WHAT IS THE DUAL OF $C[0,1]$?

1. Orientation

The space $C[0,1]$ consists of the continuous real-valued functions defined on the unit interval. It is a vector space under pointwise addition and scalar multiplication, and is infinite dimensional since $x^n \in C[0,1]$ for every $n \in \mathbb{N}$. The *uniform norm* is defined on $C[0,1]$ as

$$||f|| = \sup_{0 \le x \le 1} |f(x)|, \quad f \in C[0,1].$$

Exercise 1.1. Prove $C[0,1]$ is complete as a metric space under the metric $d(f,g) = ||f-g||$. That is, show that if $f_n \to f$ uniformly then $f \in C[0,1]$.

We summarize the facts above by saying $C[0,1]$ is a *Banach space*, i.e. a normed linear space which is complete.

It turns out to be useful to consider elements of the dual space $(C[0,1])'$ which consists of the continuous linear functionals $l : C[0,1] \to \mathbb{R}$. Linearity requires

$$l(f+g) = l(f) + l(g), \quad \forall f,g \in C[0,1]$$
$$l(\lambda f) = \lambda l(f), \quad \forall f \in C[0,1], \ \lambda \in \mathbb{R}$$

and by continuity we mean

$$f_n \to f \implies l(f_n) \to l(f).$$

The convergence on the left is uniform, i.e. $||f_n - f|| \to 0$. The convergence on the right is that of a sequence of real numbers.

If l is linear, continuity at any one $f \in C[0,1]$ implies continuity on all of $C[0,1]$. In particular, a linear functional $l : C[0,1] \to \mathbb{R}$ is in the dual space $(C[0,1])'$ if and only if

$$f_n \to 0 \implies l(f_n) \to 0.$$

A sufficient condition for continuity is the existence of a positive constant M such that

$$|l(f)| \le M||f|| \quad \forall f \in C[0,1]. \tag{1.1}$$

If a linear functional satisfies this condition we say it is *norm-bounded* and we define its norm $||l||$ to be the smallest such constant M for which (1.1) holds.

Suppose now l is continuous at zero, then there is a ball about the origin such that $|l(f)| \le \epsilon$ whenever $||f|| \le \delta$. Given any $f \in C[0,1]$ we find

$$|l\left(\delta \frac{f}{||f||}\right)| \le \epsilon$$

and hence l is norm-bounded with $||l|| \le \epsilon/\delta$. We have proved the following

Proposition 1.2. *A linear functional $l : C[0,1] \to \mathbb{R}$ belongs to the dual space $(C[0,1])'$ if and only if it is norm-bounded.*

Let's see some examples.

Example 1.3. Fix $x_0 \in [0,1]$ and define the "Dirac mass" at x_0,

$$\delta_{x_0}(f) = f(x_0).$$

This is clearly in the dual space and has $||\delta_{x_0}|| = 1$.

Example 1.4. Given a sequence of points $x_i \in [0,1]$, $i \in \mathbb{N}$ along with absolutely summable weights a_i, define

$$l(f) = \sum_i a_i f(x_i).$$

This is linear and has $||l|| \leq \sum_i |a_i|$ so it is in the dual space. In fact $||l|| = \sum_i |a_i|$ as can be seen by considering f^n with $f^n(x_i) = \text{sign}(a_i)$, $i = 1, \ldots, n$.

Example 1.5. The Riemann integral is in the dual space. That is, the mapping

$$f \mapsto I(f) = \int_0^1 f\,dx$$

is linear and has $||I|| \leq 1$ by the triangle inequality for integration

$$|\int f\,dx| \leq \int |f|\,dx.$$

By choosing $f = 1$ we see $||I|| = 1$.

The next example is more complicated and involves defining a different type of integral known as the Lebesgue-Stieljies integral. It is worth understanding well.

2. The main example: Lebesgue-Stieljies integration

Given $\alpha \in BV[0,1]$ with $\alpha(0) = 0$ we define an integral via the following recipe. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$ and make the sum

$$\sum_{k=0}^{n-1} f(x_{k+1})(\alpha(x_{k+1}) - \alpha(x_k)).$$

If $f \in C[0,1]$ then as we refine the partition and take the mesh size $\delta \downarrow 0$, the sum converges to a number

$$\int_0^1 f\,d\alpha = \lim_{\delta \downarrow 0} \sum_{k=0}^{n-1} f(x_{k+1})(\alpha(x_{k+1}) - \alpha(x_k)).$$

Indeed if P_1, P_2 are two partitions they have a common refinement P, and if the mesh sizes δ_1, δ_2 are small enough then we can write

$$|\sum_{P_1} f(x_{k+1})\Delta\alpha_k - \sum_{P_2} f(x_{k+1})\Delta\alpha_k| \leq 2\epsilon \text{Var}(\alpha).$$

Now from our definition it is clear that the map $f \mapsto \mathcal{I}(f) = \int_0^1 f\,d\alpha$ is linear. Moreover if we define the running total variation of α as

$$|\alpha|(x) = \sup\left\{\sum_{k=0}^{n-1} |\alpha(x_{k+1}) - \alpha(x_k)|,\ 0 = x_0 < x_1 < \cdots < x_n = x\right\},$$

we find

$$|\int_0^1 f\,d\alpha| \leq \int_0^1 |f|\,d|\alpha| \leq \text{Var}(\alpha)\,||f||$$

so that $\|\mathcal{I}\| \leq \mathrm{Var}(\alpha)$ and $\mathcal{I} \in (C[0,1])'$. Again this inequality is actually equality, and this is a good exercise to work out before moving on.

Exercise 2.1. Prove $\|\mathcal{I}\| = \mathrm{Var}(\alpha)$. (Hint: consider non-decreasing α first.)

Exercise 2.2. Each of the examples from the previous section can be written in the form

$$l(f) = \int_0^1 f\, d\alpha$$

for some $\alpha \in BV$, $\alpha(0) = 0$. Work this out. Are the α's uniquely determined?

3. The Riesz theorem

A theorem due to Riesz asserts that this last example is generic.

Theorem 3.1 (Riesz)**.** *Given $l \in (C[0,1])'$ there exists $\alpha \in BV$, $\alpha(0) = 0$ so that*

$$l(f) = \int f\, d\alpha \quad \forall f \in C[0,1].$$

Exercise 3.2. Check that if $\alpha, \beta \in BV$, $\alpha(0) = \beta(0) = 0$ satisfy

$$\int f\, d\alpha = \int f\, d\beta \qquad \forall f \in C[0,1]$$

then $\alpha = \beta$ except at most countably many points. In fact, if α is non-decreasing prove

$$\alpha(x) \leq \inf_{f \geq 1_{[0,x]}} \int f\, d\alpha \leq \alpha\left(x^+\right).$$

For a general $\alpha \in BV$ argue similarly to conclude $\alpha(x^+) = \lim_{\delta \downarrow 0} \alpha(x+\delta)$ is determined uniquely by knowledge of all the integrals $\left\{\int f\, d\alpha,\ f \in C[0,1]\right\}$.

Discussion: If we assume α is right-continuous then it is uniquely determined, and indeed we can assume this at $x \in (0,1]$ without changing anything we've said already. However, we may not simultaneously assume α is right-continuous at 0 and $\alpha(0) = 0$ if we wish to reproduce the entire dual space. We chose $\alpha(0) = 0$ simply to clean up the definition of the Lebesgue-Stiejies integral.

We begin the proof of the theorem by discussing the positive linear functionals, i.e. those linear functionals l with the property that $l(f) \geq 0$ if $f \geq 0$.

Exercise 3.3. Show every positive linear functional $l : C[0,1] \to \mathbb{R}$ is automatically continuous.

Given a positive linear functional l define

$$\alpha(x) = \inf\left\{l(f) \,\middle|\, 1_{[0,x]} \leq f,\ f \in C[0,1]\right\}, \quad 0 < x \leq 1$$

and set $\alpha(0) = 0$. Then α is non-decreasing and it defines a Lebesgue-Stiejies integral. We will prove

$$l(f) = \int_0^1 f\, d\alpha \quad \forall f \in C[0,1].$$

Pick a partition $0 = x_0 < x_1 < \cdots < x_n = 1$ on which f is almost constant, so $|f(x) - f(x_k)| \leq \epsilon$ whenever $x_{k-1} \leq x \leq x_{k+1}$, $k = 1, \ldots, n-1$. Using the definition of α find functions

$$1_{[0,x_i]} \leq \psi_i \leq 1_{[0,x_{i+1}]}, \quad i = 1, \ldots, n-1$$

satisfying

$$\alpha(x_i) \le l(\psi_i) \le \alpha(x_i) + \frac{\epsilon}{n}.$$

If we take $\psi_0 = 0$, $\psi_n = 1$ we may define a partition of unity

$$\phi_i = \psi_i - \psi_{i-1}, \quad i = 1, \ldots, n$$

well-suited for approximation of f on the mesh. Then

$$|l(f) - \sum_{i=1}^{n} f(x_i)\, l(\phi_i)| \le ||l||\epsilon$$

and $||l|| < \infty$ since every positive linear functional on $C[0,1]$ is automatically continuous. Also

$$\sum_{i=1}^{n} f(x_i)\, l(\phi_i) = \sum_{i=0}^{n-1} f(x_{i+1})\,(\alpha(x_{i+1}) - \alpha(x_i)) + e(n)$$

where

$$|e(n)| \le ||f||\epsilon,$$

so that

$$|l(f) - \int_0^1 f\, d\alpha| \le (||l|| + ||f||)\,\epsilon.$$

Since this holds for all $\epsilon > 0$ we have the result.

Given a general $l \in (C[0,1])'$ we have to be more sly. First we define an auxiliary linear functional $|l|$ with the property that the linear functionals $|l| \pm l$ are positive. Then since $2l = (|l| + l) - (|l| - l)$ we finish by applying the proof above twice and superimposing the results.

Now if we expect $l = \int \cdot\, d\alpha$ then it is natural to go after $|l| = \int \cdot\, d|\alpha|$. (Remember $|\alpha|$ does not signify the usual absolute value but rather the running total variation.) Therefore we define

$$|\alpha|(x) = \inf_{y>x} \sup \left\{ \sum |l(\psi_i)| \,\Big|\, 1_{[0,x]} \le \sum \psi_i \le 1_{[0,y)},\ 0 \le \psi_i \le 1,\ \psi_i \in C[0,1] \right\}, \quad 0 < x \le 1$$

along with $|\alpha|(0) = 0$. Note $|\alpha|$ is non-decreasing and finite since

$$\sum |l(\psi_i)| = l\left(\sum \pm\psi_i\right) \le ||l||$$

whenever ψ_i are admissable. Thus $|\alpha|$ defines a Lebesgue-Stieljies integral

$$|l|(f) = \int f\, d|\alpha| \qquad f \in C[0,1],$$

and we claim

$$|l(f)| \le |l|(f) \quad \forall f \in C[0,1],\ f \ge 0.$$

As before, the heart of the proof is in setting up a useful partition of unity. Given $f \in C[0,1]$, begin by finding a partition $0 = x_0 < x_1 < \cdots < x_n = 1$ such that $|f(x) - f(x_k)| \le \epsilon$ whenever $x_{k-1} \le x \le x_{k+1}$, $k = 1, \ldots, n-1$. Then find $y_k \in (x_k, x_{k+1})$ and admissable $\psi_{k,i}$ with

$$1_{[0,x_k]} \le \sum_i \psi_{k,i} \le 1_{[0,y_k)}, \quad k = 1, \ldots, n-1$$

so that

$$|\alpha|(x_k) - \frac{\epsilon}{n+1} \le \sum_i |l(\psi_{k,i})| \le |\alpha|(x_k) + \frac{\epsilon}{n+1}.$$

From the definition of $|\alpha|$ it is clear we can pick the $\psi_{k,i}$ inductively to include the previously chosen $\psi_{k-1,i}$ along with new admissable functions supported on $[x_{k-1}, y_k)$. Indeed if we have $\psi_{k-1,i}$ already and $\psi_{k,i}$ are proposed, and if we call $\Psi_{k-1} = \sum_i \psi_{k-1,i}$, we can break up each proposed function into two pieces

$$\psi_{k,i} = \psi_{k,i}\left(1_{[0,y_k)} - \Psi_{k-1}\right) + \psi_{k,i}\Psi_{k-1}$$

and make a new proposed family consisting of the individual pieces. This new family is still admissable and gets a larger value in the supremum part of the definition of $|\alpha|$. So we may assume the $\psi_{k,i}$ come to us already with support either on $[0, y_{k-1})$ or on $[x_{k-1}, y_k)$, and then we clearly get a better approximation to the supremum by replacing those $\psi_{k,i}$ supported on $[0, y_{k-1})$ by the already chosen $\psi_{k-1,i}$.

Having built up $\psi_{k,i}$ as described above we are ready to build the partition of unity. Take $\Psi_0 = 0$, $\Psi_n = 1$, and $\Psi_k = \sum_i \psi_{k,i}$ for $k = 1, \ldots, n$, then the relevant partition of unity is given by

$$\Phi_k = \Psi_k - \Psi_{k-1}, \quad k = 1, \ldots, n.$$

Note for fixed $k = 1, \ldots, n$ we have

$$l\left(\Psi_k - \Psi_{k-1}\right) = \sum_{\text{supp}(\psi_{k,i}) \subset [x_{k-1}, y_k)} l\left(\psi_{k,i}\right)$$

so that

$$\left|l\left(\Phi_k\right)\right| \le |\alpha|\left(x_k\right) - |\alpha|\left(x_{k-1}\right) + \frac{2\epsilon}{n+1}.$$

Then since

$$\left|l(f) - \sum_{i=1}^{n} f\left(x_i\right) l\left(\Phi_i\right)\right| \le ||l||\epsilon,$$

we conclude for $f \ge 0$ that

$$\begin{aligned}
|l(f)| &\le \sum_{i=0}^{n-1} f\left(x_{i+1}\right)\left|l\left(\Phi_{i+1}\right)\right| + ||l||\epsilon \\
&\le \sum_{i=0}^{n-1} f\left(x_{i+1}\right)\left(|\alpha|\left(x_{i+1}\right) - |\alpha|\left(x_i\right)\right) + \left(||l|| + 2\right)\epsilon.
\end{aligned}$$

Refining the partition and taking $\epsilon \downarrow 0$ we conclude

$$|l(f)| \le |l|(f)$$

whenever $f \ge 0$, $f \in C[0,1]$.

Now we know the linear functionals $|l| \pm l$ are positive, hence we may apply the first half of the proof to find $\alpha_+, \alpha_- \in BV$ with

$$\begin{aligned}
\frac{1}{2}\left(|l| + l\right)(f) &= \int f \, d\alpha_+ \\
\frac{1}{2}\left(|l| - l\right)(f) &= \int f \, d\alpha_-.
\end{aligned}$$

Finally we have

$$l(f) = \int f \, d\left(\alpha_+ - \alpha_-\right), \quad \forall f \in C[0,1]$$

and we're done.