investigate the second assumption for plane Couette flow. Our intuition is that the turbulent flow should look as follows:

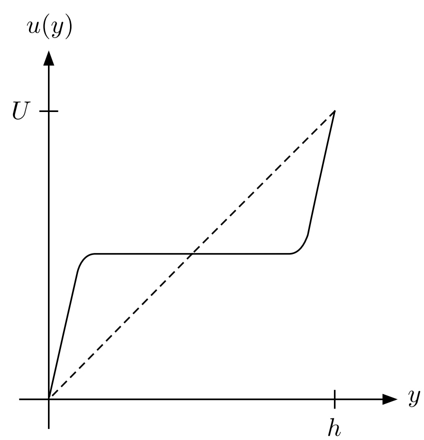



Figure 2.1: Expected turbulent Couette flow profile.

So let's assume $u'\,(y) \geq 0$. We'll only consider the bottom half of the layer (by symmetry). Then the distance to the wall ("mixing length") is just $y$. Thus,

$$\nu_e = \kappa^2 u'\,(y)\cdot y^2$$

where $\kappa$ is the "von Karman constant." Note that $\kappa \approx .4$ as per experiments.

[1] In particular, $\overline{v} = 0$ follows from the divergence-free condition and boundary conditions on $\overline{v}$.

Let's look at (2.1) and (2.2) more carefully. First, (2.2) says

$$\begin{aligned} \frac{d}{dy}\overline{p} &= \frac{d}{dy}\sigma_{22}^{\mathrm{Rey}}(y) \\ &= -\frac{d}{dy}\overline{(v_2)^2}, \end{aligned}$$

and thus

$$\overline{p}(y) = -\overline{(v_2)^2} + \text{const.}$$

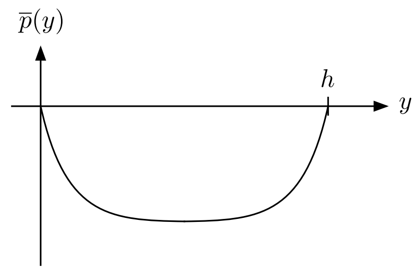



Figure 2.2: Average pressure profile.

Second, (2.1) says

$$\begin{aligned} 0 &= \nu\frac{d^2}{dy^2}\overline{u}(y) + \frac{d}{dy}\left(\nu_e(y)\frac{d\overline{u}}{dy}\right) \\ &= \nu\frac{d^2}{dy^2}\overline{u} + \frac{d}{dy}\left[\kappa^2y^2\left(\frac{d\overline{u}}{dy}\right)^2\right] \\ &= \frac{d}{dy}\left[\nu\overline{u}' + \kappa^2y^2\left(\overline{u}'\right)^2\right]. \end{aligned}$$

So integrating from the boundary up yields

$$\nu\overline{u}'(y) + \kappa^2y^2\left(\overline{u}'(y)\right)^2 = \nu\overline{u}'(0) = \frac{\text{wall shear stress}}{\rho} = u_*^2$$

once we define $u_*$ by the relation

$$\rho u_*^2 = \frac{\overline{F}}{L_xL_z}.$$

Hence

$$\begin{aligned} \kappa^2y^2\left(\overline{u}'(y)\right)^2 + \nu\overline{u}'(y) &= u_*^2 \\ \implies \overline{u}'(y) &= -\frac{\nu}{\kappa^2} \pm \sqrt{\left(\frac{\nu}{2\kappa^2y^2}\right)^2 + \frac{u_*^2}{\kappa^2y^2}}. \end{aligned}$$

Recall we expect $\overline{u}'(y) \geq 0$, so

$$\overline{u}'(y) = -\frac{\nu}{\kappa^2} + \sqrt{\left(\frac{\nu}{2\kappa^2 y^2}\right)^2 + \frac{u_*^2}{\kappa^2 y^2}}.$$

Now as $\overline{u}(0) = 0$ and as (by symmetry) $\overline{u}(h/2) = U/2$, we can determine $u_*$. After some work, one finds

$$\frac{\kappa}{u_*} u(y) = \log\left[\frac{2\kappa u_* y}{\nu} + \sqrt{1 + \left(\frac{2\kappa u_* y}{\nu}\right)^2}\right] + \frac{1 - \sqrt{1 + \left(\frac{2\kappa u_* y}{\nu}\right)^2}}{\left(\frac{2\kappa u_* y}{\nu}\right)}.$$

And demanding $U/2 = \overline{u}(h/2)$ yields

$$\frac{\kappa U}{2u_*} = \log\left[\frac{\kappa u_* h}{\nu} + \sqrt{1 + \left(\frac{\kappa u_* h}{\nu}\right)^2}\right] + \frac{1 - \sqrt{1 + \left(\frac{\kappa u_* h}{\nu}\right)^2}}{\left(\frac{\kappa u_* h}{\nu}\right)},$$

an implicit equation for $u_*$ in terms of known constants. Equivalently, one can write

$$\frac{1}{2}\kappa\left(\frac{U}{u^*}\right) = \log\left[\kappa\left(\frac{u^*}{U}\right)\mathrm{Re} + \sqrt{1 + \left(\kappa\left(\frac{u_*}{U}\right)\mathrm{Re}\right)^2}\right] + \frac{1 - \sqrt{1 + \left(\kappa\left(\frac{u_*}{U}\right)\mathrm{Re}\right)^2}}{\kappa\left(\frac{u_*}{U}\right)\mathrm{Re}}$$

with

$$\mathrm{Re} = \frac{Uh}{\nu}$$

as usual. Note that if $\mathrm{Re} \to 0$, then

$$\frac{\tau}{\rho} = u_*^2 \to \nu\frac{U}{h},$$

so then

$$\tau = \mu\frac{U}{h}.$$

And if $\mathrm{Re} \to \infty$, then

$$u_*^2 \sim \frac{\kappa^2}{4}\frac{U^2}{(\log \mathrm{Re})^2}$$

and then

$$\frac{\epsilon h}{U^3} \sim \frac{\kappa^2}{4}\frac{1}{(\log \mathrm{Re})^2}.$$

This matches the experimental data with $\kappa \approx 0.4$. In particular, experiments show $\beta \sim (\log \mathrm{Re})^{-2}$ for large enough Re.

## Lecture 19: Kolmogorov's Theory and the Energy Cascade

Today we'll discuss a general theory of turbulence, attributed to Komogorov. Others associated with this are Taylor and Richardson. The theory will describe homogenous, isotropic turbulence; the theory is a "scaling" theory.

# 1 Kolmogorov's (Scaling) Theory of (Homogenous, Isotropic) Turbulence

Consider the Navier-Stokes equations without boundaries (i.e., in a periodic box):

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \nabla p &= \nu \triangle \mathbf{u} + f(\mathbf{x}, t) \\ \nabla . \mathbf{u} &= 0. \end{aligned}$$

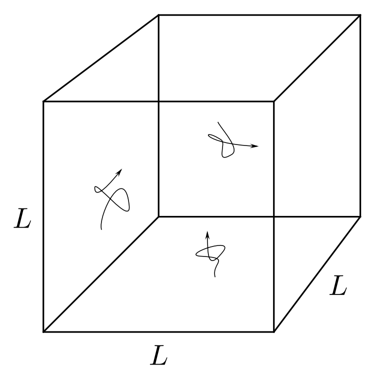



Figure 1.1: Turbulent flow in a box.

We can decompose a solution $\mathbf{u}$ into its Fourier modes with wave-numbers $\mathbf{k}$ as

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} \hat{\mathbf{u}}(\mathbf{k}, t).$$

The Fourier coeffcients satisfy

$$\hat{\mathbf{u}}(\mathbf{k}, t) = \frac{1}{L^3} \int_{\text{box}} e^{-i\mathbf{k}.\mathbf{x}} \mathbf{u}(\mathbf{x}, t)\, d\mathbf{x}$$

with

$$\mathbf{k} = \frac{2\pi}{L}\mathbf{n}, \quad \mathbf{n} = \hat{\mathbf{i}} n_1 + \hat{\mathbf{j}} n_2 + \hat{\mathbf{k}} n_3, \quad n_i \in \mathbb{Z}.$$

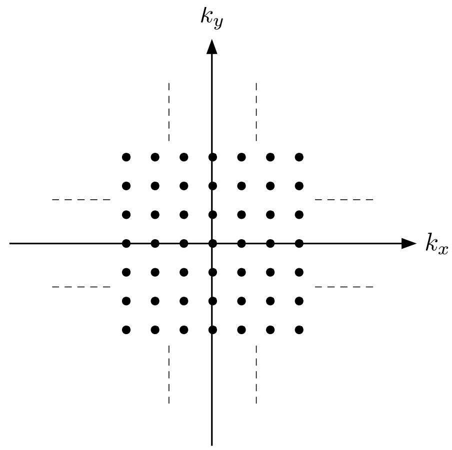



Figure 1.2: Fourier space.

In this decomposition, small scales in real space correspond to high $\|\mathbf{k}\|$.

The kinetic energy satisfies

$$\begin{aligned}
\text{Kinetic Energy} &= \frac{1}{2}\int_{\text{box}} \|\mathbf{u}(\mathbf{x},t)\|^2 \, d\mathbf{x} \\
&= \frac{1}{2}\int_{\text{box}} \left(\sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} \hat{\mathbf{u}}(\mathbf{k},t)\right) \cdot \left(\sum_{\mathbf{k}'} e^{i\mathbf{k}'.\mathbf{x}} \hat{\mathbf{u}}(\mathbf{k}',t)\right) d\mathbf{x} \\
&= \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'} \hat{\mathbf{u}}(\mathbf{k},t) . \hat{\mathbf{u}}(\mathbf{k}',t) \underbrace{\int_{\text{box}} e^{i(\mathbf{k}+\mathbf{k}').\mathbf{x}} \, d\mathbf{x}}_{L^3 \cdot \delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}} \\
&= \frac{L^3}{2}\sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k},t) . \hat{\mathbf{u}}(-\mathbf{k},t) .
\end{aligned}$$

It can be shown that $\hat{\mathbf{u}}(-\mathbf{k},t) = \hat{\mathbf{u}}(\mathbf{k},t)^*$, so

$$\text{Kinetic Energy} = \frac{L^3}{2}\sum_{\mathbf{k}} \|\hat{\mathbf{u}}(\mathbf{k},t)\|^2 .$$

Note that $\|\hat{\mathbf{u}}(\mathbf{k},t)\|^2$ is sometimes called the "spectrum" of the flow $\mathbf{u}(\mathbf{x},t)$.

Now we make a key assumption to this theory: We assume isotropy, i.e., we assume $\overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2}$ depends only on $\|\mathbf{k}\|$. Noting $\Delta k = 2\pi/L$ and also that $(\Delta k)^3 = (2\pi)^3/L^3 = \Delta k_x \cdot \Delta k_y \cdot \Delta k_z$, we can write

$$\begin{aligned}
\frac{1}{2}\int_{\text{box}} \rho \cdot \overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2} \, d\mathbf{x} &= \frac{\rho L^3}{2}\sum_{\mathbf{k}} \overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2} \cdot (\Delta k)^3 \cdot \left(\frac{L}{2\pi}\right)^3 \\
&= \frac{M}{2}\left(\frac{L}{2\pi}\right)^3 \sum_{\mathbf{k}} \overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2} \cdot (\Delta k)^3 .
\end{aligned}$$

The sum now looks like a Riemann integral, so (??)

$$\begin{aligned}\frac{1}{2}\int_{\text{box}} \rho \cdot \overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2}\, d\mathbf{x} &= \frac{M}{2}\left(\frac{L}{2\pi}\right)^3 \int \overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2}\, d\mathbf{k} \\ &= M\int_{2\pi/L}^{\infty} \underbrace{\frac{1}{2}\left(\frac{L}{2\pi}\right)^3 \cdot \overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2} \cdot 4\pi k^2}_{E(k)}\, dk.\end{aligned}$$

**Definition.** The quantity

$$E(k) = \frac{1}{2}\left(\frac{L}{2\pi}\right)^3 \cdot \overline{\|\hat{\mathbf{u}}(\mathbf{k},\cdot)\|^2} \cdot 4\pi k^2$$

is called the *energy spectrum* of the flow $\mathbf{u}(\mathbf{k},t)$.

*Remark.* In fact, $E(k)$ is the energy spectrum per unit mass; one could think of $E(k)\ dk$ as the energy per unit mass contained in a shell of thickness $dk$ about $\|\mathbf{k}\|$.

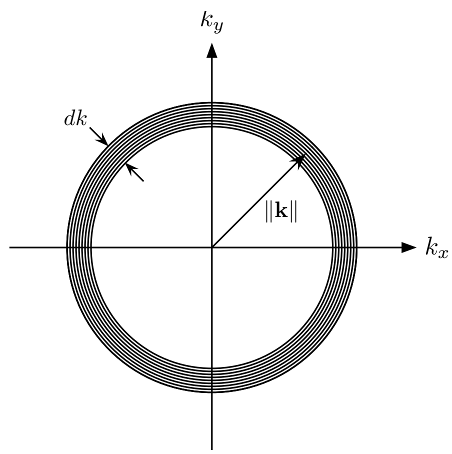



Figure 1.3: $E(k)\ dk$ is energy/mass in shells.

Now we have a famous (dimensional) argument. First, consider the flow of energy in $\mathbf{k}$-space as one stirs the flow at medium-to-large scales. We know that

$$\frac{d}{dt}\left(\frac{\rho}{2}\|\mathbf{u}\|_2^2\right) = \underbrace{-\nu\|\nabla\mathbf{u}\|_2^2}_{\text{viscous dissipation}} + \underbrace{\int \mathbf{u}.\mathbf{f}}_{\text{applied force}}.$$

Secondary flow is generated from the large scales at sizes smaller than the original scales, resulting in the well-known <u>energy cascade</u> of three-dimensional turbulence.

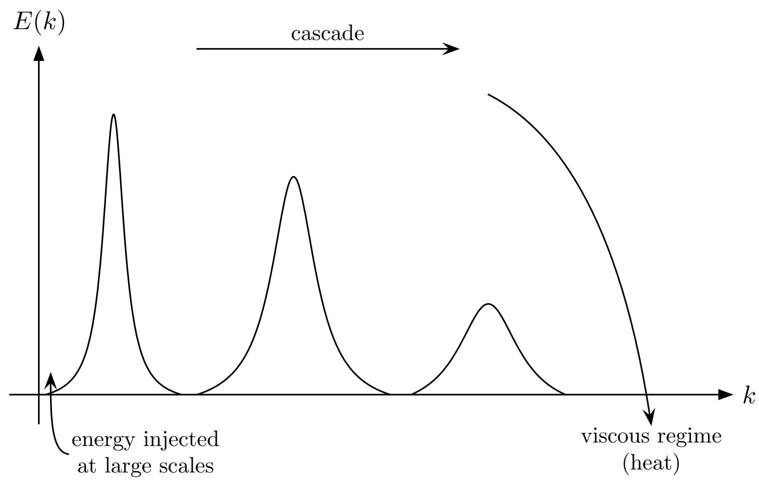



Figure 1.4: The forward energy cascade.

If viscosity were zero, the cascade would go on forever. But note that viscosity has a large effect only at large wavenumbers. To see why, consider that

$$\nabla \mathbf{u} = \sum_{\mathbf{k}} i\mathbf{k} e^{i\mathbf{k}.\mathbf{x}} \hat{\mathbf{u}}\left(\mathbf{k}, t\right),$$

and so

$$\begin{aligned} \epsilon &= \nu \int_{\text{box}} \overline{\|\nabla \mathbf{u}\|^2}\, d\mathbf{x} \\ &= \nu \int_{\text{box}} k^2 E\left(k\right)\, dk. \end{aligned}$$

So for small $k$, the dissipation is very small: $\left[\nu k^2\right] = [\text{dissipation}]$, which is the rate of dissipation. (??) If Navier-Stokes were linear, energy would stay in place; the non-linear term gives rise to the cascade.

Now we'll determine exactly how $E$ depends on $k$. The power put into the flow by stirring is given by $\epsilon M L^3$. Since $[\epsilon] = L^2/T^3$, $[k] = 1/L$, and $[E] = L^3/T^2$, a dimensional argument leads us to the well-known relation

$$E\left(k\right) = C_k \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}.$$

Here $C_k$ is known as the "Kolmogorov constant." In deriving this we've assumed:

- Scales exist;
- The energy cascade is driven by the non-linear terms;
- A steady-state exists.

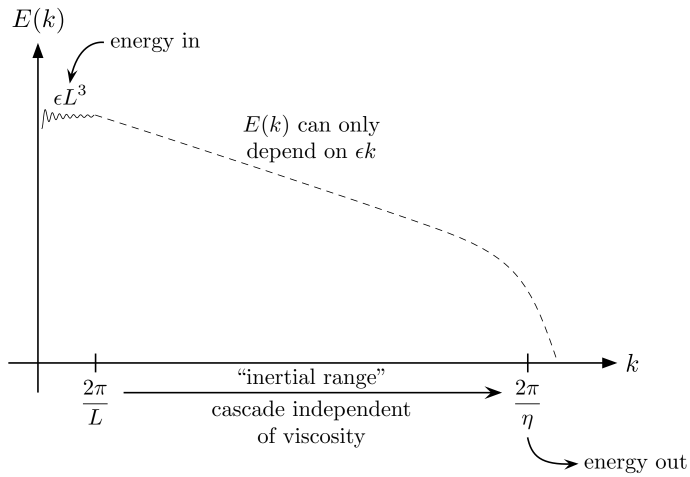



Figure 1.5: $E(k)$ in the inertial range.

The quantity $\eta$ in the diagram above is the so-called "dissipation lengthscale," and will be determined below.

For now, consider the quantity

$$U^2 = \left\langle \|\mathbf{u}\|^2 \right\rangle .$$

We have

$$\begin{aligned} U^2 &= 2\int_{\frac{2\pi}{L}}^{\frac{2\pi}{\eta}} E(k)\ dk \\ &= 2\int_{\frac{2\pi}{L}}^{\frac{2\pi}{\eta}} C_k \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}\ dk \\ &= 2C_k\epsilon^{\frac{2}{3}}\frac{3}{2}\left[\left(\frac{L}{2\pi}\right)^{\frac{2}{3}} - \left(\frac{\eta}{2\pi}\right)^{\frac{2}{3}}\right]. \end{aligned}$$

So if we neglect the contribution by the (small) $\eta$ term,

$$U^2 \approx \frac{3C_k\epsilon^{\frac{2}{3}}}{(2\pi)^{\frac{3}{2}}} L^{\frac{2}{3}}$$

and hence

$$\epsilon \approx A\frac{U^3}{L}$$

where $A = 2\pi/(3C_k)^{3/2}$. But now recall that $\beta = \epsilon L/U^3$ is approximately constant in the turbulent regime! So dissipation does not scale with the Reynolds number; it is instead universal.

Let's try to find $\eta$. We have

$$\begin{aligned} \epsilon &= \nu \int_{\frac{2\pi}{L}}^{\frac{2\pi}{\eta}} k^2 C_k \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}\, dk \\ &= \epsilon^{\frac{2}{3}} \nu C_k \int_{\frac{2\pi}{L}}^{\frac{2\pi}{\eta}} k^{\frac{1}{2}}\, dk. \end{aligned}$$

Note that the main contribution here is from high $k$. Continuing on, we find

$$\begin{aligned} \epsilon &= \frac{3}{4}\epsilon^{\frac{2}{3}} \nu C_k \left[ \left(\frac{2\pi}{\eta}\right)^{\frac{4}{3}} - \underbrace{\left(\frac{2\pi}{L}\right)^{\frac{4}{3}}}_{\approx 0} \right] \\ &= \frac{3}{4} (2\pi)^{\frac{4}{3}} C_k \nu \epsilon^{\frac{2}{3}} \eta^{-\frac{4}{3}}, \end{aligned}$$

and thus

$$\eta = 2\pi \left(\frac{3}{4} C_k\right)^{\frac{3}{4}} \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}.$$

Now we have a closed picture.

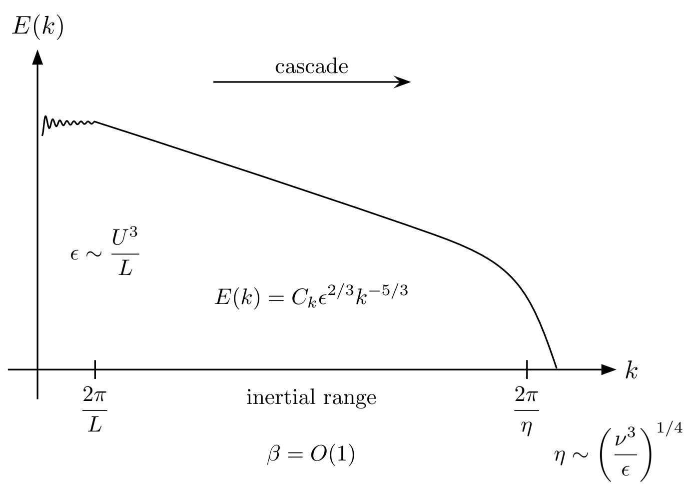



Figure 1.6: Kolmogorov's theory of turbulence.

## Lecture 20: Turbulent Energy Dissipation

The equations are

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p &= \nu\triangle\mathbf{u} + \mathbf{f}(\mathbf{x},t),\\ \nabla.\mathbf{u} &= 0.\end{aligned}$$

We decomposed the flow as

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}\hat{\mathbf{u}}(\mathbf{k},t)$$

with

$$\hat{\mathbf{u}}(\mathbf{k},t) = \frac{1}{L^3}\int e^{-i\mathbf{k}.\mathbf{x}}\mathbf{u}(\mathbf{x},t)\,d\mathbf{x}.$$

The setup here is periodic, boundaryless flow; a cube with side length $L$:

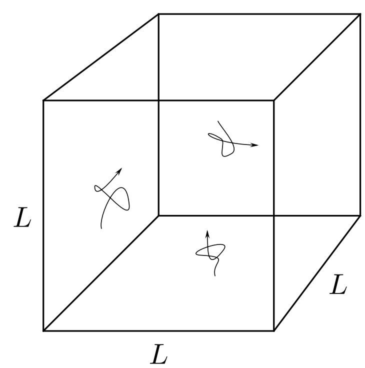



Figure 0.1: Turbulent flow in a box.

Denoting a time-space average by $\langle\cdot\rangle$, we have two experimental laws of fully developed turbulence. The first law is that

$$\left\langle\|\mathbf{u}(\cdot,\cdot) - \mathbf{u}(\cdot+\mathbf{l},\cdot)\|^2\right\rangle = C(\epsilon l)^{\frac{2}{3}}$$

where $\eta < l < L$ and viscosity $\nu$ does not matter in this regime. Dimensional analysis leads to this law as well. The Fourier version of this, assuming homogeneity and isotropy as before, is

$$E(k) \sim \epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$$

for $\eta^{-1} \ll k \ll L^{-1}$. The second law says that $\epsilon = \left\langle\nu\|\nabla\mathbf{u}\|^2\right\rangle$ stays finite and non-zero as $\nu \to 0$. So for turbulent flow, $\|\nabla\mathbf{u}\|^2 \to \infty$ as $\nu \to 0$. This is akin to energy cascading to the small scales.

## 1 Bounds on Turbulent Energy Dissipation in a Box

We'll see how to go from the control parameters $\nu, L, \mathbf{f}$ to the energy law. Suppose

$$\mathbf{f}(\mathbf{x},t) = \sum_{\frac{2\pi}{L}\le\|\mathbf{k}\|\le 7\cdot\frac{2\pi}{L}} e^{i\mathbf{k}.\mathbf{x}}\hat{\mathbf{f}}(\mathbf{k},t).$$

If we demand the force to be divergence-free, then

$$0 = \nabla.\mathbf{f} = \sum ie^{i\mathbf{k}.\mathbf{x}}\mathbf{k}.\hat{\mathbf{f}}\left(\mathbf{k},t\right),$$

and so

$$\mathbf{k} \perp \hat{\mathbf{f}}\left(\mathbf{k},t\right)$$

for all $\mathbf{k}$. Also suppose time dependence of the coefficients of $\mathbf{f} \sim e^{i\Omega_1 t}, e^{i\Omega_2 t}$ where $\Omega_1/\Omega_2 \notin \mathbb{Q}$. And suppose $\Omega_{\max} < \infty$. Then by dotting the momentum equation with $\mathbf{u}$ and integrating over the box, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\left(\cdot,t\right)\|_2^2\right) = -\nu\|\nabla\mathbf{u}\left(\cdot,t\right)\|_2^2 + \int \mathbf{f}\left(\mathbf{x},t\right).\mathbf{u}\left(\mathbf{x},t\right)\,d\mathbf{x}.$$

Now if we assume (w.l.o.g.) that $\mathbf{u}\left(\mathbf{x},0\right)$ satisfies $\int u_i\left(\mathbf{x},0\right)\,dxdydz = 0$, the Poincare lemma reads

$$\|\nabla\mathbf{u}\|_2^2 \geq \frac{4\pi^2}{L^2}\|\mathbf{u}\|_2^2.$$

To see this, write

$$\|\mathbf{u}\|_2^2 = L^3\sum_{\mathbf{k}}\|\hat{\mathbf{u}}\left(\mathbf{k}\right)\|^2$$

by Parseval's theorem, and thus

$$\begin{aligned}
\|\nabla\mathbf{u}\|_2^2 &= L^2\sum_{\mathbf{k}\neq 0}\|\mathbf{k}\|^2\|\hat{\mathbf{u}}\left(\mathbf{k}\right)\|^2 \\
&\geq L^2\sum_{\mathbf{k}\neq 0}\frac{4\pi^2}{L^2}\|\hat{\mathbf{u}}\left(\mathbf{k}\right)\|^2 \\
&= \frac{4\pi^2}{L^2}\|\mathbf{u}\|_2^2.
\end{aligned}$$

And by Cauchy-Schwarz,

$$\left|\int \mathbf{f}.\mathbf{u}\right| \leq \|f\|_2\|\mathbf{u}\|_2.$$

Setting these into the differential equation from above yields

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|_2^2\right) \leq -\frac{4\pi^2}{L^2}\nu\|u\|_2^2 + \|f\|_2\|u\|_2.$$

Since

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|_2^2\right) = \|\mathbf{u}\|_2\frac{d}{dt}\|\mathbf{u}\|_2,$$

we have

$$\frac{d}{dt}\|\mathbf{u}\|_2 \leq -\frac{4\pi^2}{L^2}\nu\|\mathbf{u}\|_2 + \mathbf{f}\left(\cdot,\mathbf{t}\right)_2.$$

By Gronwall's inequality,

$$\begin{aligned}\|\mathbf{u}(\cdot,t)\|_2 &\leq \|\mathbf{u}(\cdot,0)\|_2 e^{-\frac{4\pi^2\nu}{L^2}t} + \int_0^t \|\mathbf{f}(\cdot,t')\|_2 e^{-\frac{4\pi^2\nu}{L^2}(t-t')}\,dt' \\ &\leq \|\mathbf{u}(\cdot,0)\|_2 e^{-\frac{4\pi^2\nu}{L^2}t} + \sup_{t''}\{\|\mathbf{f}(\cdot,t'')\|_2\}\cdot\left(\frac{1-e^{-\frac{4\pi^2\nu}{L^2}t}}{\left(\frac{4\pi^2\nu}{L^2}\right)}\right).\end{aligned}$$

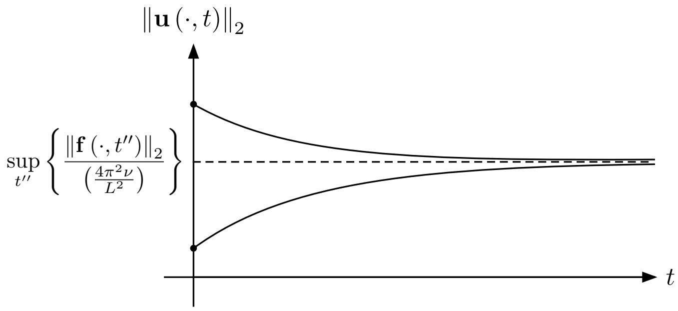



Figure 1.1: $\|\mathbf{u}\|_2$ stays bounded for all time.

Note that transient time $\sim L^2/\nu$, and in long enough time $\|\mathbf{u}(\cdot,t)\|_2 \leq \sup_{t''}\{\|\mathbf{f}(\cdot,t'')\|_2\}$, i.e., $\limsup_{t\to\infty}\|\mathbf{u}\|_2$ is bounded. This is a neccesary condition for a statistical steady state to exist given any amount of viscosity.

Now we can write

$$\frac{1}{T}\int_0^T\left[\frac{d}{dt}\left(\frac{\frac{1}{2}\|\mathbf{u}(\cdot,t)\|_2^2}{L^3}\right)\right] = \frac{1}{T}\int_0^T\left[-\frac{\nu\|\nabla\mathbf{u}(\cdot,t)\|_2^2}{L^3} + \frac{\int \mathbf{f}(\mathbf{x},t).\mathbf{u}(\mathbf{x},t)\;dxdydz}{L^3}\right]$$

and thus

$$\frac{1}{T}\left(\frac{1}{2}\|\mathbf{u}(\cdot,T)\|_2^2 - \frac{1}{2}\|\mathbf{u}(\cdot,0)\|_2^2\right) = -\left\langle\nu\|\nabla\mathbf{u}\|^2\right\rangle_T + \langle\mathbf{f}.\mathbf{u}\rangle_T.$$

Assuming the terms on the right have finite limits as $T\to\infty$ we arrive at the power balance

$$\epsilon = \langle\mathbf{f}.\mathbf{u}\rangle.$$

Now define

$$U = \left\langle\|\mathbf{u}\|^2\right\rangle^{\frac{1}{2}}.$$

Let's look for a lower bound on $\beta$:

$$\begin{aligned}\epsilon &= \left\langle\nu\|\nabla\mathbf{u}\|^2\right\rangle \\ &\geq \nu\frac{4\pi^2}{L^2}\left\langle\|\mathbf{u}\|^2\right\rangle \\ &= \frac{4\pi^2}{L^2}\nu U^2\end{aligned}$$

by Poincare's lemma. So

$$\begin{aligned}\beta &= \frac{\epsilon L}{U^3}\\ &\geq \frac{4\pi^2}{L^2}\nu U^2 \cdot \frac{L}{U^3}\\ &= 4\pi^2 \frac{\nu}{LU}\\ &= 4\pi^2 \frac{1}{\mathrm{Re}}.\end{aligned}$$

Now we look for an upper bound on $\beta$:

$$\begin{aligned}\epsilon &= \langle \mathbf{f}.\mathbf{u}\rangle\\ &\leq \left\langle \|\mathbf{f}\|^2\right\rangle^{\frac{1}{2}} \left\langle \|\mathbf{u}\|^2\right\rangle^{\frac{1}{2}}\\ &= \left\langle \|\mathbf{f}\|^2\right\rangle^{\frac{1}{2}} U.\end{aligned}$$

Dotting the momentum equation with $\mathbf{f}$ and integrating over the box yields

$$\int \mathbf{f}.\partial_t \mathbf{u} + \int \mathbf{f}.\left(\mathbf{u}.\nabla\mathbf{u}\right) = \nu \int \mathbf{f}.\triangle\mathbf{u} + \|\mathbf{f}\|_2^2.$$

After integration by parts,

$$\frac{d}{dt}\int \left(\mathbf{f}.\mathbf{u}\right) - \int \partial_t \mathbf{f}.\mathbf{u} - \int \mathbf{u}.\left(\nabla \mathbf{f}\right).\mathbf{u} = \nu \int \triangle \mathbf{f}.\mathbf{u} + \|\mathbf{f}\|_2^2.$$

Dividing by $L^3$ to get a spatial average, then applying $\frac{1}{T}\int_0^T (\cdot)\, dt$ yields

$$-\left\langle \partial_t \mathbf{f}.\mathbf{u}\right\rangle - \left\langle \mathbf{u}.\left(\nabla\mathbf{f}\right).\mathbf{u}\right\rangle - \nu\left\langle \triangle\mathbf{f}.\mathbf{u}\right\rangle = \left\langle \|\mathbf{f}\|^2\right\rangle.$$

This implies

$$\left\langle \|\mathbf{f}\|^2\right\rangle \leq \underbrace{\left\langle \|\partial_t \mathbf{f}\|^2\right\rangle^{\frac{1}{2}}}_{\Omega_{\max}\left\langle\|\mathbf{f}\|^2\right\rangle^{\frac{1}{2}}} \cdot U + \underbrace{\sup_{\mathbf{x},t}\left\{\|\nabla\mathbf{f}\left(\mathbf{x},t\right)\|\right\}}_{C_1\frac{1}{L}\left\langle\|\mathbf{f}\|^2\right\rangle^{\frac{1}{2}}} \cdot U^2 + \nu\underbrace{\left\langle \|\triangle\mathbf{f}\|^2\right\rangle^{\frac{1}{2}}}_{\frac{C_2}{L^2}\left\langle\|\mathbf{f}\|^2\right\rangle^{\frac{1}{2}}} \cdot U.$$

Note that $C_1$ depends on the spectrum of $\mathbf{f}$ but not the amplitude, and similarly for $C_2$. Hence,

$$\left\langle \|\mathbf{f}\|^2\right\rangle^{\frac{1}{2}} \leq \Omega_{\max} U + C_1 \frac{1}{L} U^2 + \frac{\nu C_2}{L^2} U,$$

and hence

$$\epsilon \leq U\left\langle \|\mathbf{f}\|^2\right\rangle^{\frac{1}{2}} \leq \Omega_{\max} U^2 + C_1\frac{1}{L}U^3 + \frac{\nu C_2}{L^2}U^2.$$

This all goes to show

$$\beta = \frac{L\epsilon}{U^3} \leq \Omega_{\max}\frac{L}{U} + C_1 + C_2\frac{1}{\mathrm{Re}}.$$

There are multiple ways to send $\mathrm{Re}\to\infty$,[1] but in any case $\beta$ is bounded above independent of viscosity.

[1] Hold $U$ fixed and send $\nu \to 0$, or hold $\nu$ fixed and send $U \to \infty$, or some combination.

The bounds we found on $\beta$ for turbulent flow in a box are depicted in the $\beta$-Re diagram below.

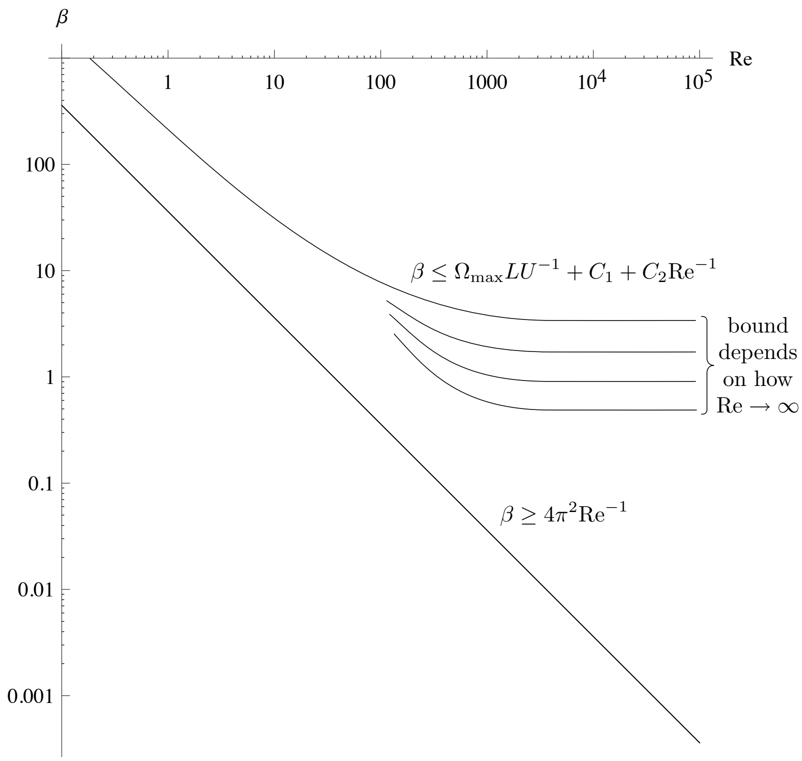



Figure 1.2: $\beta$ vs. Re.

## Lecture 21: 2D Turbulence and Introduction to Existence/Uniquenss

Last time we talked about how energy cascades to smaller scales and dissipates in turbulent flow. This has a particular form in 2D, which we'll discuss now. Then we'll switch gears and discuss some of the more mathematical questions in Navier-Stokes.

## 1 Turbulence in $\mathbb{R}^2$

Consider turbulent flow in a 2-dimensional box.

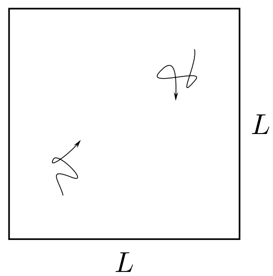



Figure 1.1: Turbulent flow in a 2D box.

The equations are

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p &= \nu\triangle\mathbf{u} + \mathbf{f}, \\ \nabla.\mathbf{u} &= 0, \end{aligned}$$

where now $\mathbf{u} = \hat{\mathbf{i}}u + \hat{\mathbf{j}}v$. Again,

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|_2^2\right) = -\nu\|\nabla\mathbf{u}\|_2^2 + \int \mathbf{f}.\mathbf{u}.$$

Recall that

$$\omega = \nabla\times\mathbf{u} = \hat{\mathbf{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \hat{\mathbf{k}}\omega$$

in $\mathbb{R}^2$, and

$$\partial_t\omega + \mathbf{u}.\nabla\omega = \nu\triangle\omega + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

We can derive an evolution equation for the enstrophy via the equation above. This turns out to be

$$\frac{d}{dt}\left(\frac{1}{2}\|\omega\|_2^2\right) = -\nu\|\nabla\omega\|_2^2 + \int \omega\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

The key quantities are

- Energy: $\frac{1}{2}\|\mathbf{u}\|_2^2$
- Enstrophy: $\|\omega\|_2^2$

- Energy dissipation rate: $\epsilon = \left\langle \nu \|\nabla \mathbf{u}\|^2 \right\rangle$
- Enstrophy dissipation rate: $\chi = \left\langle \nu \|\nabla \omega\|^2 \right\rangle$.

Let's go to spectral space. Decompose $\mathbf{u}$ as

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}, t),$$

where

$$\mathbf{k}.\hat{\mathbf{u}}(\mathbf{k}, t) = 0;$$

this is the spectral version of $\nabla.\mathbf{u} = 0$. We are in $\mathbb{R}^2$, so vorticity appears as

$$\hat{\mathbf{k}}\omega = \nabla \times \mathbf{u} = \sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{k}} \underbrace{i\mathbf{k} \times \hat{\mathbf{u}}(\mathbf{k}, t)}_{\hat{k}\hat{\omega}(\mathbf{k},t)}.$$

Thus

$$\begin{aligned} \|\omega\|_2^2 &= L^2 \sum_{\mathbf{k}} \|\hat{\omega}(\mathbf{k}, t)\|^2 \\ &= L^2 \sum_{\mathbf{k}} \|\mathbf{k}\|^2 \|\hat{\mathbf{u}}(\mathbf{k}, t)\|^2 \\ &= \|\nabla \mathbf{u}\|_2^2, \end{aligned}$$

and thus

$$\epsilon = \left\langle \nu \omega^2 \right\rangle \text{ and } \chi = \left\langle \nu \|\triangle \mathbf{u}\|^2 \right\rangle.$$

Recall that if $\nu = 0$, then energy is conserved. And in 2D, enstrophy is conserved as well. Now if we take $\delta k = 2\pi/L$ in the Fourier decomposition, then

$$\begin{aligned} \frac{1}{2} \sum_{\mathbf{k}} \|\hat{\mathbf{u}}(\mathbf{k}, t)\| &= \frac{L^2}{4\pi^2} \frac{1}{2} \sum_{\mathbf{k}} \|\hat{\mathbf{u}}(\mathbf{k}, t)\|^2 (\delta k)^2 \\ &\approx \int E(k)\, dk \end{aligned}$$

where we've assumed homogenous isotropic turbulence. This shows that the energy spectrum in 2D is

$$E(k) = \frac{L^2}{8\pi^2} \|\hat{\mathbf{u}}(\mathbf{k})\|^2 2\pi k.$$

The important quantities are

- Kinetic Energy: $\int E(k)\, dk$
- Enstrophy: $\int k^2 E(k)\, dk$
- Energy dissipation rate: $\int \nu k^2 E(k)\, dk$

- Enstrophy dissipation rate: $\int \nu k^4 E(k)\, dk$.

Observe that in 2D, we can't simultaneously conserve kinetic energy and enstrophy if energy only cascades to larger wave numbers (smaller scales). So instead, 2D flows have both an inverse cascade of energy to larger scales and a forward cascade to smaller scales.

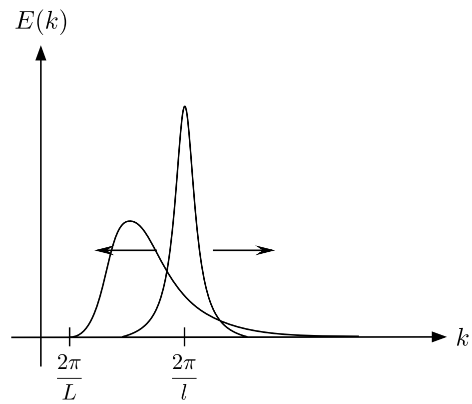



Figure 1.2: The forward/inverse cascade.

Can we see the form of $E(k)$? Notice that $[\epsilon] = L^2/T^3$, $[\chi] = 1/T^3$, $[k] = 1/L$, and $[E(k)] = L^3/T^2$. Dimensional reasoning gives us two choices: $\epsilon^{2/3}k^{-5/3}$ and $\chi^{2/3}k^{-3}$. It has been argued that nature follows the smaller of these two choices, for a given $k$.

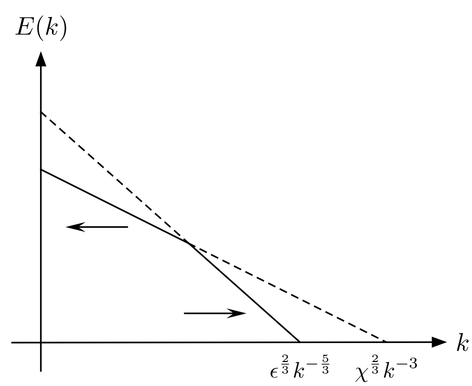



Figure 1.3: The 2D energy spectrum.

In particular if we say

$$E(k) \sim \chi^{2/3}k^{-3}$$

for large $k$, then

$$\begin{aligned}\chi &= \int^{2\pi/\eta} \nu k^4 E(k)\, dk \\ &\sim \int^{2\pi/\eta} \nu k^4 \cdot \chi^{2/3} k^{-3}\, dk \\ &\sim \chi^{2/3} \frac{\nu}{\eta^2}\end{aligned}$$

for large $k$. Solving for $\eta$ tells us

$$\eta = \left(\frac{\nu^3}{\chi}\right)^{\frac{1}{6}}$$

is the smallest length scale at which energy-driven structures are observable before viscosity dominates. But the most noticeable feature of 2D turbulence is the cascade of energy to large scales. This is featured on the cover of the text for this course!

## 2 Introduction to Existence and Uniqueness

Let's begin with a simple example. Consider the following ode for $X(t)$:

$$\frac{dX}{dt} = X^2,\ X(0) = x_0 \in \mathbb{R}.$$

Questions we might ask:

1. Does a solution exist for $t > 0$?
2. Can we find smooth solutions?
3. Will these solutions exist for all $t > 0$ or just some $t > 0$?

So there are two types of solutions: "global" solutions which exist for all time, and "local" solutions which exist for some time $t > 0$. In general, the answer to the questions above depend on the initial data. From algebraic considerations we know a solution is

$$X(t) = \frac{x_0}{1 - x_0 t}.$$

If $x_0 < 0$, then $X(t)$ exists for all $t > 0$, and $X(t) \to 0$ as $t \to \infty$. If $x_0 > 0$, then $X(t) \to \infty$ as $t \to 1/x_0$. What if we try to continue $X(t)$ beyond the singularity at $1/x_0$? Suppose we allow smoothness to fail at $t = 1/x_0$. For any $0 \leq a \leq \infty$ we can produce such a solution:

$$X(t) = \begin{cases} \frac{x_0}{1-x_0 t} & 0 \leq t < \frac{1}{x_0} \\ \frac{-|a|}{1+|a|\left(t - \frac{1}{x_0}\right)} & t > \frac{1}{x_0} \end{cases}.$$

This is horribly non-unique!

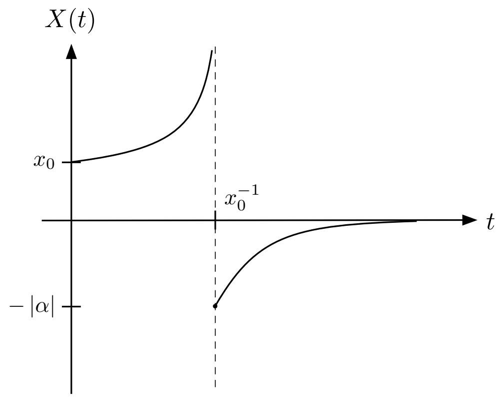



Figure 2.1: Finite time blow-up.

Now we'll discuss existence and uniqueness of solutions to systems of ode. Consider $z(t) = \left(z^1(t), \ldots, z^N(t)\right) \in \mathbb{C}^N$ with

$$|z|^2 = \sqrt{|z^1|^2 + \cdots + |z^N|^2}.$$

The setup is

$$\frac{dz}{dt} = F(z)$$

with $F : \mathbb{C}^N \to \mathbb{C}^N$, so

$$\begin{aligned} \frac{dz^1}{dt} &= F^1\left(z^1, z^2, \ldots, z^N\right) \\ &\vdots \\ \frac{dz^N}{dt} &= F^N\left(z^1, z^2, \ldots, z^N\right). \end{aligned}$$

The initial data is

$$z(0) = z_0 \in \mathbb{C}^N.$$

**Definition.** $F : \mathbb{C}^N \to \mathbb{C}^N$ is said to be *Lipschitz continuous* if there exists $K \in \mathbb{R}$ so that for all $z, \tilde{z} \in \mathbb{C}^N$,

$$|F(z) - F(\tilde{z})| \leq K |z - \tilde{z}|.$$

Here $K$ is called the *Lipschitz constant.*

*Remark.* An alternate characterization of $K$ is

$$K = \sup_{z, \tilde{z} \in \mathbb{C}^N} \frac{|F(z) - F(\tilde{z})|}{|z - \tilde{z}|}.$$

**Example.** $F(x) = x^2$ is not Lipschitz, for

$$\begin{aligned} |F(x) - F(\tilde{x})| &= \left|x^2 - \tilde{x}^2\right| \\ &= |x + \tilde{x}| \, |x - \tilde{x}|, \end{aligned}$$

which is unbounded on $\mathbb{R}$. But $F(x) = \frac{x^2}{1+\epsilon|x|}$ with $0 < \epsilon \ll 1$ is Lipschitz. Here, $K \sim O\left(\epsilon^{-1}\right)$.

**Theorem.** *If $F$ is Lipschitz continuous, then there is a unique global solution to the initial value problem*

$$\frac{dz}{dt} = F(z), \ z(0) = z_0 \in \mathbb{C}^N.$$

*Proof.* The idea is to write the ode in integral form, then iterate towards a solution. So we explicitly construct solutions to

$$z(t) = z_0 + \int_0^t F(z(t'))\, dt'.$$

Start from $n = 0$ and define "Picard iterates"

$$z_{n+1}(t) = z_0 + \int_0^t F(z_n(t'))\, dt'.$$

So

$$\begin{aligned} z_1(t) &= z_0 + \int_0^t F(z_0)\, dt' = z_0 + F(z_0) \cdot t \\ z_2(t) &= z_0 + \int_0^t F(z_1(t'))\, dt' \\ &\vdots \end{aligned}$$

and so forth. We need to demonstrate convergence. Let $K$ be the Lipschitz constant for $F$, and consider $t \in [0, T]$ for $T < K^{-1}$. Each $z_n$ is continuous. We'll show that the sequence of continuous functions $\{z_n(t)\}$ converges in the space

$$C\left([0, T]; \mathbb{C}^N\right) = \left\{f : [0, T] \to \mathbb{C}^N \text{ s.t.} f \text{ is continuous}\right\}$$

with the norm

$$\|f\| = \sup_{t \in [0,T]} |f(t)|.$$

We'll continue this next time. □

## Lecture 22: Existence and Uniqueness (ctd.)

Recall the following definition:

**Definition 1.** $F : \mathbb{C}^N \to \mathbb{C}^N$ is *Lipschitz continuous* if there exists $K \in \mathbb{R}$ so that

$$|F(z) - F(z')| \leq K\,|z - z'|\,.$$

We were in the middle of proving a theorem giving sufficient conditions for existence and uniqueness in ode.

## 1 Existence and Uniqueness for ODE

**Theorem 1.** *If $F$ is Lipshitz, then the initial value problem*

$$\dot{z} = F(z)\,,\ \ z(0) = z_0 \in \mathbb{C}^N$$

*has a unique global (smooth) solution.*

*Proof.* We proceed by Picard iteration. Define a sequence of continuous functions $\{z_{n+1}\}$ via

$$z_{n+1}(t) = z_0 + \int_0^t F(z_n(t'))\,dt'$$

for $t \in [0,T]$ where $T < K^{-1}$. Now we'll show the $z_n(t)$ converge in $C\left([0,T]\,;\mathbb{C}^N\right)$ in the norm

$$\|z\| = \sup_{t\in[0,T]} |z(t)|\,.$$

Consider that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \sup_{t\in[0,T]} \left|\int_0^t \left[F(z_n(t')) - F(z_{n-1}(t'))\right] dt'\right| \\
&\leq \sup_{t\in[0,T]} \int_0^t \left|F(z_n(t')) - F(z_{n-1}(t'))\right| dt' \\
&\leq \sup_{t\in[0,T]} K \int_0^t \left|z_n(t') - z_{n-1}(t')\right| dt' \\
&\leq \sup_{t\in[0,T]} \left\{K \cdot \sup_{t'\in[0,T]} \left|z_n(t') - z_{n-1}(t')\right| \cdot t\right\} \\
&= KT\|z_n - z_{n-1}\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|z_k - z_{k-1}\| &\leq (KT)\,\|z_{k-1} - z_{k-2}\| \\
&\leq (KT)^2\,\|z_{k-2} - z_{k-3}\| \\
&\leq (KT)^{k-1}\,\|z_1 - z_0\|,
\end{aligned}$$

and as $T < K^{-1}$, we have

$$\|z_k - z_{k-1}\| \to 0$$

as $k \to \infty$. Now write

$$\begin{aligned} z_n(t) &= z_0 + (z_1 - z_0) + (z_2 - z_1) + \cdots + (z_n - z_{n-1}) \\ \implies |z_n(t)| &\le |z_0| + \sum_{k=1}^{n} |z_k - z_{k-1}| \\ \implies \|z_n\| &\le \|z_0\| + \|z_1 - z_0\| \sum_{k=1}^{n} (KT)^{k-1}. \end{aligned}$$

Since

$$\sum_{k=1}^{n} (KT)^{k-1} \le \sum_{k=1}^{\infty} (KT)^{k-1} = \frac{1}{1-KT},$$

we can conclude that $\{z_n(t)\}$ is a uniformly convergent sequence of continuous functions.[1] Thus,

$$z(t) = \lim_{n\to\infty} z_n(t)$$

is a continuous function on $[0, T]$.

To see $z(t)$ satisfies the differential equation, we must show

$$\int_0^t F(z_n(t'))\, dt' \longrightarrow \int_0^t F(z(t'))\, dt'$$

as $t \to \infty$ and in the given norm. So write

$$\begin{aligned} \sup_{t\in[0,T]} \left| \int_0^t F(z_n(t'))\, dt' - \int_0^t F(z(t'))\, dt' \right| &\le \sup_{t\in[0,T]} \int_0^t |F(z(t')) - F(z_n(t'))|\, dt' \\ &\le KT\|z - z_n\| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. To finish the proof, we must extend the solution to $t > 0$. Just apply the same process with new initial conditions at $t = T$ and go forward in steps. But if we had uniqueness, then we could restart the process at $T - \delta$ and smoothness (differentiability) would be guaranteed.

So to prove uniqueness, suppose $z(t)$ and $\tilde{z}(t)$ both satisfy the ivp:

$$\frac{dz}{dt} = F(z) \text{ and } \frac{d\tilde{z}}{dt} \text{ with } z(0) = z_0 = \tilde{z}(0).$$

Then define $y(t) = z(t) - \tilde{z}(t)$. We have

$$\frac{dy}{dt} = F(z(t)) - F(\tilde{z}(t)) = F(z(t)) - F(z(t) - y(t)),$$

[1] Proof of this is by completeness of $\mathbb{C}$ and the Weierstrass M-test.

so

$$\begin{aligned}\frac{d}{dt}\left|y(t)\right|^2 &= y^*\left(F(z)-F(z-y)\right)+y\left(F^*(z)-F(z-y)^*\right)\\ &\le 2\left|y\right|\left|F(z)-F(z-y)\right|\\ &\le 2K\left|y\right|^2\end{aligned}$$

by the Lipshitz condition. By Gronwall,

$$\left|y(t)\right|^2 \le \left|y(0)\right|^2 e^{2Kt} = 0$$

which proves uniqueness. □

How can we generalize the idea of Lipshitz continuous to problems like Navier-Stokes?

**Definition 2.** We say $F:\mathbb{C}^N\to\mathbb{C}^N$ is *locally Lipshitz* if for each bounded set $A\subset\mathbb{C}^N\times\mathbb{C}^N$ there exists a bounded $K(z,z')\le K_A<\infty$ so that

$$\left|F(z)-F(z')\right|\le K(z,z')\left|z-z'\right|$$

on $A$.

**Theorem 2.** *If $F$ is locally Lipshitz, then the initial value problem*

$$\dot z = F(z)\,,\ \ z(0)=z_0\in\mathbb{C}^N$$

*has a unique local (smooth) solution.*

*Proof.* Picard iteration, but locally. Choose $c\in\mathbb{R}$ so that $c>|z_0|$ and let

$$K'=\sup_{|z|,|\tilde z|<2c} K(z,\tilde z)\,.$$

Let $T=\left(K'+\left|F(z_0)\right|/c\right)^{-1}$, then construct a solution on $[0,T]$ via Picard.

Now show that all iterates stay in a ball of radius $2c$. So we proceed via induction. For the base case, observe that

$$|z_0|<c<2c,$$

and since

$$z_1(t)=z_0+F(z_0)\,t,$$

we also have

$$\begin{aligned}\left|z_1(t)\right| &\le |z_0|+\left|F(z_0)\right|T\\ &= |z_0|+\frac{\left|F(z_0)\right|}{K'+\frac{|F(z_0)|}{c}}\\ &\le c+\frac{c}{1+\frac{K'}{|F(z_0)|}}\\ &< 2c.\end{aligned}$$

For the inductive step, suppose $\|z_1\|, \ldots, \|z_n\| < 2c$, and argue that $\|z_{n+1}\| < 2c$. Write

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \sup_{t \in [0,T]} \left| \int_0^t \left[ F\left(z_n\left(t'\right)\right) - F\left(z_{n-1}\left(t'\right)\right)\right] dt' \right| \\
&\leq K'T \|z_n - z_{n-1}\| \\
&\leq \cdots \\
&\leq \left(K'T\right)^n \|z_1 - z_0\| \\
&\leq \left(K'T\right)^n \left|F\left(z_0\right)\right| T,
\end{aligned}$$

then write

$$\begin{aligned}
\|z_{n+1}\| &= \|z_0 + \sum_{m=0}^{n} \left(z_{m+1} - z_m\right)\| \\
&\leq \|z_0\| + \sum_{m=0}^{n} \|z_{m+1} - z_m\| \\
&\leq \|z_0\| + \sum_{m=0}^{n} \left(K'T\right)^m \left|F\left(z_0\right)\right| T \\
&\leq \|z_0\| + \frac{\left|F\left(z_0\right)\right| T}{1 - K'T} \\
&\leq 2c
\end{aligned}$$

by choice of $T$.

Now proceed as before: iterate forward in $T$, then prove uniqess. □

But the solution is not global in general. The following example shows why.

**Example 1.** Consider the differential equation

$$\frac{dx}{dt} = x^2$$

which has local solution

$$x\left(t\right) = \frac{x_0}{1 - x_0 t}.$$

This has finite-time blow-up. But $x^2$ is locally Lipshitz, so the theorem guarantees the uniqueness of this solution locally.

However, there are global solutions to locally Lipshitz differential equations.

**Example 2.** Consider

$$\frac{dx}{dt} = -\alpha x^3$$

with $\alpha > 0$. Say $x\left(0\right) = x_0$, then a solution is

$$x\left(t\right) = \frac{x_0}{\sqrt{1 + 2x_0^2 \alpha t}}.$$

This is certainly defined for all $t \geq 0$, but $-\alpha x^3$ is only locally Lipshitz.

How would we demonstrate the existence of the global solution in the previous example, even though we only have local Lipshitz? The idea is as follows: If we know a priori that

$$|x(t)| \leq C(x_0) < \infty,$$

then there is no problem constructing solutions for $t \geq 0$. So if we have a priori bounds on solutions, then local Lipshitz is effectively global Lipshitz, and we have continuous and unique solutions for all time.

But note that local Lipshitz is necessary. The canonical example is as follows:

**Example 3.** Consider the differential equation

$$\frac{dx}{dt} = \sqrt{|x|}.$$

This is not locally Lipshitz. And for $x(0) = 0$, we can exhibit more than one solution. One solution is $x(t) = 0$, and another is $x(t) = \frac{1}{4}t^2$. In fact there are an infinite number of solutions:

$$x(t) = \begin{cases} 0 & 0 \leq t \leq t' \\ \frac{1}{4}(t - t')^2 & t \geq t' \end{cases}.$$

So we need at least local Lipshitz for any sort of uniqueness.

## Lecture 23: Navier-Stokes in Fourier Space and Galerkin Truncations

Last time we gave a proof of the classic existence and uniqueness theorem for the ode $\dot{z} = F(z)$ with Lipshitz continuous $F$. To do so, we constructed "Picard iterates"

$$z_{n+1}(t) = z_0 + \int_0^t F(z_n(s))\, ds$$

and demonstrated the existence of a limiting function $z_\infty$ which solved the ode in question.

We know the following facts concerning the ode $\dot{z} = F(z)$ with $z \in \mathbb{C}^N$:

1. If $F$ is Lipshitz continuous then the ode has a unique global solution.
2. If $F$ is locally Lipshitz continuous then there exists a unique local solution.
3. If $F$ is locally Lipshitz and we have an a priori bound on the solution then there exists a global solution.
4. Without the local Lipshitz condition, there may not be a unique solution.

Today we'll construct a sequence of approximations to Navier-Stokes, each of which falls under the third case.

### 1 The Fourier Representation of Navier-Stokes

The equations are

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p &= \nu\triangle\mathbf{u} + \mathbf{f}(\mathbf{x}) \\ \nabla.\mathbf{u} &= 0 \end{aligned}$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$. The domain is a cube with periodic boundary conditions.

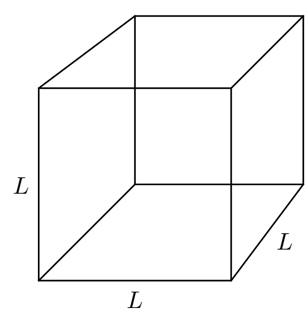



Figure 1.1: The domain of interest.

W.l.o.g., we can demand

$$\nabla.\mathbf{f} = \nabla.\mathbf{u}_0 = 0.$$

The Fourier decomposition of $\mathbf{u}$ is

$$\mathbf{u}(\mathbf{x},t)=\sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}\hat{\mathbf{u}}(\mathbf{k},t)$$

with

$$\mathbf{k}=\frac{2\pi}{L}\left(\hat{\mathbf{i}}n_1+\hat{\mathbf{j}}n_2+\hat{\mathbf{k}}n_3\right),\quad n_i\in\mathbb{Z}.$$

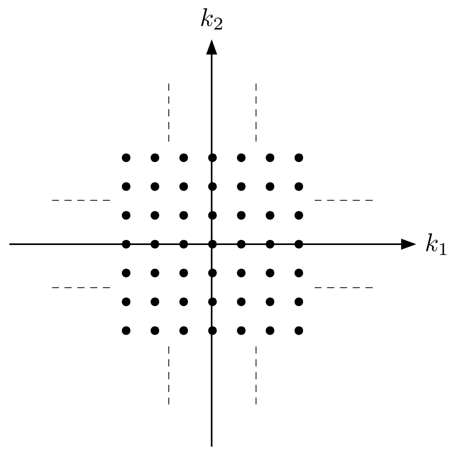



Figure 1.2: Wavenumber space.

As usual,

$$\hat{\mathbf{u}}(\mathbf{k},t)=\frac{1}{L^3}\int e^{-i\mathbf{k}.\mathbf{x}}\mathbf{u}(\mathbf{x},t)\ d\mathbf{x}.$$

We can write the initial condition as

$$\mathbf{u}_0(\mathbf{x})=\sum_{\mathbf{k}}\hat{\mathbf{u}}_0(\mathbf{k})\,e^{i\mathbf{k}.\mathbf{x}},$$

and reinterpret the divergence-free condition on $\mathbf{u}_0$ as

$$0=\nabla.\mathbf{u}_0=\sum_{\mathbf{k}} i\mathbf{k}.\hat{\mathbf{u}}_0(\mathbf{k})\,e^{i\mathbf{k}.\mathbf{x}}\iff\mathbf{k}.\hat{\mathbf{u}}_0(\mathbf{k})=0.$$

And because $\mathbf{u}(\mathbf{x},t)\in\mathbb{R}^3$, we must have

$$\hat{\mathbf{u}}(\mathbf{k},t)^*=\hat{\mathbf{u}}(-\mathbf{k},t)\,.$$

Now we'll rewrite Navier-Stokes in Fourier spaces, i.e., we'll find a set of odes involving the Fourier coefficients $\hat{\mathbf{u}}(\mathbf{k},t)$. The first step is to multiply Navier-Stokes by $e^{-i\mathbf{k}.\mathbf{x}}$ and integrate: we do this term by term. The $\partial_t\mathbf{u}$ term becomes

$$\frac{1}{L^3}\int e^{-i\mathbf{k}.\mathbf{x}}\partial_t\mathbf{u}(\mathbf{x},t)\ d\mathbf{x}=\frac{d}{dt}\left[\frac{1}{L^3}\int e^{-i\mathbf{k}.\mathbf{x}}\mathbf{u}(\mathbf{x},t)\ d\mathbf{x}\right]=\frac{d}{dt}\hat{\mathbf{u}}(\mathbf{k},t)\,.$$

The $\triangle\mathbf{u}$ term becomes

$$\frac{1}{L^3}\int e^{-i\mathbf{k}.\mathbf{x}}\triangle\mathbf{u}(\mathbf{x},t)\ d\mathbf{x} \quad = \quad \frac{1}{L^3}\int\left[\nabla.\left(e^{-i\mathbf{k}.\mathbf{x}}\nabla\mathbf{u}(\mathbf{x},t)\right)+i\mathbf{k}.\nabla\mathbf{u}(\mathbf{x},t)\,e^{-i\mathbf{k}.\mathbf{x}}\right]\ d\mathbf{x};$$

the first integral is zero due to periodic boundary conditions, so

$$\begin{aligned}\frac{1}{L^3}\int e^{-i\mathbf{k}.\mathbf{x}}\triangle\mathbf{u}(\mathbf{x},t)\ d\mathbf{x} &= \frac{i}{L^3}\mathbf{k}.\int e^{-i\mathbf{k}.\mathbf{x}}\nabla\mathbf{u}(\mathbf{x},t)\ d\mathbf{x}\\ &= \frac{i}{L^3}\mathbf{k}.i\mathbf{k}\int e^{-i\mathbf{k}.\mathbf{x}}\mathbf{u}(\mathbf{x},t)\ d\mathbf{x}\\ &= -\|\mathbf{k}\|^2\hat{\mathbf{u}}(\mathbf{k},t)\,.\end{aligned}$$

The non-linear term becomes

$$\begin{aligned}\frac{1}{L^3}\int e^{-i\mathbf{k}.\mathbf{x}}\mathbf{u}.\nabla\mathbf{u}\,d\mathbf{x} &= \frac{1}{L^3}\int\left[e^{-i\mathbf{k}.\mathbf{x}}\left(\sum_{\mathbf{k}'}e^{i\mathbf{k}'.\mathbf{x}}\hat{\mathbf{u}}(\mathbf{k}',t)\right).\nabla\left(\sum_{\mathbf{k}''}e^{i\mathbf{k}''.\mathbf{x}}\hat{\mathbf{u}}(\mathbf{k}'',t)\right)\right]d\mathbf{x}\\ &= \sum_{\mathbf{k}',\mathbf{k}''}\hat{\mathbf{u}}(\mathbf{k}',t)\,.i\mathbf{k}''\hat{\mathbf{u}}(\mathbf{k}'',t)\underbrace{\frac{1}{L^3}\int e^{i\left(\mathbf{k}'+\mathbf{k}''-\mathbf{k}\right).\mathbf{x}}\,d\mathbf{x}}_{\delta_{\mathbf{k}'+\mathbf{k}'',\mathbf{k}}}\\ &= \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}i\mathbf{k}''.\hat{\mathbf{u}}(\mathbf{k}',t)\,\hat{\mathbf{u}}(\mathbf{k}'',t)\,.\end{aligned}$$

Now we'll deal with the pressure term. Taking the divergence of Navier-Stokes and manipulating the result yields

$$-\triangle p=\nabla.(\mathbf{u}.\nabla\mathbf{u})\,;$$

taking the Fourier transform then yields

$$\|\mathbf{k}\|^2\hat{p}(\mathbf{k},t)=i\mathbf{k}.i\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\mathbf{k}''.\hat{\mathbf{u}}(\mathbf{k}',t)\,\hat{\mathbf{u}}(\mathbf{k}'',t)$$

once we recall

$$\widehat{\nabla.\mathbf{v}}(\mathbf{k})=i\mathbf{k}.\hat{\mathbf{v}}(\mathbf{k})\,.$$

So

$$\hat{p}(\mathbf{k},t)=-\frac{\mathbf{k}}{\|\mathbf{k}\|^2}\cdot\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\mathbf{k}''.\hat{\mathbf{u}}(\mathbf{k}',t)\,\hat{\mathbf{u}}(\mathbf{k}'',t)\,,$$

and since

$$\widehat{\nabla p}(\mathbf{k})=i\mathbf{k}\hat{p}(\mathbf{k})$$

we find

$$\widehat{\nabla p}(\mathbf{k})=-i\frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}\cdot\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\mathbf{k}''.\hat{\mathbf{u}}(\mathbf{k}',t)\,\hat{\mathbf{u}}(\mathbf{k}'',t)\,.$$

We can interpret the dot as a matrix-vector product where

$$\mathbf{k}\mathbf{k}=\begin{pmatrix}k_1k_1 & k_1k_2 & k_1k_3\\ k_2k_1 & \ddots & \\ k_3k_1 & & k_3k_3\end{pmatrix}.$$

So we have arrived at an infinite set of odes constituting Navier-Stokes in Fourier space:

$$\frac{d}{dt}\hat{\mathbf{u}}(\mathbf{k},t) = -\nu\|\mathbf{k}\|^2\hat{\mathbf{u}}(\mathbf{k},t) - i\left(I - \frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}\right)\cdot\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\mathbf{k}''.\hat{\mathbf{u}}(\mathbf{k}',t)\,\hat{\mathbf{u}}(\mathbf{k}'',t) + \hat{\mathbf{f}}(\mathbf{k},t).$$

This relation holds for $\mathbf{k} \neq 0$; we'll deal with this case in a bit.

Observe that the operator $\mathcal{P} = I - \frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}$ is a projection operator onto vectors perpendicular to $\mathbf{k}$. Indeed $\mathcal{P}$ is a projector, as

$$\begin{aligned}
\mathcal{P}^2 &= \left(I - \frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}\right)\left(I - \frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}\right)\\
&= I - 2\frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2} + \frac{\mathbf{k}\|\mathbf{k}\|^2\mathbf{k}}{\|\mathbf{k}\|^4}\\
&= I - \frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}\\
&= \mathcal{P}.
\end{aligned}$$

Notice also that $\mathcal{P}$ is symmetric. Finally note that since

$$\mathbf{k}.\left(I - \frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}\right) = \mathbf{k} - \frac{\|\mathbf{k}\|^2\mathbf{k}}{\|\mathbf{k}\|^2} = 0,$$

our set of odes is self-consistent.

For $\mathbf{k} = 0$, we look at the time-evolution of

$$\hat{\mathbf{u}}(0,t) = \frac{1}{L^3}\int \mathbf{u}(\mathbf{x},t)\;d\mathbf{x}.$$

Since

$$\int \nabla p\, d\mathbf{x} = \int \nabla.(\mathbf{u}\mathbf{u})\;d\mathbf{x} = 0,$$

$\hat{\mathbf{u}}(0,t)$ evolves in time via

$$\frac{d}{dt}\hat{\mathbf{u}}(0,t) = \hat{\mathbf{f}}(0,t).$$

We can solve this forward in time trivially:

$$\hat{\mathbf{u}}(0,t) = \hat{\mathbf{u}}_0(0) + \int_0^t \hat{\mathbf{f}}(0,s)\;ds.$$

## 2 Galerkin Truncation to Navier-Stokes

From here on, we'll assume $\hat{\mathbf{u}}_0(0) = 0$ and $\hat{\mathbf{f}}(0,t) \equiv 0$ for simplicity, otherwise we'll complicate things without gaining any understanding. To build approximate solutions to Navier-Stokes, we could keep only the Fourier modes of $\mathbf{u}$ with $\|\mathbf{k}\| < K$ for positive $K$. This would yield $\sim K^3$ modes. Or we could keep the modes with $k_i < K$, again yielding $\sim K^3$ modes. Both of these truncations are pictured on the next page.

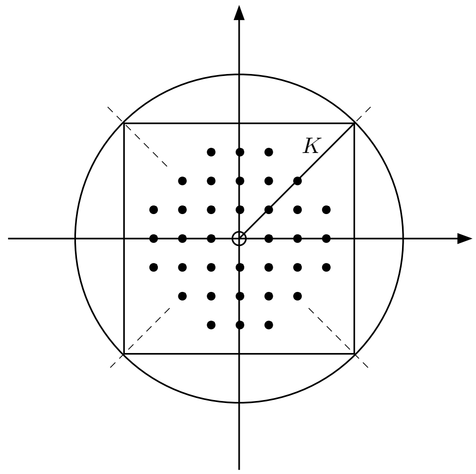



Figure 2.1: Two truncation methods.

The first method of truncation is called "Galerkin truncation", and is what we'll pursue throughout the remainder of the course.

Consider the Galerkin projection operator $\mathbb{P}^N$ which projects onto modes into the set $\{\mathbf{k} \text{ s.t.} \|\mathbf{k}\| < K\}$:

$$\mathbb{P}^N\left(\hat{\mathbf{u}}\left(\mathbf{k},t\right)\right) = \begin{cases} \hat{\mathbf{u}}\left(\mathbf{k},t\right) & \|\mathbf{k}\| < K \\ 0 & \|\mathbf{k}\| \geq K \end{cases}.$$

We'll build the $N^{\text{th}}$ approximate to $\mathbf{u}$ by requiring that the $\hat{\mathbf{u}}^N\left(\mathbf{k},t\right)$ satisfy Navier-Stokes in Fourier space whenever $\|\mathbf{k}\| < K$. Otherwise, we'll require $\hat{\mathbf{u}}^N\left(\mathbf{k},t\right) \equiv 0$. Now the odes are

$$\frac{d}{dt}\hat{\mathbf{u}}^N\left(\mathbf{k},t\right) = -\nu\|\mathbf{k}\|^2\hat{\mathbf{u}}^N\left(\mathbf{k},t\right) - i\left(I - \frac{\mathbf{k}\mathbf{k}}{\|\mathbf{k}\|^2}\right)\cdot\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\mathbf{k}''.\hat{\mathbf{u}}^N\left(\mathbf{k}',t\right)\hat{\mathbf{u}}^N\left(\mathbf{k}'',t\right) + \underbrace{\hat{\mathbf{f}}^N\left(\mathbf{k},t\right)}_{\mathbb{P}^N\left(\hat{\mathbf{f}}(\mathbf{k},t)\right)}.$$

Note the sum here is now a finite sum, due to the truncation. The initial conditions are given by $\hat{\mathbf{u}}^N\left(\mathbf{k},0\right) = \mathbb{P}^N\left(\hat{\mathbf{u}}_0\left(\mathbf{k}\right)\right)$. This is a set of approximately $K^3$ odes, and is locally Lipshitz in time. Next time, we'll show that

$$\sum_{\mathbf{k}}\|\hat{\mathbf{u}}^N\left(\mathbf{k},t\right)\|^2 < M$$

for some $M \in \mathbb{R}$ which is independent of $N$, so establishing an a priori bound.

## Lecture 24: Global Existence of Solutions to Galerkin Truncated NSE

We have

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \nabla p &= \nu \triangle \mathbf{u} + \mathbf{f} \\ \nabla.\mathbf{u} &= 0 \end{aligned}$$

in a periodic box with side length $L$.

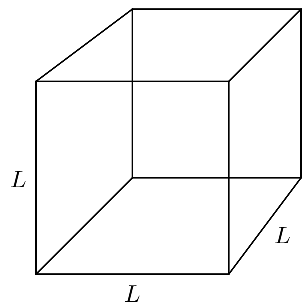



Figure 0.1: The domain of interest.

W.l.o.g., we demand

$$\nabla.\mathbf{f} = 0 \text{ and } \int \mathbf{f}(\mathbf{x})\, d\mathbf{x} = 0.$$

The initial condition is $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ and satisfies

$$\nabla.\mathbf{u}_0 = 0 \text{ and } \int \mathbf{u}_0(\mathbf{x})\, d\mathbf{x} = 0.$$

We wrote

$$\hat{\mathbf{u}}(\mathbf{k}, t) = \frac{1}{L^3} \int e^{-i\mathbf{k}.\mathbf{x}} \mathbf{u}(\mathbf{x}, t)\, d\mathbf{x}$$

with

$$\mathbf{k} = \hat{\mathbf{i}} \frac{2\pi n_x}{L} + \hat{\mathbf{j}} \frac{2\pi n_y}{L} + \hat{\mathbf{k}} \frac{2\pi n_z}{L}$$

where $n_i = \ldots, -2, -1, 0, 1, 2, \ldots$. With this we showed NSE is equivalent to

$$\frac{d}{dt} \hat{\mathbf{u}}(\mathbf{k}, t) = -\nu k^2 \hat{\mathbf{u}}(\mathbf{k}, t) - i \left( I - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}', t) . \mathbf{k}'' \hat{\mathbf{u}}(\mathbf{k}'', t) + \hat{\mathbf{f}}(\mathbf{k}).$$

Also we showed $\nabla.\mathbf{u} = 0$ is equivalent to

$$\mathbf{k}.\hat{\mathbf{u}}(\mathbf{k}, t) = 0.$$

This is an infinite set of ordinary differential equations, and as they are all coupled it is truly an infinite set.

Recall we discussed two ways to truncate the equations; the first of these methods was the "Galerkin truncation," pictured below.

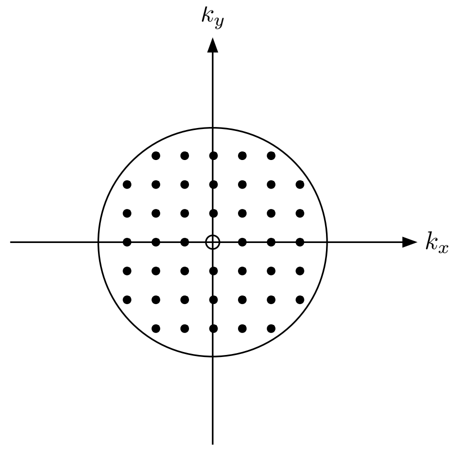



Figure 0.2: Galerkin truncation of Fourier modes.

Note that $\mathbf{k} = 0$ is not in the set. Now let $N$ be the order of a Galerkin truncation, i.e., the number of wave-numbers considered. Then the Galerkin truncated ode is

$$\frac{d}{dt}\hat{\mathbf{u}}^{(N)}(\mathbf{k},t) = -\nu k^2 \hat{\mathbf{u}}^{(N)} - i\mathcal{P}\left[\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \hat{\mathbf{u}}^{(N)}(\mathbf{k}',t)\,.\mathbf{k}''\hat{\mathbf{u}}^{(N)}(\mathbf{k},t)\right] + \hat{\mathbf{f}}^{(N)}(\mathbf{x}) \tag{0.1}$$

where we've set

$$\hat{\mathbf{u}}^{(N)}(\mathbf{k},t) \equiv 0$$

whenever $\mathbf{k}$ is outside the collection and

$$\hat{\mathbf{f}}^{(N)}(\mathbf{k}) = \begin{cases} \hat{\mathbf{f}}(\mathbf{k},t) & \mathbf{k} \text{ in the set} \\ 0 & \text{otherwise} \end{cases}.$$

As initial conditions,

$$\hat{\mathbf{u}}^{(N)}(\mathbf{k},t) = \begin{cases} \hat{\mathbf{u}}_0(\mathbf{k}) & \mathbf{k} \text{ in the set} \\ 0 & \text{otherwise} \end{cases}.$$

This is a set of $N$ ivps. set is locally Lipshetz, hence there exists a local unique solution. But to get global existence, we need an a priori bound.

## 1 A Priori Bound for Galerkin Approximations

Define

$$\mathbf{u}^{(N)}(\mathbf{x},t) = \sum_{\mathbf{k} \text{ in the set}} e^{i\mathbf{k}.\mathbf{x}} \hat{\mathbf{u}}^{(N)}(\mathbf{k},t)\,.$$

Also define a new projection operator $\mathbb{P}^N$ on periodic functions $g(\mathbf{x})$ by

$$\left(\mathbb{P}^N g\right)(\mathbf{x}) = \sum_{\mathbf{k} \text{ in the set}} e^{i\mathbf{k}.\mathbf{x}} \hat{g}(\mathbf{k})\,.$$

This is projection onto the $N$ retained Fourier modes. Note that

$$\left(\mathbb{P}^N\right)^2 = \mathbb{P}^N,$$

so indeed $\mathbb{P}^N$ is a projector. Also note

$$\begin{aligned}
\int g(\mathbf{x})\left(\mathbb{P}^N h(\mathbf{x})\right) d\mathbf{x} &= L^3 \sum_{\mathbf{k}} \hat{g}(\mathbf{k})^* \, \mathbb{P}^{\hat{N}} h(\mathbf{k}) \\
&= L^3 \sum_{\mathbf{k} \text{ in the set}} \hat{g}(\mathbf{k})^* \, \mathbb{P}^{\hat{N}} h(\mathbf{k}) \\
&= L^3 \sum_{\mathbf{k}} \mathbb{P}^{\hat{N}} g^* (\mathbf{k}) \, \hat{h}(\mathbf{k}) \\
&= \int \left(\mathbb{P}^N g\right)(\mathbf{x}) \, h(\mathbf{x}) \, d\mathbf{x},
\end{aligned}$$

so $\mathbb{P}^N$ is self-adjoint.

Now multiply (0.1) by $e^{i\mathbf{k}.\mathbf{x}}$:

$$e^{i\mathbf{k}.\mathbf{x}} \frac{d}{dt} \hat{\mathbf{u}}^{(N)}(\mathbf{k},t) = -\nu k^2 \hat{\mathbf{u}}^{(N)} e^{i\mathbf{k}.\mathbf{x}} - i\mathcal{P}\left[\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} e^{i\mathbf{k}'.\mathbf{x}} \hat{\mathbf{u}}^{(N)}(\mathbf{k}',t) . \mathbf{k}'' e^{i\mathbf{k}''.\mathbf{x}} \hat{\mathbf{u}}^{(N)}(\mathbf{k},t)\right] + \mathbf{f}^{(N)}(\mathbf{x}) \, e^{i\mathbf{k}.\mathbf{x}}.$$

So then write

$$\frac{\partial}{\partial t} \mathbf{u}^{(N)}(\mathbf{x},t) = \nu \triangle \mathbf{u}^{(N)}(\mathbf{x},t) - \mathbb{P}^N \left(\mathbf{u}^{(N)} . \nabla \mathbf{u}^{(N)} + \nabla p^{(N)}\right) + \left(\mathbb{P}^N \mathbf{f}\right)(\mathbf{x})$$

and after rearranging,

$$\frac{\partial}{\partial t} \mathbf{u}^{(N)}(\mathbf{x},t) + \mathbb{P}^N \left(\mathbf{u}^{(N)} . \nabla \mathbf{u}^{(N)} + \nabla p^{(N)}\right) = \nu \triangle \mathbf{u}^{(N)}(\mathbf{x},t) \left(\mathbb{P}^N \mathbf{f}\right)(\mathbf{x}) . \tag{1.1}$$

The initial conditions are

$$\mathbf{u}(\mathbf{x},0) = \left(\mathbb{P}^N \mathbf{u}_0\right)(\mathbf{x}) .$$

This is almost NSE (in a rough sense, $\mathbb{P}^N \to I$ as $N \to \infty$). Note that

$$\begin{aligned}
\left(\mathbb{P}^N g\right)(\mathbf{x}) &= \sum_{\mathbf{k} \text{ in the set}} e^{i\mathbf{k}.\mathbf{x}} \hat{g}(\mathbf{k}) \\
&= \sum_{\mathbf{k} \text{ in the set}} e^{i\mathbf{k}.\mathbf{x}} \frac{1}{L^3} \int e^{-i\mathbf{k}.\mathbf{x}'} g(\mathbf{x}') \, d\mathbf{x}' \\
&= \int \hat{\mathbb{P}}(\mathbf{x},\mathbf{x}') \, g(\mathbf{x}') \, d\mathbf{x}'
\end{aligned}$$

with

$$\hat{\mathbb{P}}(\mathbf{x},\mathbf{x}') = \frac{1}{L^3} \sum_{\mathbf{k} \text{ in the set}} e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}.$$

As we take more and more wave numbers in the set, $\hat{\mathbb{P}}$ looks more and more like a delta function.

Now dot $\mathbf{u}^N(\mathbf{x},t)$ into (1.1) and integrate over the box:

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}^{(N)}(\cdot,t)\|_2^2\right) = -\nu\|\nabla\mathbf{u}^{(N)}(\cdot,t)\|_2^2 + \int \mathbf{u}^{(N)}(\mathbf{x},t)\,.\mathbf{f}(\mathbf{x})\;d\mathbf{x}.$$

By Parseval and Poincare,

$$\begin{aligned}\|\nabla\mathbf{u}^{(N)}\|_2^2 &= \sum_{\mathbf{k}\text{ in the set}} k^2\|\hat{\mathbf{u}}^{(N)}(\mathbf{k},t)\|^2\\ &\geq \frac{4\pi^2}{L^2}\sum_{\mathbf{k}\text{ in the set}}\|\hat{\mathbf{u}}(\mathbf{k},t)\|^2\\ &= \frac{4\pi^2}{L^2}\|\mathbf{u}^{(N)}\|_2^2.\end{aligned}$$

So by Cauchy-Schwarz,

$$\frac{d}{dt}\|\mathbf{u}^{(N)}\|_2 \leq -\nu\frac{4\pi^2}{L^2}\|\mathbf{u}^{(N)}\|_2 + \|f\|_2.$$

Now by Gronwall:

$$\|\mathbf{u}^{(N)}(\cdot,t)\|_2 \leq \|\mathbf{u}_0^{(N)}\|_2 e^{-\frac{4\pi^2}{L^2}\nu t} + \|f\|_2\left(\frac{1-e^{-\frac{4\pi^2}{L^2}\nu t}}{\frac{4\pi^2}{L^2}\nu}\right).$$

Note that

$$\begin{aligned}\|\mathbf{u}_0^{(N)}\|_2 &\leq \left(L^3\sum_{\mathbf{k}\text{ in the set}}\|\hat{\mathbf{u}}_0(\mathbf{k})\|^2\right)^{\frac{1}{2}}\\ &\leq \left(L^3\sum_{\mathbf{k}}\|\hat{\mathbf{u}}_0(\mathbf{k})\|^2\right)^{\frac{1}{2}}\\ &= \|\mathbf{u}_0\|_2,\end{aligned}$$

and hence

$$\|\mathbf{u}^{(N)}(\cdot,t)\|_2 \leq \|\mathbf{u}_0\|_2 e^{-\frac{4\pi^2}{L^2}\nu t} + L^2\|f\|_2\left(\frac{1-e^{-\frac{4\pi^2}{L^2}\nu t}}{4\pi^2\nu}\right).$$

We've obtained a uniform bound for all time and all $N$.

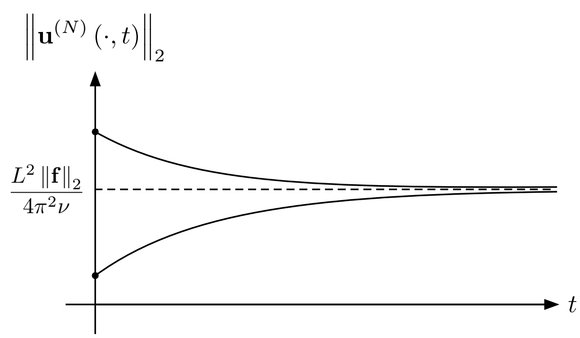



Figure 1.1: A priori bound on $\|\mathbf{u}^{(N)}(\cdot,t)\|_2$.

All this has brought us to the following questions:

- Does $\mathbf{u}^{(N)}(\mathbf{x}, t)$ possess a limit as $N \to \infty$?
- Does the limit satisfy NSE?
- Is the limit unique?
- Is the limit smooth?

Next time we'll discuss how difficult these questions really are.

## Lecture 25: Convergence of Galerkin Approximations and Weak NSE

The setup:

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p &= \nu\triangle\mathbf{u} + \mathbf{f}\\ \nabla.\mathbf{u} &= 0\end{aligned}$$

with $\nabla.\mathbf{f} = 0$ and initial conditions $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$. The domain:

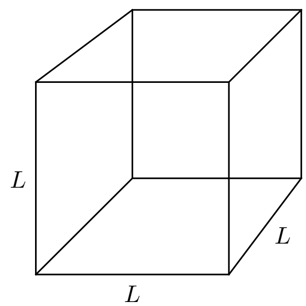



Figure 0.1: The domain of interest.

And there is no flow on average, i.e., $\int \mathbf{u}_0\, d\mathbf{x} = 0$ and $\int \mathbf{f}\, d\mathbf{x} = 0$. To approximate we projected onto $N$ Fourier modes via $\mathbb{P}^N$ to give Galerkin approximations

$$\mathbb{P}^N\left(\mathbf{u}^N\right) = \mathbf{u}^N(\mathbf{x},t).$$

Initial conditions truncate as

$$\mathbf{u}^N(\mathbf{x},0) = \left(\mathbb{P}^N\mathbf{u}_0\right)(\mathbf{x}).$$

Thus we arrive at the (approximated) conservation laws

$$\begin{aligned}\partial_t\mathbf{u}^N + \mathbb{P}^N\left(\mathbf{u}^N.\nabla\mathbf{u}^N\right) + \nabla p^N &= \nu\triangle\mathbf{u}^N + \left(\mathbb{P}^N\mathbf{f}\right)(\mathbf{x})\\ \nabla.\mathbf{u}^N &= 0.\end{aligned}$$

The first is really an integral-differential equation, as the $\mathbb{P}^N$ operator acts like an integral against a Dirichlet kernel.

Next, given

$$\|\mathbf{u}^N\|_2^2 = L^3\sum_{\mathbf{k}\in\mathcal{S}}\|\hat{\mathbf{u}}^N(\mathbf{k},t)\|^2,$$

where $\mathcal{S}$ is the set of retained wave numbers in the truncation, we derived the evolution equation

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}^N\|_2^2\right) = -\nu\|\nabla\mathbf{u}^N\|_2^2 + \int\mathbf{u}^N.\mathbf{f}.$$

This gave an a priori energy bound and thus global existence of (approximate) solutions. Today we look at convergence of these solutions; in particular, we'll ask if they converge at all.

# 1 Convergence of Approximate Solutions

We'd like to write $\mathbf{u}^N \to \mathbf{u}(\mathbf{x},t)$ in some sense. Recall that given a sequence of real numbers in a compact set there must exist a convergent subsequence. But the limit is not unique. In our situation, we have $\mathbf{u}^N \in L^2\left([0,L]^3\right)$. And given the a priori energy bound, we know the $\mathbf{u}^N$ are all contained in a closed (and bounded) subset of $L^2\left([0,L]^3\right)$. But $L^2\left([0,L]^3\right)$ is an infinite dimensional space! So this is not compactness.

Suppose we integrate the energy evolution equation on $[0,T]$:

$$\frac{1}{2}\|\mathbf{u}^N(\cdot,T)\|_2^2 + \nu\int_0^T \|\nabla\mathbf{u}^N(\cdot,t)\|_2^2\,dt = \frac{1}{2}\|\mathbf{u}_0^N\|_2^2 + \int_0^T \left(\int \mathbf{u}^N.\mathbf{f}\,d\mathbf{x}\right)dt. \tag{1.1}$$

Because $\mathbb{P}^N$ is a projection operator, and because of Cauchy-Schwarz, we can write

$$\frac{1}{2}\|\mathbf{u}^N(\cdot,T)\|_2^2 + \nu\int_0^T \|\nabla\mathbf{u}^N(\cdot,t)\|_2^2\,dt \le \frac{1}{2}\|\mathbf{u}_0\|_2^2 + C\cdot T$$

for some $C<\infty$ uniform in $N,T$. Thus

$$\nu\int_0^T \|\nabla\mathbf{u}^N(\cdot,t)\|_2^2\,dt \le \frac{1}{2}\|\mathbf{u}_0\|_2^2 + C\cdot T,$$

a uniform bound in $N$. This is extra information: $\mathbf{u}^N$ lies in a compact subset of $L^2$ in an appropriate topology. Recall that in the norm topology (the usual topology), we say $\psi_n \to 0$ iff $\|\psi_n\|_2 \to 0$.

**Definition 1.** In the *weak topology*, we say $\psi_n \to 0$ iff for all smooth $\phi$ we have $\int \phi\psi_n\,dx \to 0$, and then we write $\psi_n \rightharpoonup 0$.

The fact is that there are weakly convergent subsequences of $\{\mathbf{u}^N\}$ which satisfy the "weak form" of NSE. Suppose $\mathbf{u}(\mathbf{x},t)$ solves NSE. Then given a smooth, periodic, divergence-free $\mathbf{v}(\mathbf{x},t)$, we can write

$$\begin{aligned}\int_0^T\int_{\text{box}} \mathbf{v}.\partial_t\mathbf{u}\,d\mathbf{x}dt &+ \int_0^T\int_{\text{box}} \mathbf{v}.\left(\mathbf{u}.\nabla\mathbf{u}\right)\,d\mathbf{x}dt \\ &= \nu\int_0^T\int_{\text{box}} \mathbf{u}.\triangle\mathbf{v}\,d\mathbf{x}dt + \int_0^T\int \mathbf{v}.\mathbf{f}\,d\mathbf{x}dt,\end{aligned}$$

after dotting NSE by $\mathbf{v}$ and integrating. Integrating by parts, we get

$$\begin{aligned}\int \mathbf{v}(\mathbf{x},T).\mathbf{u}(\mathbf{x},T)\,d\mathbf{x} &- \int_0^T\int \partial_t\mathbf{v}.\mathbf{u}\,d\mathbf{x}dt - \int_0^T\int \mathbf{u}.\left(\nabla\mathbf{v}\right).\mathbf{u}\,d\mathbf{x}dt \\ &= \int \mathbf{v}(\mathbf{x},0).\mathbf{u}_0(\mathbf{x})\,d\mathbf{x} + \nu\int_0^T\int \mathbf{u}.\triangle\mathbf{v}\,d\mathbf{x}dt + \int_0^T\int \mathbf{v}.\mathbf{f}\,d\mathbf{x}dt.\end{aligned} \tag{1.2}$$

Also if we have a smooth, periodic $q(\mathbf{x})$, we can write

$$\int \mathbf{u}.\nabla q\,d\mathbf{x} = 0 \tag{1.3}$$

by the divergence-free condition.

**Definition 2.** We say $\mathbf{u}(\mathbf{x},t)$ satisfies the *weak form of NSE* if, given any smooth, periodic, and divergence-free $\mathbf{v}(\mathbf{x},t)$ and any smooth and periodic $q(\mathbf{x})$, we have (1.2) and (1.3).

It's a fact that "weak solutions" satisfy the energy equality (and subsequent inequality) (1.1). We'd like to ask if a weak limit has the same characteristic. First, consider functions which are periodic on $[0,L]$ in the following example.

**Example 1.** Consider the sequence $\{\phi_N\}$ given by

$$\phi_N(x) = \sqrt{2}\sin\left(\frac{2\pi N x}{L}\right).$$

Then we have

$$\|\phi_N\|_2^2 = L,$$

so $\phi_N \in \overline{B}(0,L)$ for all $N$. And the $\phi_N$ do not converge in the norm topology. Nevertheless,

$$\begin{aligned}\left|\int_0^L \psi(x)\,\phi_N(x)\,dx\right| &= \frac{L}{2\pi N}\left|\int_0^L \psi'(x)\sqrt{2}\cos\left(\frac{2\pi N x}{L}\right)dx\right| \\ &\leq \frac{L}{2\pi N}\left(\int_0^L (\psi'(x))^2\,dx\right)^{\frac{1}{2}} L^{\frac{1}{2}} \\ &\to 0\end{aligned}$$

as $N\to\infty$. So $\phi_N \to 0$ in the weak topology! Let $\phi$ be the weak-limit of $\phi_N$. We've found

$$\phi = 0 \text{ and } \|\phi\|_2 = 0,$$

but

$$\lim_{N\to\infty}\|\phi_N\| = \sqrt{L} \neq 0.$$

We've come to a general fact. Let $\mathbf{u}$ denote the weak limit of the Galerkin approximations, $\mathbf{u}^N$. Then

$$\|\mathbf{u}\|_2 \leq \lim_{N\to\infty}\|\mathbf{u}^N\|_2.$$

So what was the energy equality, (1.1), becomes an inequality in the weak limit. What would happen if this inequality is strict? This would indicate that some part of the energy dissapeared into the fine scales when we smear the $\mathbf{u}^N$ with a test function and integrate. But really we don't know if this inequality is strict or not. This is unsettling indeed.

It's even worse: the weak form of NSE is non-predictive. Consider the next example, first due to Leray in 1932.

**Example 2.** Suppose $\mathbf{x}\in\mathbb{R}^3$. Define $\mathbf{u}$ by

$$\mathbf{u}(\mathbf{x},t) = a(t)\,\nabla\phi(\mathbf{x})$$

for some $\phi$ with $\triangle\phi(\mathbf{x}) = 0$. We claim $\mathbf{u}$ is a weak solution of the (unforced) NSE, so long as $a$ satisfies

$$\int_0^T (a(t))^2\,dt < \infty.$$

It's clear from the definition that $\mathbf{u}$ satisfies (1.3), where now test functions have compact support. What about (1.2)? Since $\mathbf{v}$ is divergence-free (it is a test function), each term in (1.2) dissapears immediately expect for the third term on the left. But

$$\begin{aligned}\int \mathbf{u}.\nabla\mathbf{v}.\mathbf{u}\,d\mathbf{x} &= (a(t))^2 \int \phi_{,i} v_{j,i} \phi_{,j}\, d\mathbf{x} \\ &= -(a(t))^2 \int v_j \phi_{,i} \phi_{,ij}\, d\mathbf{x} \\ &= -(a(t))^2 \int \mathbf{v}.\nabla \left(\frac{1}{2}\|\nabla\phi\|^2\right) d\mathbf{x} \\ &= 0,\end{aligned}$$

for again $\mathbf{v}$ is divergence-free. So indeed $\mathbf{u}$ is a weak solution. But $\mathbf{u}$ lives on an infinite domain, and has infinite energy! This is entirely unphysical.

## Lecture 26: Uniqueness and Enstrophy

We had been considering uniqueness of solutions to Navier-Stokes. We'll continue that today.

## 1 Uniqueness of Solutions

Suppose we have $\mathbf{u}, \tilde{\mathbf{u}}$ satisfying

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p &= \nu\triangle\mathbf{u} + \mathbf{f}, \\ \partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}}.\nabla\tilde{\mathbf{u}} + \nabla p &= \nu\triangle\tilde{\mathbf{u}} + \mathbf{f} \end{aligned}$$

and the divergence-free condition. Then we define $\mathbf{v}$ by

$$\mathbf{v}(\mathbf{x}, t) = \tilde{\mathbf{u}} - \mathbf{u}.$$

Clearly $\mathbf{v}$ is divergence-free, and now

$$\partial_t \mathbf{v} + \mathbf{v}.\nabla\mathbf{v} + \mathbf{u}.\nabla\mathbf{v} + \mathbf{v}.\nabla\mathbf{u} + \nabla\tilde{p} = \nu\triangle\mathbf{v}.$$

Dotting with $\mathbf{v}$ and integrating by parts yields

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{v}\|_2^2\right) + \int \mathbf{v}.(\nabla\mathbf{u}).\mathbf{v}\, d\mathbf{x} = -\nu\|\nabla\mathbf{v}\|_2^2 \leq 0.$$

Then

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|v\|_2^2\right) &\leq -\frac{\left[\int \mathbf{v}.(\nabla\mathbf{u}).\mathbf{v}\right]}{\mathbf{v}_2^2}\|\mathbf{v}\|_2^2 \\ &\leq \max_{\mathbf{u}}\left[-\frac{\int \mathbf{v}.(\nabla\mathbf{u}).\mathbf{v}}{\mathbf{v}_2^2}\right]\|\mathbf{v}\|_2^2 \\ &\leq \|\nabla\mathbf{u}(\cdot,t)\|_\infty\|\mathbf{v}\|_2^2 \end{aligned}$$

where

$$\|\nabla\mathbf{u}(\cdot,t)\|_\infty = \sup_{\mathbf{x}}\|\nabla\mathbf{u}(\mathbf{x},t)\|.$$

So by Gronwall,

$$\|\mathbf{v}(\cdot,t)\|_2 \leq \|\mathbf{v}(\cdot,0)\|_2 \exp\left(\int_0^t \|\nabla\mathbf{u}(\cdot,\mathbf{t}')\|_\infty\, dt'\right).$$

We need more regularity to guarantee a bound.

Here are some sufficient conditions for "everything to be nice"...

- $\int_0^t \|\nabla\mathbf{u}(\cdot,t')\|_2^4\, dt' < \infty$, or
- $\|\nabla\mathbf{u}(\cdot,t)\|_2^2 < \infty$.

If we could know either of these, then we could demonstrate both that the solution is $C^\infty((0,t))$ and that it is unique. Recall that $\|\nabla\mathbf{u}(\cdot,t)\|_2^2 = \|\omega\|_2^2$, so we could say the second condition is an "enstrophy condition." Let's look at the second condition some more.

## 2 The Enstrophy Condition

Navier-Stokes again:

$$\partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \nabla p = \nu \triangle \mathbf{u} + \mathbf{f}.$$

Now if we take the curl of Navier-Stokes we get

$$\partial_t \omega + \mathbf{u}.\nabla \omega = \nu \triangle \omega + \omega.\nabla \mathbf{u} + \nabla \times \mathbf{f}.$$

Of course we could do all this with Galerkin approximations, then we'd get

$$\partial_t \omega^N + \mathbf{u}^N.\nabla \omega^N = \nu \triangle \omega^N + \omega^N.\nabla \mathbf{u}^N + \mathbb{P}^N \left(\nabla \times \mathbf{f}\right)$$

and similarly for Navier-Stokes. As always, we could dot $\omega$ into the vorticity equation and integrate. Or we could dot $-\triangle \mathbf{u}$ into Navier-Stokes and integrate. (They are "morally" the same.) The second results in the equation

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_2^2 = -\nu \|\triangle \mathbf{u}\|_2^2 + \int \mathbf{u}.\left(\nabla \mathbf{u}\right).\triangle \mathbf{u},$$

after forgetting about the forcing term. Equivalently, we could write (after integration by parts)

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 = -\nu \|\nabla \omega\|_2^2 + \int \omega.\left(\nabla \mathbf{u}\right).\omega.$$

We'll continue with the first of these.

The analysis goes as follows. First,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_2^2 &= -\nu \|\triangle \mathbf{u}\|_2^2 + \int \mathbf{u}.\left(\nabla \mathbf{u}\right).\triangle \mathbf{u} \\
&\leq -\nu \|\triangle \mathbf{u}\|_2^2 + \|\mathbf{u}\|_\infty \|\nabla \mathbf{u}\|_2 \|\triangle \mathbf{u}\|_2
\end{aligned}$$

by Cauchy-Schwarz. We look to express $\|\mathbf{u}\|_\infty$ in terms of $\|\nabla \mathbf{u}\|_2, \|\triangle \mathbf{u}\|_2$.

**Proposition 1.** *If $\mathbf{u}$ is mean-zero, we have*

$$\|\mathbf{u}\|_\infty \leq C \|\nabla \mathbf{u}\|_2^{\frac{1}{2}} \|\triangle \mathbf{u}\|_2^{\frac{1}{2}}$$

*in $\mathbb{R}^3$.*

*Proof.* We write

$$\begin{aligned}
|\mathbf{u}(\mathbf{x})| &= \left| \sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}.\mathbf{x}} \hat{\mathbf{u}}(\mathbf{k}) \right| \\
&\leq \sum_{\mathbf{k}\neq 0} |\hat{\mathbf{u}}(\mathbf{k})| \\
&= \sum_{0<|\mathbf{k}|\leq K} |\hat{\mathbf{u}}(\mathbf{k})| + \sum_{|\mathbf{k}|>K} |\hat{\mathbf{u}}(\mathbf{k})|
\end{aligned}$$

for some $K$. Then

$$\begin{aligned}
|\mathbf{u}(\mathbf{x})| &\le \sum_{0<|\mathbf{k}|\le K} \frac{|\mathbf{k}|}{|\mathbf{k}|}|\hat{\mathbf{u}}(\mathbf{k})| + \sum_{|\mathbf{k}|>K} \frac{|\mathbf{k}|^2}{|\mathbf{k}|^2}|\hat{\mathbf{u}}(\mathbf{k})| \\
&\le \left(\sum_{0<|\mathbf{k}|\le K} \frac{1}{|\mathbf{k}|^2}\right)^{\frac{1}{2}} \left(\sum_{\mathbf{k}\neq 0} |\mathbf{k}|^2 |\hat{\mathbf{u}}(\mathbf{k})|^2\right)^{\frac{1}{2}} + \left(\sum_{|\mathbf{k}|>K} \frac{1}{|\mathbf{k}|^4}\right)^{\frac{1}{2}} \left(\sum_{\mathbf{k}\neq 0} |\mathbf{k}|^4 |\hat{\mathbf{u}}(\mathbf{k})|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{0<|\mathbf{k}|\le K} \frac{1}{|\mathbf{k}|^2}\right)^{\frac{1}{2}} \left(\frac{1}{L^3}\|\nabla\mathbf{u}\|_2^2\right)^{\frac{1}{2}} + \left(\sum_{|\mathbf{k}|>K} \frac{1}{|\mathbf{k}|^4}\right)^{\frac{1}{2}} \left(\frac{1}{L^3}\|\triangle\mathbf{u}\|_2^2\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{(2\pi)^3} \sum_{\frac{2\pi}{L}<|\mathbf{k}|\le K} \frac{1}{|\mathbf{k}|^2}\frac{(2\pi)^3}{L^3}\right)^{\frac{1}{2}} \|\nabla\mathbf{u}\|_2 + \left(\frac{1}{(2\pi)^3} \sum_{|\mathbf{k}|>K} \frac{1}{|\mathbf{k}|^4}\frac{(2\pi)^3}{L^3}\right)^{\frac{1}{2}} \|\triangle\mathbf{u}\|_2 \\
&\le \left(\frac{1}{(2\pi)^3} 4\pi \int_0^K \frac{k^2 dk}{k^2}\right)^{\frac{1}{2}} \|\nabla\mathbf{u}\|_2 + \left(\frac{1}{(2\pi)^2} 4\pi \int_K^\infty \frac{k^2 dk}{k^4}\right) \|\triangle\mathbf{u}\|_2 \\
&\le \frac{1}{\sqrt{2}\pi}\left(K^{\frac{1}{2}}\|\nabla\mathbf{u}\|_2 + \frac{1}{K^{\frac{1}{2}}}\|\triangle\mathbf{u}\|_2\right).
\end{aligned}$$

Note that in our setup "$dk$" $= 2\pi/L$, since we are in a box of side length $L$. Now we are free to choose positive $K$. So pick $K = \|\triangle\mathbf{u}\|_2/\|\nabla\mathbf{u}\|_2$, then

$$|\mathbf{u}(\mathbf{x})| \le \frac{2}{\sqrt{2}\pi}\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\triangle\mathbf{u}\|_2^{\frac{1}{2}}.$$

So take $C = 2/\left(\sqrt{2}\pi\right)$ to finish the proof. □

Since $2/\left(\sqrt{2}\pi\right) < 1$ we can now write

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\nabla\mathbf{u}\|_2^2\right) &\le -\nu\|\triangle\mathbf{u}\|_2^2 + \|\triangle\mathbf{u}\|_2^{\frac{3}{2}}\|\triangle\mathbf{u}\|_2^{\frac{3}{2}} \\
&\le \nu\left[-\|\triangle\mathbf{u}\|_2^2 + \frac{1}{\nu}\|\triangle\mathbf{u}\|_2^{\frac{3}{2}}\|\triangle\mathbf{u}\|_2^{\frac{3}{2}}\right] \\
&\le C\nu\frac{\|\nabla\mathbf{u}\|_2^6}{\nu^4} \\
&= \frac{C}{\nu^3}\|\nabla\mathbf{u}\|_2^6
\end{aligned}$$

for some constant $C$. Setting $X(t) = \|\nabla\mathbf{u}(\cdot,t)\|_2^2$, we have thus

$$\frac{dX}{dt} \le \frac{2C}{\nu^3}X^3.$$

Going forward, we write

$$\frac{d}{dt}X \le \left[\frac{2C}{\nu^3}(X(t))^2\right]\cdot X(t)$$

and hence

$$X(t) \le X(0) \cdot \exp\left(\frac{2C}{\nu^3}\int_0^t X(t')^2 \, dt'\right).$$

Hence

$$\|\nabla \mathbf{u}(\cdot,t)\|_2^2 \le \|\nabla \mathbf{u}(\cdot,0)\|_2^2 \cdot \exp\left(\frac{2C}{\nu^3}\int_0^t \|\mathbf{u}(\cdot,t')\|_2^4 \, dt'\right).$$

So if we knew $\|\mathbf{u}(\cdot,t')\|_2^4$ had only time-integrable singularities, then in fact $\|\nabla \mathbf{u}(\cdot,t)\|_2^2$ would be finite for all time.

We could also write

$$\begin{aligned} \frac{dX}{dt} &\le \frac{2C}{\nu^3}X^3 \\ \implies \int_{X(0)}^{X(t)} \frac{1}{X^3}\, dX &\le \int_0^t \frac{2C}{\nu^3}\, dt' \end{aligned}$$

and hence

$$\begin{aligned} -\frac{1}{2}\left(\frac{1}{X(t)}\right)^2 + \frac{1}{2}\left(\frac{1}{X(0)}\right)^2 &\le \frac{2C}{\nu^3}t \\ \implies -\left(\frac{X(0)}{X(t)}\right)^2 + 1 &\le \frac{4C}{\nu^3}(X(0))^2 t \\ \implies \left(\frac{X(0)}{X(t)}\right)^2 &\ge 1 - \frac{4C}{\nu^3}(X(0))^2 t \\ \implies \left(\frac{X(t)}{X(0)}\right)^2 &\le \frac{1}{1-\frac{4C}{\nu^3}(X(0))^2 t}. \end{aligned}$$

Therefore,

$$X(t) \le \frac{X(0)}{\sqrt{1-\frac{4C}{\nu^3}(X(0))^2 t}}.$$

Thus $X(t)$ is finite until time $t_* = \nu^3 / \left(4C(X(0))^2\right)$. We've just shown the following:

**Proposition 2.** *If $\|\nabla \mathbf{u}(\cdot,0)\|_2^2 < \infty$, then there exists $T > 0$ so that $\|\nabla \mathbf{u}(\cdot,t)\|_2^2 < \infty$ for $t < T$.*

We said earlier that $\|\nabla \mathbf{u}(\cdot,t)\|_2^2 < \infty$ is enough to guarantee smoothness of solutions, so if enstrophy is bounded at $t = 0$ then a smooth and unique solution exists for finite time. See §8 of the text for more details.