INTRO TO MATH ANALYSIS I

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Topics for the Course. Functions of one variable. Limits and continuity. Derivatives. Riemann integral. Infinite series and integrals. Absolute and uniform convergence. Taylor series. Infinite series of functions. Fourier series.

Reference Texts.

- *Advanced Calculus*, Friedman
- *An Introduction to Analysis*, Wade
- *Principles of Mathematical Analysis*, Rudin
- *Differential and Integral Calculus*, Courant

Recitations.

(1) Number systems: $\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R}$. Irrationality of $\sqrt{2}$. Rational roots theorem. Induction. Examples included

$$1+2+3+\cdots+n=\frac{n\left(n+1\right)}{2}$$
$$1+z+z^{2}+\cdots+z^{n}=\frac{z^{n+1}-1}{z-1}.$$

Density of rationals via

$$\left|\theta-\frac{m}{n}\right|\leq\frac{1}{nN}.$$

(2) Review of Dedekind cuts. Definition of $\sqrt{2}$ as

$$\sqrt{2}=\sup\left\{ q\in\mathbb{Q}\,\middle|\,q^{2}<2\right\} .$$

Countable and uncountable infinity. Exercises:

(a) $|\mathbb{N}|=|\text{evens}|=|\text{odds}|$
(b) $|\mathbb{N}|=|\mathbb{Q}|$
(c) $|\mathbb{N}|<|\mathbb{R}|$ (diagonal argument)
(d) $|X|=|X\times X|=|X\times\cdots\times X|$
(e) $|X|<|X\times X\times\cdots|$
(f) $|X|<|2^{X}|$ (diagonal argument)

Proof of Bolzano-Weierstrass by bisection and finite intersection property. Notions of open, closed sets. Homework: Let $a_n\geq0$ and prove

$$\liminf\,\frac{a_{n}}{a_{n-1}}\leq\liminf\,\left(a_{n}\right)^{1/n}\leq\limsup\,\left(a_{n}\right)^{1/n}\leq\limsup\,\frac{a_{n}}{a_{n-1}}. \tag{0.1}$$

(3) Discussed homework 2. Example problem: Prove that

$$\sup\left\{ \sin\left(x^{-1}\right)\,\middle|\,x\in\mathbb{Q}\backslash\left\{ 0\right\} \right\} =1.$$

Solution: We can relabel and consider

$$E=\left\{ \sin\left(x\right)\,\middle|\,x\in\mathbb{Q}\backslash\left\{ 0\right\} \right\} .$$

Since $\sin(x) \leq 1$ for all x we have $\sup E \leq 1$. Now for any $n \in \mathbb{N}$ we can find $x \in E$ with $|x - \pi/2| \leq n^{-1}$. Then

$$|1 - \sin(x)| \leq |x - \pi/2| \leq n^{-1}$$

which proves $\sup E \geq 1$. The inequality used above follows from the inequality $|\sin(x)| \leq |x|$ whose proof is self-evident from a picture of a right triangle with angle x inside a circle. Hints for problem (0.1): taking the log and relabeling as $\log a_n - \log a_{n-1} = b_n$ the statement is equivalent to

$$\liminf b_n \leq \liminf \frac{1}{n}\sum_{i=1}^{n} b_i \leq \limsup \frac{1}{n}\sum_{i=1}^{n} b_i \leq \limsup b_n.$$

(4) Discussed homework 3. Practice problems:
 (a) Suppose E is continuous and satisfies

$$E(a)\,E(b) = E(a+b)$$

 for all $a, b \in \mathbb{R}$. What is E?
 (b) Let f be given by

$$f(x) = \begin{cases} \frac{1}{n} & x = m/n,\ (m,n) = 1,\ x \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

 What is the largest subset of $\mathbb{R}$ on which f is continuous?

Solutions:
 (a) Claim:

$$E(x) = a^x,\ a = E(1).$$

 Let $m/n \in \mathbb{Q}_+$ then

$$E\left(\frac{m}{n}\right) = E\left(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ terms}}\right) = E\left(\frac{1}{n}\right)^m.$$

 We can take $m = n$ here to find

$$E\left(\frac{1}{n}\right) = E(1)^{1/n},$$

 hence

$$E(m/n) = E(1)^{m/n}.$$

 Let $m_k/n_k \to x$, then by continuity

$$E(x) = E(1)^x$$

 for $x \in \mathbb{R}_+ \cup \{0\}$. Since

$$E(-x)\,E(x) = E(0) = 1$$

 we have

$$E(-x) = E(x)^{-1} = E(1)^{-x}$$

 for $x \in \mathbb{R}_+$, hence

$$E(x) = E(1)^x$$

 for all x.

(b) f is discontinuous on $\mathbb{Q}\setminus\{0\}$, for the irrationals are dense in $\mathbb{R}$. Claim: f is continuous on $\mathbb{Q}^c\cup\{0\}$. Let us prove it at $x=0$, then it will be clear how to do the rest. W.l.o.g. take a test sequence $x_k = m_k/n_k$, $(m_k, n_k) = 1$. If $f(x_k) \not\to 0$ then $|1/n_k| \geq 1/N$ for some $N \in \mathbb{N}$ and infinitely many k. Then $|x_k| \geq 1/N$ for infinitely many k, which proves $x_k \not\to 0$. The contrapositive is what we're after.

Extra exercise: Prove $x_n \to x$ iff for every $\epsilon > 0$ only finitely many n have $|x_n - x| \geq \epsilon$. Conclude if $x_n \to x$ then for every permutation $\pi : \mathbb{N} \to \mathbb{N}$ we have $x_{\pi(n)} \to x$. It is well-known that if $\sum_n |x_n| < \infty$ and $\sum_n x_n = x$ then $\sum_n x_{\pi(n)} = x$ for every permutation π. Deduce this from the result above. We'll see later that if $\sum_n x_n = x$ but $\sum_n |x_n| = \infty$, then for every $y \in \mathbb{R}\cup\{\pm\infty\}$ there exists a permutation π with $\sum_n x_{\pi(n)} = y$.

(5) Discussed homework 4.

(6) Discussed homework 5. Intro to convexity. A function is *convex* if for all $x, y \in \mathbb{R}$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

whenever $t \in [0,1]$. Examples: lines, $|x|$, x^2. If $f'' \geq 0$ then f is convex. Pf: $f'' \geq 0$ implies f' is non-decreasing, then by mean value theorem

$$f(y) - f(x) = f'(\xi)(y-x) \geq f'(x)(y-x)$$

and hence

$$f(y) \geq f(x) + f'(x)(y-x) \quad \forall\, x, y \in \mathbb{R}.$$

Exercise: this is convexity. Now check $-\log(x)$ is convex, and deduce the arithmetic-geometric inequality

$$(a_1 \cdots a_n)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$

whenver $a_i \geq 0$. As a result, the largest volume box with fixed total edge length is a cube. If $B = \Pi_{i=1}^{n}[x_i, x_i + l_i]$ then

$$(\mathrm{Vol}\, B)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} l_i$$

with equality iff $l_1 = \cdots = l_n$. Exercise: deduce the claim about equality from the fact that $-\log(x)$ is strictly convex, i.e.

$$\log\left(\frac{x+y}{2}\right) = \frac{1}{2}(\log x + \log y) \implies x = y.$$

(7) The practice midterm is included at the end along with solutions.

(8) Suppose f is conditionally but not absolutely integrable on $(0,1]$, e.g. $f = \sin(x)/x$. Let $f^+ = f \cdot 1_{f>0}$ and $f^- = f \cdot 1_{f<0}$, and note $f = f^+ - f^-$. Fact: Under the given assumptions, for every $\alpha \in \mathbb{R}$ there exist sequences $a_n, b_n \downarrow 0$ such that

$$\lim_n \int_{a_n}^{1} f^+ - \int_{b_n}^{1} f^- = \alpha.$$

Interpretation: The notion of (signed) area under the graph of a non-absolutely integrable f does not make sense. This is simply a restatement of the following theorem. Hints are given for the proof.

Theorem. (Riemann) Suppose $\sum_n x_n = x \in \mathbb{R}$ but $\sum_n |x_n| = \infty$. Then for every $y \in \mathbb{R}$ there is a permutation π with $\sum_n x_{\pi(n)} = y$.

Proof. Suppose we want a rearragement whose partial sums $S_N = \sum_1^N x_{\pi(n)}$ satisfy

$$\alpha = \liminf S_n \leq \limsup S_n = \beta$$

for given $\alpha, \beta \in \mathbb{R}$.

Exercise 1. Deduce $\sum_n x_n^+ = \sum_n x_n^- = \infty$ from the hypotheses, so we have infinite positive and negative mass to play with.

W.l.o.g. suppose no $x_n = 0$, and relabel the positive and negative parts as their own sequences, $p_i > 0$ and $q_i < 0$. Now pick up just enough positive mass to get

$$\sum_1^{n_1} p_i > \beta, \text{ but } \sum_1^{n_1 - 1} p_i \leq \beta.$$

The positive terms $p_1, p_2, \ldots, p_{n_1}$ form the first n_1 terms in the rearranged sum. Next find m_1 such that

$$S_{n_1} + \sum_1^{m_1} q_i < \alpha, \text{ but } S_{n_1} + \sum_1^{m_1 - 1} q_i \geq \alpha.$$

The negative terms $q_1, \ldots, q_{m_1}$ form the $n_1 + 1$ through $n_1 + m_1$ terms in the rearranged sum. Continue in this way, repeatedly adding just enough positive mass to get above β and then just enough negative mass to get below α. You'll use up all the original x_n in this process.

Exercise 2. Check that the estimates

$$\begin{aligned} \alpha + q_{m_k} \leq S_{\sum_{i=1}^k n_i + m_i} &< \alpha \\ \beta < S_{n_{k+1} + \sum_{i=1}^k n_i + m_i} &\leq \beta + p_{n_{k+1}} \end{aligned}$$

hold for $k \in \mathbb{N}$. Observe $x_n \to 0$ as $n \to \infty$ and deduce the result.

Exercise 3. Strengthen the result to allow $\alpha = -\infty$, $\beta = +\infty$.

Exercise 4. On the other hand, if $\sum |x_n| < \infty$ then every rearrangement gives the same sum.

Likewise, if f is absolutely integrable then the signed area of its graph can be properly defined via limits of integrals. The point is that now it doesn't matter what limits you take.

The next result gives a complete characterization of Riemann integrability. Follow the exercises to discover a proof.

Theorem. (Lebesgue) A bounded function f on $[0, 1]$ is Riemann integrable iff $\big|\{x \mid f \text{ is discontinuous at } x\}\big| = 0$.

Exercise 5. (warmup to measure zero) Recall $A \subset \mathbb{R}$ is said to be *measure zero*, written $|A| = 0$, if for every $\epsilon > 0$ there is a collection of open intervals $\{(t_i, t_{i+1})\}$ with lengths $l_i = t_{i+1} - t_i$ such that $\sum_i l_i < \epsilon$. Check that

(a) points, finite collections of points, and countable collections of points are all measure zero
(b) countable unions of measure zero sets are measure zero
(c) subsets of measure zero sets are measure zero
(d) the set of discontinuities of a monotone function is measure zero
(e) the set of discontinuities of a BV function is measure zero

Proof of the theorem. Define the oscillation on a set $[a,b]$ by

$$\operatorname*{osc}_{[a,b]} f = \sup_{[a,b]} f - \inf_{[a,b]} f,$$

and the oscillation at a point as

$$O_f(x) = \lim_{\epsilon \downarrow 0} \operatorname*{osc}_{[x-\epsilon, x+\epsilon]} f.$$

Exercise 6. Prove $O_f(x) = 0$ iff f is continuous at x.

Now suppose $|\{O_f \neq 0\}| = 0$ then by exercise 5c, $|\{O_f \geq 1/n\}| = 0$ for each n. The next step is to exhibit a partition $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that

$$\sum_i \operatorname*{osc}_{[t_i, t_{i+1}]} f \cdot \Delta t_i \leq 2|f|_\infty \frac{1}{n} + \frac{1}{n}.$$

This is possible because

Exercise 7. $\{O_f \geq 1/n\}$ is compact. Hint: On $\mathbb{R}$ compact = closed + bounded. Check $\{O_f < 1/n\}$ is open.

So first cover the compact set $\{O_f \geq 1/n\}$ with open intervals with total length $\leq 1/n$, and then pass to a finite sub-cover. Disjointize the cover and make a partition out of it, and the above inequality follows immediately. Finish by taking $n \to \infty$.

On the other hand, by exercise 5b if $|\{O_f \neq 0\}| \geq \alpha > 0$ then $\{O_f \geq 1/n\} \geq \alpha > 0$ for some $n \in \mathbb{N}$. Then given any partition $t_0 = 0 < t_1 < \cdots < t_n = 1$ we have

$$\sum_i \operatorname*{osc}_{[t_i, t_{i+1}]} f \cdot \Delta t_i \geq \frac{\alpha}{n} > 0$$

so that f is not Riemann integrable.

Practice Midterm

Problem 1. Prove
$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

Problem 2. Let $\{a_n\}$ be a non-negative sequence satisfying
$$a_{m+n} \le a_m + a_n \quad \forall m,n \in \mathbb{N}.$$
Prove
$$\lim_{n\to\infty}\frac{a_n}{n} = \inf_{n\in\mathbb{N}}\left\{\frac{a_n}{n}\right\}.$$
(Hint: $m = pn + r$.)

Problem 3. Consider the circle $S^1 = \left\{(x,y) \,\middle|\, x^2+y^2 = 1,\ x,y \in \mathbb{R}\right\}$ along with the distance
$$d\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) = \sqrt{\left(x_2-x_1\right)^2 + \left(y_2-y_1\right)^2}.$$
We call $f : S^1 \to \mathbb{R}$ *continuous* if
$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. } d\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) < \delta \implies \left|f\left(x_1,y_1\right) - f\left(x_2,y_2\right)\right| < \epsilon.$$
We call $g : \mathbb{R} \to S^1$ *continuous* if
$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. } |x-y| < \delta \implies d\left(g\left(x\right), g\left(y\right)\right) < \epsilon.$$

a) Let $f : S^1 \to \mathbb{R}$ and $g : \mathbb{R} \to S^1$ be continuous. Prove $f \circ g : \mathbb{R} \to \mathbb{R}$ is continuous.

b) We can write $g : \mathbb{R} \to S^1$ as
$$g\left(x\right) = \left(g_1\left(x\right), g_2\left(x\right)\right)$$
where $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ and $g_1^2 + g_2^2 = 1$. Prove g is continuous iff g_1, g_2 are continuous.

c) Let $f : S^1 \to \mathbb{R}$ be continuous. Prove there exists $(x,y) \in S^1$ such that
$$f\left(x,y\right) = f\left(-x,-y\right).$$

Problem 4. If $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$\left|f\left(x\right) - f\left(y\right)\right| \le \alpha|x-y| \quad \forall x,y \in \mathbb{R}$$
for some $\alpha \in (0,1)$, then we call f a *contraction*. Let f be a contraction and prove there exists a unique fixed point
$$f\left(x\right) = x$$
by completing the outline below.

a) Let $x_0 \in \mathbb{R}$ and define the recursive sequence
$$x_{n+1} = f\left(x_n\right), \quad \forall n \in \mathbb{N}.$$
Prove
$$|x_{n+1} - x_n| \le \alpha |x_n - x_{n-1}|.$$

b) Let $m > n$ and prove
$$|x_m - x_n| \le \sum_{k=n}^{m-1} \alpha^k |x_1 - x_0|.$$

c) Deduce $\{x_n\}$ is Cauchy and conclude the result.

Solutions

Problem 1. Taking logs the general term is

$$n\log\left(1+\frac{x}{n}\right)=x\cdot\frac{\log\left(1+\frac{x}{n}\right)-\log(1)}{\frac{x}{n}}\to x\,\frac{d}{dt}\log t\bigg|_{t=1}=x.$$

Problem 2. For $m>n$ write $m=pn+r$ with $0\leq r<n$, then

$$\frac{a_m}{m}\leq\frac{p}{m}a_n+\frac{a_r}{m}\leq\frac{a_n}{n}+\frac{M}{m}$$

where $M=\max_{0\leq r<n}\{a_r\}$. Thus

$$\limsup\frac{a_m}{m}\leq\frac{a_n}{n}\quad\forall n$$

and

$$\limsup\frac{a_m}{m}\leq\inf\frac{a_n}{n}\leq\liminf\frac{a_n}{n}.$$

Problem 3. a) Fix $\epsilon>0$, and find $\delta_1>0$ so that

$$d\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)<\delta_1\implies\left|f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right|<\epsilon.$$

Find $\delta_2>0$ so that

$$|x-y|<\delta_2\implies d\left(g\left(x\right),g\left(y\right)\right)<\delta_1.$$

Then

$$|x-y|<\delta_2\implies\left|f\left(g\left(x\right)\right)-f\left(g\left(y\right)\right)\right|<\epsilon.$$

b) Define $\pi_x\left(x,y\right)=x$ and $\pi_y\left(x,y\right)=y$. These are continuous as maps $\pi_x,\pi_y:S^1\to\mathbb{R}$. Since

$$g_1=\pi_x\circ g\text{ and }g_2=\pi_y\circ g$$

part (a) gives the result. Similarly the equation

$$g\left(x\right)=\left(g_1\left(x\right),0\right)+\left(0,g_2\left(x\right)\right)$$

shows us how to prove the converse.

c) Define $\phi:[0,\pi]\to S^1$ by

$$\phi\left(\theta\right)=\left(\cos\theta,\sin\theta\right)$$

and check it is continuous. Then the composition $f\circ\phi:[0,\pi]\to\mathbb{R}$ is continuous and therefore so is $f\circ\phi\left(\theta\right)-f\circ\phi\left(\theta+\pi\right)$. If

$$0=f\circ\phi\left(0\right)-f\circ\phi\left(0+\pi\right)$$

then we are done. If not, then either

$$0<f\circ\phi\left(0\right)-f\circ\phi\left(\pi\right)\text{ and }0>f\circ\phi\left(\pi\right)-f\circ\phi\left(0\right)$$

or

$$0>f\circ\phi\left(0\right)-f\circ\phi\left(\pi\right)\text{ and }0<f\circ\phi\left(\pi\right)-f\circ\phi\left(0\right).$$

In either case intermediate value theorem gives the result.

Problem 4. a) Compute

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq \alpha |x_n - x_{n-1}|.$$

b) Repeated application of (a) gives

$$|x_{n+1} - x_n| \leq \alpha^n |x_1 - x_0|.$$

Therefore by triangle inequality

$$|x_m - x_n| \leq \sum_{k=n}^{m-1} |x_{k+1} - x_k| \leq \sum_{k=n}^{m-1} \alpha^k |x_1 - x_0|.$$

c) We have

$$\sum_{k=n}^{m-1} \alpha^k = \alpha^n \sum_{k=0}^{m-n-1} \alpha^k = \alpha^n \frac{1 - \alpha^{m-n}}{1 - \alpha} = \frac{\alpha^n - \alpha^m}{1 - \alpha}.$$

Therefore

$$|x_m - x_n| \leq \frac{\alpha^n - \alpha^m}{1 - \alpha} |x_1 - x_0| \to 0$$

as $n, m \to \infty$ which proves $\{x_n\}$ is Cauchy. Hence $x_n \to x \in \mathbb{R}$ and then since f is continuous

$$f(x) = f\left(\lim_n x_n\right) = \lim_n f(x_n) = \lim_n x_{n+1} = x.$$

This proves the existence of a fixed point. For uniqueness, if x, y are two fixed points then since f is a contraction,

$$|x - y| \leq \alpha |x - y|$$

with $\alpha < 1$. This can only happen if $x = y$.